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ABOUT THE FIRST AND THE SECOND CONSTELLATIONS BELONGING TO THE TOPOLOGY U.

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INTRODUCTION

Let us suppose that a mapping $u: 2^P \to 2^P$ is given such that the image of the empty set is the empty set, $M \subset uM$ for each $M \subset P$ and $uM_1 \subset uM_2$ for each $M_1 \subset M_2 \subset P$. Then the pair (P, u) is called a topological space (Čech [1]). Let $\mathcal{T}(P)$ be the set of all topologies in P. The set $\mathcal{T}(P)$ is partially ordered in a natural way (cf. section 2.1.). If $u, v \in \mathcal{T}(P)$, let us take $(uv)\ M = u(vM)$ for $M \subset P$. Then $uv \in \mathcal{T}(P)$ and with regard to the partial ordering and the multiplication introduced in this way $\mathcal{T}(P)$ is a partially ordered semigroup with the identity.

In [2], for each $u \in \mathcal{T}(P)$ two new topologies s_u, $\sigma_u \in \mathcal{T}(P)$ are constructed, which are called the first and the second constellations belonging to the topology u. The topologies s_u, σ_u were investigated in the paper [2] of Koutský and Sekanina.

Let us suppose that u is a fixed element of the set $\mathcal{T}(P)$. In this paper we investigate the relations between the topologies u, s_u, σ_u with regard to the ordering. Further we investigate the conditions for the validity of the equations $r = s$ and $r \cdot s = t$ where $\{r, s, t\}$ is an arbitrary subset of the set $\{u, s_u, \sigma_u\}$. One of the considered relations (the equality $\sigma_u = u$) was examined in the paper [2].

§1. BASIC NOTIONS

In this paragraph we shall introduce some definitions and lemmas which can be found in the paper [2] and which we shall need.

1.1 By a topological space a topological space in Čech's sense from the year 1937 is meant, i.e. such a space (P, u), in which following axioms for the topology u are satisfied:

(1) $u\emptyset = \emptyset$,

(2) $M \subset P \Rightarrow M \subset uM$,

(3) $M_1 \subset M_2 \subset P \Rightarrow uM_1 \subset uM_2$.

1.2. A set $0 \subset P$ is said to be a neighborhood of a set $M \subset P$, if $M \cap u(P - 0) = \emptyset$. For every neighborhood 0 of the set M it is $M \subset 0$. By a neighborhood of a point x a neighborhood of the set (x) is meant.

If a topological space (P, u) is given, we shall denote the system of all neighborhoods of a point x and of the set M in (P, u) as $\mathscr{D}_u(x)$ and $\mathscr{D}_u(M)$, respectively.

Following theorem is valid:

$x \in uM$ *if and only if* $M \cap 0 \neq \emptyset$ *for each* $0 \in \mathscr{D}_u(x)$ $(x \in P,\ M \subset P)$.

Proof. Let $x \notin uM$. Then $P - M \in \mathscr{D}_u(x)$ and $M \cap (P - M) = \emptyset$. Let there exists $0 \in \mathscr{D}_u(x)$ such that $M \cap 0 = \emptyset$. Then $0 \subset P - M$, hence $P - M \in \mathscr{D}_u(x)$, i.e. $x \notin uM$.

1.3. We can introduce a topology u into a set P by assigning to each $x \in P$ such a system $\mathscr{D}(x)$ of subsets of P that following axioms are satisfied:

(1) $x \in P \Rightarrow \mathscr{D}(x) \neq \emptyset$, (2) $x \in P,\ 0 \in \mathscr{D}(x) \Rightarrow x \in 0$.

The closures we define then as follows:

$$x \in uM \Leftrightarrow M \cap 0 \neq \emptyset \text{ for every set } 0 \in \mathscr{D}(x).$$

1.4. If a topological space (P, u) is given, the intersection of all neighborhoods of M in (P, u) is said to be the first constellation of M in (P, u) and is denoted by $s_u M$. The first constellation of a set (x) is also called the first constellation of a point x.

Following theorems for the first constellation are valid:

(1) $s_u \emptyset = \emptyset$,

(2) $M \subset P \Rightarrow M \subset s_u M$,

(3) $M_i \subset P$ for every $i \in I \Rightarrow \bigcup_{i \in I} s_u M_i = s_u(\bigcup_{i \in I} M_i)$,

(4) if $x \in P$, $M \subset P$, then $x \in s_u M \Leftrightarrow M \cap u(x) \neq \emptyset$.

From the conditions (1) — (3) it follows that (P, s_u) is a topological space. The condition (4) implies for example: $x \in s_{\hat{u}}(y) \Leftrightarrow y \in u(x)$ $(x, y \in P)$.

1.5. The intersection of the closures of all neighborhoods of M in (P, u) is called the second constellation of M in (P, u) and is denoted as $\sigma_u M$. Instead of a second constellation of a set (x) we speak about a second constellation of the point x.

Following theorems are valid (cf. [2]):

(1) $\sigma_u \emptyset = \emptyset$,

(2) $M \subset P \Rightarrow uM \subset \sigma_u M$,

(3) $M_1 \subset M_2 \subset P \Rightarrow \sigma_u M_1 \subset \sigma_u M_2$,

(4) if $x \in P$, $M \subset P$, then $x \in \sigma_u M \Leftrightarrow M \cap u0 \neq \emptyset$ for every $0 \in \mathscr{D}_u(x)$.

From the conditions (1) — (3) it follows that (P, σ_u) is a topological space.

§2. THE PARTIAL ORDERING AND THE EQUALITIES $r = s$.

We shall introduce a partial ordering into the set $\mathcal{T}(P)$ of all topologies in P. Afterwards we shall investigate the relations between the topologies u, s_u, σ_u, for a fixed topology u, in the partially ordered set $\mathcal{T}(P)$ and the conditions for the topology u under which these relations between u, s_u, σ_u hold.

2.1. Into the set $\mathcal{T}(P)$ of all topologies in P one can introduce the partial ordering as follows:

if u, v are topologies in P, we put $u \leqq v$ iff $uM \subset vM$ for each set $M \subset P$.

2.2. It holds:

(1) $u \leqq \sigma_u$, (2) $s_u \leqq \sigma_u$,

(3) *for the topologies u, s_u either $s_u \leqq u$ or $s_u \parallel u$ takes place.*

Proof. From the property (2) of the topology σ_u (cf. section 1.5.) $u \leqq \sigma_u$ follows. The inequality $s_u \leqq \sigma_u$ is evident. Let us suppose that $s_u > u$, i. e. $s_u M \supset uM$ for each set $M \subset P$ but there exists a set M_1 so that $s_u M_1 \supsetneqq uM_1$. Then there exists $x_1 \in s_u M_1 - uM_1$. Since $x_1 \in s_u M_1 = \bigcup\limits_{m \in M_1} s_u(m)$, there exists $m_1 \in M_1$ so that $x_1 \in s_u(m_1)$. From $x_1 \notin uM_1$ we have $x_1 \notin u(m_1)$. Hence $x_1 \in s_u(m_1) - u(m_1)$. According to 1.4. $m_1 \in u(x_1) - s_u(x_1)$, which is a contradiction.

2.3. *For the topologies u, s_u there holds: $s_u \leqq u$ iff $u(x) \subset s_u(x)$ for each $x \in P$.*

Proof. Let $s_u \leqq u$. Let us take $0 \in \mathcal{D}_u(x)$. It is necessary to prove that $u(x) \subset 0$. For each $y \in P$ and $M \subset P$ there holds that if $y \notin uM$, then $y \notin s_u M$, hence if there exists a neighborhood U of the point y so that $U \cap M = \emptyset$, then $M \cap u(y) = \emptyset$. Let us put $y = x$, $M = P - 0$. It is $0 \cap (P - 0) = \emptyset$, hence $(P - 0) \cap u(x) = \emptyset$. From this we get $u(x) \subset 0$.

Let conversely be $u(x) \subset s_u(x)$ for each $x \in P$. Let us take $y \in P$, $M \subset P$ and let $y \notin uM$, hence there exists a neighborhood U of y so that $U \cap M = \emptyset$. Since $u(y) \subset U$, we have also $u(y) \cap M = \emptyset$, i.e. $y \notin s_u M$.

From 2.3. we get as a consequence:

2.4. For the topologies u, s_u the following conditions are equivalent:

(1) $s_u \leqq u$,

(2) $s_u(x) = u(x)$ for each $x \in P$,

(3) for each $x \in P$ the intersection of all neighborhoods of the point x is $u(x)$.

2.5. A topology u, for which $s_u(x) = u(x)$ holds for each $x \in P$, is called in the paper [2] B^*-topology.

It is easy to verify that the following conditions are equivalent (cf. [2]):

(1) u is a B^*-topology,

(2) $y \in s_u(x) \Leftrightarrow x \in s_u(y)$,

(3) $y \in u(x) \Leftrightarrow x \in u(y)$.

From 2.4. we get as a further consequence:

2.6. It holds: $u \parallel s_u \Leftrightarrow u$ is not a B^*-topology.

2.7 *The equality $s_u = u$ is valid if and only if u is a B^*-topology and $u\left(\bigcup_{i \in I} M_i\right) = \bigcup_{i \in I} uM_i \; (M_i \subset P)$.*

Proof. Let $s_u = u$. Then by 2.4. we obtain that u is a B^*-topology. The inclusion $\bigcup_{i \in I} uM_i \subset u\left(\bigcup_{i \in I} M_i\right)$ evidently holds. Let conversely $x \in u\left(\bigcup_{i \in I} M_i\right)$. Then $x \in s_u\left(\bigcup_i M_i\right)$ and from there $x \in s_u(m)$ for some $m \in \bigcup M_i$. From the relation $x \in s_u(m)$ we get $x \in u(m) \subset uM_i$ for some $i \in I$.

Let conversely u be a B^*-topology so that $u\left(\bigcup_i M_i\right) = \bigcup_i uM_i$. Then for each $x \in P$ it is $s_u(x) = u(x)$. Let M ($\subset P$) contain more than one element. Then $s_uM = \bigcup_{m \in M} s_u(m) = \bigcup_{m \in M} u(m) = uM$.

We are going to prove a lemma:

2.8. *For the topology u the following conditions are equivalent:*

(1) *Each $x \in P$ has the least neighborhood U_x in (P, u).*

(2) *For each system $\{M_i\}_{i \in I}$ of subsets of P $u\left(\bigcup_i M_i\right) = \bigcup_i uM_i$ holds.*

Proof. Let the condition (1) be valid. Evidently for each system $\{M_i\}_{i \in I}$ of subsets of P it is $\bigcup_i uM_i \subset u\left(\bigcup_i M_i\right)$. Let conversely $x \in u\left(\bigcup_i M_i\right)$. Then the neighborhood U_x contains some $m \in M_i$ $(i \in I)$. But then all neighborhoods of x contain m, hence $x \in uM_i$. Let the condition (2) be valid. Let us take $x \in P$. It is sufficient to prove that $s_u(x)$ is a neighborhood of x, i.e. $x \notin u(P - s_u(x))$. Since $u(P - s_u(x)) = \bigcup_{z \in P - s_u(x)} u(z)$, it is sufficient to prove that $x \notin u(z)$ for each $z \in P - s_u(x)$. Hence let $z \notin s_u(x)$. Then evidently $x \notin u(z)$.

In view of the preceding lemma we obtain the following statement:

2.9. The equality $s_u = u$ is valid iff for each $x \in P$ $u(x)$ is the least neighborhood of x.

In the paper [2] the notion of the R-topology is defined.

2.10. The topology u in P is called an R-topology, if for each point $x \in P$ and for every neighborhood 0 of x in (P, u) there exists a neighborhood 0_1 of x in (P, u) such that $u0_1 \subset 0$.

In the paper [2] also the following statement is proved:

2.11. The topology u is an R-topology if and only if $M \subset P \Rightarrow uM = \sigma_u M$, i.e. $u = \sigma_u$.

2.12. *If u is an R-topology, then u is a B^*-topology.*

Proof. On account of 2.5. it is sufficient to show that $y \in s_u(x)$ implies $x \in s_u(y)$. Let $y \in s_u(x)$ and let us suppose that $x \notin s_u(y)$. Then there exists $0 \in \mathscr{D}_u(y)$ so that $x \notin 0$. Since u is an R-topology, there exists $0_1 \in \mathscr{D}_u(y)$ so that $u0_1 \subset 0$. Then $x \notin u0_1$ from where we obtain that $P - 0_1 \in \mathscr{D}_u(x)$. From $0_1 \in \mathscr{D}_u(y)$ we have $y \notin u\,(P - 0_1)$. Hence $y \notin P - 0_1$ which is a contradiction.

From the last statement we obtain:

2.13. If $u = \sigma_u$, then $s_u \leqq u$.

2.14. The following conditions are equivalent:

(1) $s_u = \sigma_u$,

(2) $s_u = \sigma_u = u$,

(3) *u is an R-topology and* $u(\bigcup\limits_i M_i) = \bigcup\limits_i uM_i$ $(M_i \subset P)$,

(4) *For each $x \in P$ there exists the least neighborhood U_x and for this neighborhood it is* $uU_x = U_x$.

Proof. Since $\sigma_u \geqq u$, from $s_u = \sigma_u$ we obtain $s_u = \sigma_u = u$, hence the conditions (1), (2) are equivalent. From (2) we get (3) immediately. If the condition (3) is valid, then u is a B^*-topology, hence $s_u = u = \sigma_u$. The equivalence of the conditions (3), (4) is evident.

§3. EQUALITIES $r\,.\,s = t$.

We are going to introduce an operation of multiplication into the set $\mathscr{T}(P)$ of all topologies in P and we shall show that with regard to this operation and the partial ordering above introduced, $\mathscr{T}(P)$ is a partially ordered monoid. Then we shall investigate the conditions on the given topology u for the validity of an arbitrary from the equalities $r\,.\,s = t$, where $r, s, t \in \{u, s_u, \sigma_u\}$. For a given topology u let us denote $s_u = u_1$, $u = u_2$, $\sigma_u = u_3$. First of all we shall investigate the equalities $u_i\,.\,u_j = u_k$ $(i, j, k \in \{1, 2, 3\})$ for $k \geqq \max\{i, j\}$ in the following sequence: The set $\{(i, j, k)$: $i, j, k \in \{1, 2, 3\}, k \geqq \max\{i, j\}\}$ we order lexicographicaly from the left. Then if $(i, j, k) < (i_1, j_1, k_1)$, the equality $u_i\,.\,u_j = u_k$ we examine before the equality $u_{i_1}\,.\,u_{j_1} = u_{k_1}$ with the exception of $u_1\,.\,u_1 = u_2$ (i.e. $s_u\,.\,s_u = u$) and $u_2\,.\,u_2 = u_2$ (i. e. $u\,.\,u = u$). Then we shall show that the equalities $u_i\,.\,u_j = u_1$, **max** $\{i, j\} > 1$ and $u_1\,.\,u_1 = u_2$ are equivalent. At the end the equalities $u_i\,.\,u_j = u_2$, **max** $\{i, j\} > 2$ remain unsolved. Those we shall divide into two groups. First we shall indicate that the equalities where **min** $\{i, j\} \geqq 2$ are equivalent. At the end the equalities where **min** $\{i, j\} < 2$ (i. e. $s_u\,.\,\sigma_u = u$, $\sigma_u\,.\,s_u = u$) we investigate one by one.

3.1 *Let r, s be arbitrary topologies in P. Let us define $r \,.\, s$ as follows*: $(r.s)\, M = r(sM)$, $M \subset P$. *With regard to the operation defined in this way and to the partial ordering introduced above, the set $\mathscr{T}(P)$ of all topologies in P is a partially ordered monoid.*

Proof. $r \,.\, s$ is evidently a topology, the multiplication is associative. The identity in the set $\mathscr{T}(P)$ is the topology u_0 such that $u_0 M = M$ for each set $M \subset P$. At the end if $r, s, t \in \mathscr{T}(P)$, $r \leqq s$, then evidently $r \,.\, t \leqq s \,.\, t$, $t \,.\, r \leqq t \,.\, s$.

3.2. *The equality $s_u \,.\, s_u = s_u$ is valid if and only if the relation R defined as follows*:

$$aRb \Leftrightarrow a \in u(b)$$

is transitive.

Proof. Let $s_u \,.\, s_u = s_u$, $a \in u(b)$, $b \in u(c)$. Then by 1.4. $c \in s_u(b)$, $b \in s_u(a)$, from where we obtain $c \in s_u(s_u(a)) = s_u(a)$, i. e. $a \in u(c)$. Let conversely R be a transitive relation, let $x \in s_u(s_u M)$. Then there exists $z \in s_u M$ so that $x \in s_u(z)$. Further there exists $m \in M$ so that $z \in s_u(m)$. Hence it holds $m \in u(z)$, $z \in u(x)$, from where we get $m \in u(x)$. Then $x \in s_u(m) \subset s_u M$.

From this statement we obtain the consequences:

3.3. If $u \,.\, u = u$, then $s_u \,.\, s_u = s_u$.

3.4 *If u is an R-topology, then $s_u \,.\, s_u = s_u$.*

Proof. Let $a \in u(b)$, $b \in u(c)$ and let us suppose that $a \notin u(c)$. Then there exists $0 \in \mathscr{D}_u(a)$ so that $c \notin 0$. Further there exists $0_1 \in \mathscr{D}_u(a)$ so that $u0_1 \subset 0$, hence $c \notin u0_1$, from where we get $P - 0_1 \in \mathscr{D}_u(c)$. It is $b \in u(c) = s_u(c)$, hence $b \in P - 0_1$, which is a contradiction to the assumption $b \in s_u(a)$.

3.5 *The relation R defined in 3.2. is an equivalence iff u is a B^*-topology and $s_u \,.\, s_u = s_u$.*

Proof. The relation R is reflexive. The relativon R is symmetric if and only if u is a B^*-topology (cf. 2.5.) and transitive if and only if $s_u \,.\, s_u = s_u$.

Now two lemmas follow:

3.6. *If $x \in P$, $M \subset P$, then $x \notin \sigma_u M$ if and only if x, M have disjoint neighborhoods.*

Proof. Let $x \notin \sigma_u M$. Then by 1.5. there exists $0 \in \mathscr{D}_u(x)$ so that $M \cap u0 = \emptyset$. From this it follows that $P - 0$ is a neighborhood of M, for which $O \cap (P - 0) = \emptyset$. Conversely, let us suppose that there exists $0 \in \mathscr{D}_u(x)$, $U \in \mathscr{D}_u(M)$ so that $0 \cap U = \emptyset$. Then $0 \subset P - U$, hence $u0 \subset u(P - U)$. Since $M \cap u(P - U) = \emptyset$, it is also $M \cap u0 = \emptyset$ and from there we get $x \notin \sigma_u M$.

3.7. The following conditions are equivalent:

(1) $\sigma_u(\bigcup_{i \in I} M_i) = \bigcup_{i \in I} \sigma_u M_i$ *for each system* $\{M_i\}_{i \in I}$ *of subsets of* P.

(2) $\sigma_u M = \bigcup\limits_{m \in M} \sigma_u(m)$ *for each set* $M \subset P$.

(3) *For each* $x \in P$ *there exists* $0_1 \in \mathscr{D}_u(x)$ *so that for each* $0 \in \mathscr{D}_u(x)$ $u0_1 \subset u0$ *holds.*

(4) *For each* $x \in P$ *there exists* $0_1 \in \mathscr{D}_u(x)$ *so that* $\sigma_u(x) = u0_1$.

Proof. Evidently the conditions (1), (2) are equivalent and also the conditions (3), (4) are equivalent. It is sufficient to prove that (2), (3) are equivalent. Let the condition (2) be valid. Let us take $x \in P$, $M = \bigcup\limits_{0 \in D_u(x)} (P - u0) = P - \bigcap\limits_{0 \in D_u(x)} u0$. If $m \in M$, then there exists $0_m \in \mathscr{D}_u(x)$ so that $m \notin u0_m$. Hence $x \notin \bigcup\limits_{m \in M} \sigma_u(m) = \sigma_u M$. Then there exists $0_1 \in \mathscr{D}_u(x)$ so that $u\, 0_1 \bigcap (P - \bigcap\limits_{0 \in D_u(x)} u0) = \emptyset$, hence $u0_1 \subset \bigcap\limits_{0 \in D_u(x)} u0$. Let the condition (3) be valid and let $x \notin \bigcup\limits_{m \in M} \sigma_u(m)$. Then for each $m \in M$ there exists $0_m \in \mathscr{D}_u(x)$ so that $m \notin u0_m$. Let us take 0_1 from (3). It holds $M \bigcap u0_1 = \emptyset$, hence $x \notin \sigma_u M$.

3.8. By 2.8. and 3.7. we obtain immediately:
If for each system $\{M_i\}_{i \in I}$ of subsets of P $u(\bigcup\limits_i M_i) = \bigcup\limits_i uM_i$ holds, then it is $\sigma_u(\bigcup\limits_i M_i) = \bigcup\limits_i \sigma_u M_i$.

3.9. *The following conditions are equivalent*:

(1) $s_u \,.\, s_u = \sigma_u$

(2) ($a \in \sigma_u(b) \Leftrightarrow$ *there exists* $c \in u(a)$ *so that* $b \in u(c)$) *and* $\sigma_u(\bigcup\limits_i M_i) = \bigcup\limits_i \sigma_u M_i$ $(M_i \subset P)$.

(3) (a, b *have disjoint neighborhoods* $\Leftrightarrow$ $b \notin \bigcup\limits_{c \in u(a)} u(c)$) *and for each* $x \in P$ *there exists* $0_1 \in \mathscr{D}_u(x)$ *so that for each* $0 \in \mathscr{D}_u(x)$ *it is* $u0_1 \subset u0$.

Proof. The conditions (2), (3) are evidently equivalent. Let (1) hold. It is $a \in \sigma_u(b) \Leftrightarrow a \in s_u(s_u(b)) \Leftrightarrow$ there exists $c \in s_u(b)$ so that $a \in s_u(c) \Leftrightarrow$ there exists $c \in u(a)$ so that $b \in u(c)$. Let $x \in \sigma_u(\bigcup\limits_i M_i) = s_u(s_u(\bigcup\limits_i M_i))$. Then it is $x \in s_u(y)$, $y \in s_u(m)$ for some $m \in \bigcup\limits_i M_i$. From there we obtain $m \in u(y)$, $y \in u(x)$ and with regard to what we have proved already it is $x \in \sigma_u(m) \subset \sigma_u M_i \subset \bigcup\limits_i \sigma_u M_i$. Let the condition (2) hold. Then $y \in s_u(s_u(x)) \Leftrightarrow$ there exists $z \in s_u(x)$ so that $y \in s_u(z) \Leftrightarrow$ there exists $z \in u(y)$ so that $x \in u(z) \Leftrightarrow y \in \sigma_u(x)$. Further it holds: $y \in s_u(s_u M) \Leftrightarrow$ there exists $m \in M$ and $z \in s_u(m)$ so that $y \in s_u(z) \Leftrightarrow$ there exists $m \in M$ and $z \in u(y)$ so that $m \in u(z) \Leftrightarrow$ there exists $m \in M$ so that $y \in \sigma_u(m) \Leftrightarrow y \in \bigcup\limits_{m \in M} \sigma_u(m) = \sigma_u M$.

3.10. *The following conditions are equivalent*:

(1) $s_u \,.\, u = u$.

(2) *For each* $x \in P$ *and* $M \subset P$ $x \notin uM$ *implies* $u(x) \bigcap uM = \emptyset$.

(3) *For each* $x \in P$ 0 *is a neighborhood of* x *implies* 0 *is a neighborhood of* $u(x)$.

Proof. Since it is $s_u \,.\, u \geqq u$, the condition (1) is equivalent with the condition: for each $M \subset P$ it is $s_u(uM) \subset uM$ i. e. for each $M \subset P$ $x \notin uM$ implies $x \notin s_u(uM)$. The last relation is equivalent with the equality $uM \cap u(x) = \emptyset$. Hence (1) is equivalent with (2). Let the condition (2) hold, let us take $x \in P$, $0 \in \mathscr{D}_u(x)$. Then it is $x \notin u(P - 0)$, from where by (2) we have $u(x) \cap u(P - 0) = \emptyset$. The last equality says that 0 is a neighborhood of $u(x)$. Let (3) hold, let us take $x \in P$, $M \subset P$, let $x \notin uM$. Then $P - M \in \mathscr{D}_u(x)$, hence $P - M \in \mathscr{D}_u(u(x))$. From there it follows $u(x) \cap uM = \emptyset$.

According to the last theorem we have:

3.11. *If u is a B^*-topology and $u \,.\, u = u$, then $s_u \,.\, u = u$.*

Proof. Let $x \in P$, $M \subset P$, $x \notin uM$. Then $P - uM$ is a neighborhood of x and since $s_u(x) = u(x)$, it is $u(x) \subset P - uM$, hence $u(x) \cap uM = \emptyset$.

We are going to introduce a definition:

3.12. Let (P, u) be a topological space. Let us denote A and U the following properties of the topology u:

$M_1 \subset P, M_2 \subset P \Rightarrow u(M_1 \cup M_2) = uM_1 \cup uM_2$ and

$M \subset P \Rightarrow u(uM) = uM$ respectively.

If u has the property A, U, or the both properties, we shall call the topology u an A-topology, U-topology, or AU-topology and (P, u) we shall call an A-(topological) space, U- or AU-(topological) space, respectively. Evidently AU-topological space is the topological space in the usual meaning.

3.13. *If (P, u) is an AU-space, then $s_u \,.\, u = u$ holds if and only if every open set is a union of some closed sets.*

Proof. Let $s_u \,.\, u = u$, let A be an open set. For each $x \in A$ A is a neighborhood of x, hence for each $x \in A$ it is $u(x) \subset A$, from where we have $A = \bigcup_{x \in A} u(x)$. Conversely, let each open set be a union of some closed sets. Let $x \in P$, $0 \in \mathscr{D}_u(x)$. There exists an open neighborhood 0_1 of x such that $0_1 \subset 0$. By the assumption it is $0_1 = \bigcup_{\lambda \in \Delta} F_\lambda$. where F_λ are closed sets. Then $x \in F_\lambda$ for some $\lambda \in \Delta$, hence $u(x) \subset F_\lambda \subset 0_1 \subset 0$, from where we get that 0 is a neighborhood of $u(x)$.

3.14. *The equality $s_u \,.\, u = \sigma_u$ is valid if and only if for each $x \in P$ U is a neighborhood of $u(x)$ iff there exists $0 \in \mathscr{D}_u(x)$ so that $P - 0$ is a neighborhood of $P - U$.*

Proof. Let $s_u \,.\, u = \sigma_u$. Let us take $x \in P$. Then U is a neighborhood of $u(x) \Leftrightarrow$ $\Leftrightarrow u(x) \cap u(P - U) = \emptyset \Leftrightarrow x \notin s_u(u(P - U)) = \sigma_u(P - U) \Leftrightarrow$ there exists $0 \in \mathscr{D}_u(x)$ so that $(P - U) \cap u0 = \emptyset \Leftrightarrow$ there exists $0 \in \mathscr{D}_u(x)$ so that $P - 0$ is a neighborhood of $P - U$. Conversely, let U be a neighborhood of $u(x)$ iff there exists $0 \in \mathscr{D}_u(x)$ so that $P - 0$ is a neighborhood of $P - U$. It holds: $x \in s_u(uM) \Leftrightarrow uM \cap u(x) \neq \emptyset \Leftrightarrow$ $\Leftrightarrow P - M$ is not a neighborhood of $u(x) \Leftrightarrow$ for each $0 \in \mathscr{D}_u(x)$ $P - 0$ is not a neighborhood of $M \Leftrightarrow$ for each $0 \in \mathscr{D}_u(x)$ it is $M \cap u0 \neq \emptyset \Leftrightarrow x \in \sigma_u M$.

3.15. If (P, u) is an AU-space, then the following conditions are equivalent:

(1) $s_u \,.\, u = \sigma_u$.

(2) For each $x \in P$ U is a neighborhood of $u(x)$ iff there exists a closed set B so that $x \notin B$, $U \supset P - B^\circ$ (B° is the interior of B).

(3) For each $x \in P$ U is a neighborhood of $u(x)$ if and only if there exists an open set A so that $x \in A$, $U \supset uA$.

(4) For each $x \in P$ U is a neighborhood of $u(x)$ if and only if there exists $0 \in \mathscr{D}_u(x)$ so that $U \supset u0$.

This statement is evident.

3.16. *The equality $s_u \,.\, \sigma_u = \sigma_u$ is valid if and only if for each $x \in P$ and $M \subset P$ it holds:*

(*) *if x, M have disjoint neighborhoods, then for each $y \in u(x)$ y, M have disjoint neighborhoods.*

Proof. Let $s_u \,.\, \sigma_u = \sigma_u$ and let x, M have disjoint neighborhoods. Then $x \notin \sigma_u M$ and from there $x \notin s_u(\sigma_u M)$. From the latest relation it follows that $\sigma_u M \cap u(x) = \emptyset$, hence if $y \in u(x)$, then $y \notin \sigma_u M$, i.e. y, M have disjoint neighborhoods. Conversely, let (*) hold for each $x \in P$ and $M \subset P$. Evidently $\sigma_u M \subset s_u(\sigma_u M)$. Now, let $x \notin \sigma_u M$. Then x, M have disjoint neighborhoods, hence for each $y \in u(x)$ y, M have disjoint neighborhoods, i.e. $y \notin \sigma_u M$. Then $u(x) \cap \sigma_u M = \emptyset$, from where we have $x \notin s_u(\sigma_u M)$.

From this statement we obtain the following consequence:

3.17. *If u is a B^*-topology and U-topology (cf. 3.12), then $s_u \,.\, \sigma_u = \sigma_u$.*

Proof. Let $x \notin \sigma_u M$. Then there exists $U \in \mathscr{D}_u(M)$ so that $x \notin uU$. Since $u \,.\, u = u$, $P - uU$ is a neighborhood of x. It holds $(P - uU) \cap uU = \emptyset$ and from there we get $(P - uU) \cap \sigma_u M = \emptyset$. Since $s_u(x) = u(x)$, it holds $u(x) \subset P - uU$, from where we obtain $u(x) \cap \sigma_u M = \emptyset$, i.e. $x \notin s_u(\sigma_u M)$.

3.18. *The equality $u \,.\, s_u = u$ is valid iff for each $x \in P$ and $0 \in \mathscr{D}_u(x)$ there exists $0_1 \in \mathscr{D}_u(x)$ so that for each $z \in 0_1$ it is $u(z) \subset 0$.*

Proof. Let $u \,.\, s_u = u$, $x \in P$, $0 \in \mathscr{D}_u(x)$. Then $x \notin u(P - 0)$, from where we get $x \notin u(s_u(P - 0))$. Let us take $P - s_u(P - 0) = 0_1$. Evidently $0_1 \in \mathscr{D}_u(x)$. Let $z \in 0_1$. Then $z \notin s_u(P - 0)$, hence $(P - 0) \cap u(z) = \emptyset$, from where we have $u(z) \subset 0$. Conversely, let for each $x \in P$ and $0 \in \mathscr{D}_u(x)$ there exist 0_1 with the desired property. Let $x \notin uM$, we are going to prove that $x \notin u(s_u M)$. If $x \notin uM$, then $P - M$ is a neighborhood of x. There exists 0_1 so that for each $z \in 0_1$ it is $u(z) \subset P - M$ i.e. $u(z) \cap M = \emptyset$. Then $z \notin s_u M$, hence $0_1 \cap s_u M = \emptyset$. From the latest equality we obtain $0_1 \subset P - s_u M$, hence $P - s_u M \in \mathscr{D}_u(x)$, from where we have $x \notin u(s_u M)$.

3.19. Evidently if u is an R-topology, it holds $u \,.\, s_u = u$.

3.20. *If u is a B^*-topology and U-topology, then $u \,.\, s_u = u$.*

Proof. Let $x \in P$, $0 \in \mathscr{D}_u(x)$. Let us denote $0_1 = \{z \in 0 : 0 \in \mathscr{D}_u(z)\}$. Evidently $x \in 0_1$, $0_1 \subset 0$ holds. From there we have $u(P - 0) \subset u(P - 0_1)$. We shall show, that the inverse inclusion holds, too. It is $P - 0_1 \subset u(P - 0)$ for if $y \notin u(P - 0)$, then 0 is a neighborhood of y, hence $y \in 0_1$. Since $u \,.\, u = u$, it is $u(P - 0_1) \subset u(P - 0)$. Hence the equality $u(P - 0_1) = u(P - 0)$ is valid. Since $x \notin u(P - 0)$, it is $x \notin u(P - 0_1)$, hence 0_1 is a neighborhood of x. Let us take $z \in 0_1$. Then $0 \in \mathscr{D}_u(z)$ and since $s_u(z) = u(z)$, it is $u(z) \subset 0$.

3.21. *The equality* $u \,.\, s_u = \sigma_u$ *is valid iff for each* $x \in P$ *and* $0 \in \mathscr{D}_u(x)$ *there exists* $0_1 \in \mathscr{D}_u(x)$ *such that* $u0_1 \subset \bigcup_{z \in 0} u(z)$.

Proof. Let $u \,.\, s_u = \sigma_u$, $x \in P$, $0 \in \mathscr{D}_u(x)$. Let us put $M = P - \bigcup_{z \in 0} u(z)$. For each $z \in 0$ it is $M \cap u(z) = \emptyset$, hence for every $z \in 0$ it is $z \notin s_u M$. From there we have $0 \cap s_u M = \emptyset$. Then $x \notin u(s_u M) = \sigma_u M$, i.e. there exists $0_1 \in \mathscr{D}_u(x)$ so that $M \cap u0_1 = \emptyset$. From the latest equality we have $u0_1 \subset \bigcup_{z \in 0} u(z)$. We shall prove the inverse implication. Let $x \in u(s_u M)$ and let us take $0 \in \mathscr{D}_u(x)$. Then there exists $z \in 0$ so that $z \in s_u M$, i.e. $M \cap u(z) \neq \emptyset$. It holds $M \cap u0 \supset M \cap u(z)$, hence $M \cap u0 \neq \emptyset$. From it we have $x \in \sigma_u M$. Let $x \notin u(s_u M)$. Then there exists $0_1 \in \mathscr{D}_u(x)$ so that $0_1 \cap s_u M = \emptyset$, hence for every $z \in 0_1$ it is $z \notin s_u M$. Then $M \cap \bigcup_{z \in 0_1} u(z) = \emptyset$. By the assumption there exists $0_2 \in \mathscr{D}_u(x)$ so that $u0_2 \subset \bigcup_{z \in 0_1} u(z)$. We obtain $M \cap u0_2 = \emptyset$, hence $x \notin \sigma_u M$.

3.22. Remark. From the proof of the last theorem one can see that it holds $u \,.\, s_u \leqq \sigma_u$ for an arbitrary topology u. There need not be $s_u \,.\, u \leqq \sigma_u$ (cf. example in te section 4.10.).

3.23. *The equality* $u \,.\, u = \sigma_u$ *holds true iff for each* $x \in P$ *and* $M \subset P$ x, M *have disjoint neighborhood if and only if there exists* $0 \in \mathscr{D}_u(x)$ *so that* $0 \cap uM = \emptyset$.

Proof. Let the equality $u \,.\, u = \sigma_u$ be valid, $x \in P$, $M \subset P$. x, M have disjoint neighborhoods if and only if $x \notin \sigma_u M$, this is equivalent to $x \notin u(uM)$, the last relation is equivalent to the following: there exists $0 \in \mathscr{D}_u(x)$ so that $0 \cap uM = \emptyset$. We are going to prove the inverse implication. It is $x \notin u(uM)$ iff there exists $0 \in \mathscr{D}_u(x)$ such that $0 \cap uM = \emptyset$. According to the assumption the last predicate is equivalent with the statement: x, M have disjoint neighborhoods, i.e. $x \notin \sigma_u M$.

3.24. The equality $u \,.\, \sigma_u = \sigma_u$ is valid if and only if the following holds: x, M have disjoint neighborhoods ($x \in P$, $M \subset P$) iff there exists $0 \in \mathscr{D}_u(x)$ so that for every $y \in 0$ y, M have disjoint neighborhoods.

This statement is evident.

3.25. *For a U-topology* u $u \,.\, \sigma_u = \sigma_u$ *holds.*

Proof. Let $x \notin \sigma_u M$. Then there exists $U \in \mathscr{D}_u(M)$ so that $x \notin uU$. Then $P - uU$ is a neighborhoods of x, which is disjoint with $\sigma_u M$. Hence $x \notin u(\sigma_u M)$ holds. The inclusion $\sigma_u M \subset u(\sigma_u M)$ is evident.

3.26. The equality $\sigma_u \,.\, s_u = \sigma_u$ holds true if and only if the following is valid: if x, M have disjoint neighborhoods ($x \in P$, $M \subset P$), then there exists $0 \in \mathscr{D}_u(x)$ so that $M \bigcap \bigcup_{y \in u0} u(y) = \emptyset$.

This statement is evident.

3.27. *If u is a U-topology, then $\sigma_u \,.\, s_u = \sigma_u$.*

Proof. First of all it is $\bigcup_{y \in u0} u(y) = u0$. Let $x(\in P)$, $M(\subset P)$ have disjoint neighborhoods 0, U. Then obviously $M \bigcap u0 = \emptyset$ for $P - 0$ is a neighborhood of M.

3.28. The equality $\sigma_u \,.\, u = \sigma_u$ is valid if and only if the following holds: If x, M have disjoint neighborhoods ($x \in P$, $M \subset P$), then x, uM have disjoint neighborhoods.

The predicate 'x, uM have disjoint neighborhoods' is equivalent with the predicate 'there exists $0 \in \mathscr{D}_u(x)$ so that $u0 \cap uM = \emptyset$'.

This statement is evident.

3.29. *Let u be an AU-topology. The equality $\sigma_u \,.\, u = \sigma_u$ is valid if and only if an arbitrary regularly closed set and a point out of it can be separated by open sets.*

Proof. Let $\sigma_u \,.\, u = \sigma_u$ and let A be a regularly closed set, i.e. such a set that $A = uA^\circ$ (A° is the interior of A), $x \notin A$. A°, x have disjoint neighborhoods, hence also $uA^\circ = A$, x have disjoint neighborhoods. Further we shall prove the opposite implication. Let x, M have disjoint neighborhoods. Let 0 and U be an open neighborhood of x and M, respectively, where $0 \cap U = \emptyset$. Let us take $A = uU$. A is a regularly closed set, $x \notin A$, hence by the assumption there exist disjoint open sets $0_1, U_1$ so that $x \in 0_1$, $A \subset U_1$. They are the desired disjoint neighborhoods of x and uM.

3.30. The equality $\sigma_u \cdot \sigma_u = \sigma_u$ is valid if and only if the following holds: if x, M have disjoint neighborhoods ($x \in P$, $M \subset P$), then there exists $0 \in \mathscr{D}_u(x)$ so that if $y \in u0$, then y, M have disjoint neighborhoods.

This statement is evident.

3.31. *If for a topology u the equality $\sigma_u \cdot \sigma_u = \sigma_u$ is valid, then the relation R defined as follows:*

$$aRb \Leftrightarrow a \in \sigma_u(b)$$

is an equivalence.

Proof. The relation R is always reflexive, symmetric and if $\sigma_u \cdot \sigma_u = \sigma_u$ holds, also transitive.

3.32. *If for a topology u $\sigma_u \cdot \sigma_u = \sigma_u$ is valid, then $s_u \cdot \sigma_u = \sigma_u \cdot s_u = u \,.\, \sigma_u = \sigma_u \,.\, u = \sigma_u$.*

Proof. If the equality $\sigma_u \cdot \sigma_u = \sigma_u$ is valid, then, since it is $\sigma_u \leqq s_u \cdot \sigma_u$, $\sigma_u \cdot s_u$, $u \,.\, \sigma_u$, $\sigma_u \,.\, u \leqq \sigma_u \cdot \sigma_u$, the assertion of the theorem holds.

3.33. *The following conditions are equivalent:*

(1) $s_u \,.\, u = s_u$

(2) $s_u \, . \, s_u = u$

(3) $s_u \, . \, \sigma_u = s_u$

(4) $u \, . \, s_u = s_u$

(5) $u \, . \, u = s_u$

(6) $u \, . \, \sigma_u = s_u$

(7) $\sigma_u \, . \, s_u = s_u$

(8) $\sigma_u \, . \, u = s_u$

(9) $\sigma_u \, . \, \sigma_u = s_u$

(10) $s_u = u = u \, . \, u$

(11) $s_u = \sigma_u = u = u \, . \, u$

(12) $s_u = \sigma_u = u$.

Proof. First of all we are going to show that the conditions (10), (11), (12) are equivalent. (11) follows from the condition (10): Let $x \in P$, $0 \in \mathscr{D}_u(x)$. Since $s_u = u$, x has the least neighborhood $u(x)$ and for it $u[(u(x)] = u(x) \subset 0$ holds. Hence u is an R-topology, i.e. $u = \sigma_u$. From (11) the condition (12) follows immediately. From (12) follows (10). Namely for an R-topology the equality $u \, . \, s_u = u$ holds (cf. 3.19.) and furthermore, if $s_u = u$, we have $u \, . \, u = u$. Evidently, the conditions (1) — (9) follow from (11). We are going to verify that from each of the conditions (1) — (9) (10) and (12) follow, respectively. If (1), (4) or (5) holds, then, since $s_u \, . \, u$, $u \, . \, s_u$, $u \, . \, u \geqq u$ and the inequality $s_u > u$ cannot be valid, it is $s_u = u$ and evidently $u \, . \, u = u$. If (3), (6), (7), (8) or (9) holds, then, since $s_u \, . \, \sigma_u$, $u \, . \, \sigma_u$, $\sigma_u \, . \, s_u$, $\sigma_u \, . \, u$, $\sigma_u \, . \, \sigma_u \geqq \sigma_u$ and always $s_u \leqq \sigma_u$, it is $\sigma_u = s_u$. Then according to the statement in the section 2.14. it holds (12). Finally, let the equality $s_u \, . \, s_u = u$ be valid. It is sufficient to prove that $s_u = u$. Since $s_u \, . \, s_u = u \geqq s_u$, u is a B^*-topology and according to the statement in the section 2.7. it is sufficient to prove that $u(\bigcup_i M_i) = \bigcup_i uM_i$ $(M_i \subset P)$. Let $x \in u(\bigcup_i M_i) = s_u[s_u(\bigcup_i M_i)]$. Then there exists $y \in s_u(\bigcup_i M_i)$ so that $x \in s_u(y)$. Further there exists $m \in \bigcup_i M_i$ so that $y \in s_u(m)$. Then $x \in s_u[s_u(m)] = u(m)$, from which we have $x \in uM_i \subset \bigcup_i uM_i$.

Further conditions equivalent to (1) — (12) are introduced in the section 2.14.

3.34. *The following conditions are equivalent:*

(1) $u \, . \, \sigma_u = u$

(2) $\sigma_u \, . \, u = u$

(3) $\sigma_u \, . \, \sigma_u = u$

(4) $\sigma_u = u = u \, . \, u$

(5) *For each* $x \in P$ *and* $0 \in \mathscr{D}_u(x)$ *there exists* $0_1 \in \mathscr{D}_u(x)$ *so that* 0 *is a neighborhood of* $u0_1$.

Proof. First of all we shall prove the equivalence of (4) and (5). Let (4) be valid, let us take $x \in P$, $0 \in \mathscr{D}_u(x)$. Since $x \notin u(P - 0) = u[u(P - 0)]$, there exists $0' \in \mathscr{D}_u(x)$ such that $0' \cap u(P - 0) = \emptyset$. u is an R-topology, hence there exists $0_1 \in \mathscr{D}_u(x)$ so that $u0_1 \subset 0'$. Then $u0_1 \cap u(P - 0) = \emptyset$, from where we have $0 \in \mathscr{D}_u(u0_1)$. In contrary, let the condition (5) hold. Then evidently u is an R-topology. It remains to prove the equality $u \,.\, u = u$. Let $M \subset P$, $x \notin uM$. Then $P - M \in \mathscr{D}_u(x)$ and by the assumption there exists $0_1 \in \mathscr{D}_u(x)$ so that $u0_1 \cap uM = \emptyset$. Then $0_1 \cap uM = \emptyset$, from where we have $x \notin u(uM)$. Evidently the condition (4) implies (1), (2), (3). We are going to prove that each of the conditions (1), (2), (3) implies (4). It holds: $u \,.\, \sigma_u$, $\sigma_u \,.\, u$, $\sigma_u \,.\, \sigma_u \geqq \sigma_u$ and since $\sigma_u \geqq u$, it must be $\sigma_u = u$.

3.35. *The equality $\sigma_u \,.\, s_u = u$ is valid if and only if u is an R-topology.*

Proof. Let $\sigma_u \,.\, s_u = u$. Then, since $\sigma_u \,.\, s_u \geqq \sigma_u$, it is $u = \sigma_u$. In contrary, let u be an R-topology. Then $\sigma_u \,.\, s_u = u \,.\, s_u$ and according to the statement in 3.19. it is $u \,.\, s_u = u$.

3.36. *The equality $s_u \,.\, \sigma_u = u$ is valid if and only if it is fulfiled one of the following equivalent conditions:*

(1) $\sigma_u = u = s_u \,.\, u$

(2) *u is such an R-topology that for each $x \in P$ every neighborhood of x is a neighborhood of the set $u(x)$.*

Proof. The equivalence of (1), (2) is evident (cf. section 3.10.). Let $s_u \,.\, \sigma_u = u$. Then, since $s_u \,.\, \sigma_u \geqq \sigma_u$, it is $\sigma_u = u$. Then $s_u \,.\, u = s_u \,.\, \sigma_u = u$. Let (1) hold. Then $s_u \,.\, \sigma_u = s_u \,.\, u = u$.

§4. THE SEMIGROUP GENERATED BY u, s_u, σ_u.

In this paragraph we shall show what is the subsemigroup $\mathscr{P}(P)$ of the semigroup $\mathscr{T}(P)$ of all topologies defined in the set P generated by the topologies u, s_u, σ_u for some special topologies u. This problem fails to be completely solved. In the text the most topical questions are posed.

4.1. If u is a topology with the property $s_u = u = \sigma_u$, then the subsemigroup $\mathscr{P}(P)$ contains the element u only.

This statement is evident (cf. 3.33.).

An example of a such topology for $P = \{a, b, c\}$ is given by this table:

M	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	P
uM	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{a, b\}$	P	P	P

4.2. *Let u be a topology with the property $s_u = u < \sigma_u$, i.e. each point $x \in P$ has the least neighborhood $u(x)$ and u is not an R-topology. Then the subsemigroup $\mathscr{P}(P)$ consists of all natural powers of u, i.e. $\mathscr{P}(P) = \{u^n : n \, n \in N\}$.*

Proof. Let $x \in P$, $0 \in \mathscr{D}_u(x)$. It is $u(u(x)) \subset u0 = \bigcup_{z \in 0} u(z)$ hence according to the statement in the section 3.21. it holds $u \cdot s_u = \sigma_u$. Then it is $u \cdot u = \sigma_u$.

Besides it can happen that for each natural n it is $u^{n+1} > u^n$. This shows the following example.

4.3. Let P be an arbitrary partially ordered set, α ($\geqq 1$) a cardinal number. For $M \subset P$ let us put $uM = \bigcup_{\lambda \in \Lambda} I_\lambda$, where I_λ is an arbitrary interval any end point of which belongs to M and such that card $I_\lambda \leqq \alpha$.

Then evidently (P, u) is a topological space for which it is $s_u = u$. If P is the set of all integers, $\alpha = 2$, then for each natural number n it is $u^{n+1} > u^n$.

If $P = \{1, 2, \ldots, n + 1\}$, $\alpha = 2$, then $u < u^2 < u^3 < \ldots < u^n = u^{n+1}$.

4.4. *Let u be a toppology with the property $s_u < u = \sigma_u$, i.e. u is an R-topology and there exists $x \in P$ so that x has not the least neighborhood. Then the subsemigroup $\mathscr{P}(P)$ consists of the tolopologies $s_u^\varepsilon \cdot u^n$, where $\varepsilon \in \{0, 1\}$, n runs over the set of all nonnegative integers, under the exception $\varepsilon = n = 0$. (s_u^0 means the identity of the monoid $\mathscr{T}(P)$).*

Proof. According to the statement in the section 3.4. and 3.19. it holds $s_u \cdot s_u = s_u$ and $u \cdot s_u = u$, respectively.

4.5. For the topology u mentioned in 4.4. the subsemigroup $\mathscr{P}(P)$ is a chain in the partially ordered monoid $\mathscr{T}(P)$.

Proof. It holds: $s_u < u \leqq s_u \cdot u \leqq u^2 \leqq s_u \cdot u^2 \leqq u^3 \leqq \ldots$

4.6. In the paper [2] there is given an example of a topology u with $s_u < u = \sigma_u$. For this topology it is $s_u < u = s_u \cdot u < u^2 = s_u \cdot u^2 < u^3 = s_u \cdot u^3 < u^4 = s_u \cdot u^4 < \ldots$ (for each natural n it is $u^n < u^{n+1}$, $u^n = s_u \cdot u^n$).

Problem 1. To find the topologies (if they exist), for which it holds:

a) $s_u < u = \sigma_u < s_u \cdot u < u^2 < s_u \cdot u^2 < u^3 < \ldots$

b) $s_u < u = \sigma_u < s_u \cdot u < u^2 < \ldots < u^n (= s_u \cdot u^n) = u^{n+1}$, $n \geqq 2$

c) $s_u < u = \sigma_u < s_u \cdot u < u^2 < \ldots < s_u \cdot u^n = u^{n+1}$, $n \geqq 1$.

If $P = \{a, b, c\}$, then one can easily verify that the topology u given by the table:

M	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	P
uM	$\{a\}$	$\{b\}$	$\{c\}$	P	$\{a, c\}$	$\{b, c\}$	P

is a topology with $s_u < u = \sigma_u (= s_u \cdot u) = u^2$.

4.7 *Let u be a U-topology with the property $s_u < u < \sigma_u$, i.e. u is a U-, B^*-topology and no R-topology and there exists $x \in P$ so that x has not the least neighborhood. Then the subsemigroup $\mathscr{P}(P)$ contains s_u, $\sigma_u^n (n \geqq 1)$, $\sigma_u^n \cdot u (n \geqq 0)$.*

Proof. $\mathscr{P}(P)$ contains topologies $v = v_1 . v_2 \dots v_k, v_i \in \{s_u, u, \sigma_u\}$. Now the statement can be proved by induction on k using the statements in the sections 3.3., 3.11., 3.17., 3.20., 3.25., 3.27., with the aid of which one can compound the table:

	s_u	u	σ_u
s_u	s_u	u	σ_u
u	u	u	σ_u
σ_u	σ_u	$\sigma_u . u$	$\sigma_u . \sigma_u$

4.8. *For the topology u mentioned in 4.7. the subsemigroup $\mathscr{P}(P)$ is a chain in the partially ordered monoid $\mathscr{T}(P)$.*

Proof. It holds $s_u < u < \sigma_u \leqq \sigma_u . u \leqq \sigma_u^2 \leqq \sigma_u^2 . u \leqq \sigma_u^3 \leqq \dots$

4.9. Example of the topology mentioned in 4.7.:
Let P be the set of all real numbers. Let us define $u\emptyset = \emptyset$, $u(x) = (x)$ $(x \in P)$,

$$uM = \begin{cases} (-\infty, 0> & \text{if } M \cap (0, \infty) = \emptyset \\ <0, \infty) & \text{if } M \cap (-\infty, 0) = \emptyset \\ P & \text{if } M \cap (0, \infty) \neq \emptyset \ \& \ M \cap (-\infty, 0) \neq \emptyset \end{cases} \quad \left.\right\} \text{and } M (\subset P) \text{ contains more than one element.}$$

One can easily verify that s_u is the least topology in P, *i.e.* the identity of the monoid $\mathscr{T}(P)$ and σ_u is a such topology that
$\sigma_u\emptyset = \emptyset$, if $0 \in M$, then $\sigma_u M = P$,

$$\text{if } 0 \notin M, \text{ then } \sigma_u M = \begin{cases} (-\infty, 0> & \text{if } M \subset (-\infty, 0), \\ <0, \infty) & \text{if } M \subset (0, \infty), \\ P & \text{if } M \cap (-\infty, 0) \neq \emptyset \ \& \ M \cap (0, \infty) \neq \emptyset. \end{cases}$$

Then evidently $s_u < u < \sigma_u$, $u . u = u$ hold. Since $\sigma_u(u(0, \infty)) = \sigma_u <0, \infty) = P$' $\sigma_u(0, \infty) = <0, \infty)$, it is $\sigma_u . u > \sigma_u$. Further we have $\sigma_u . u < \sigma_u^2$, σ_u^2 is the greatest topology in P, i.e. $\sigma_u^2 M = P (\emptyset \neq M \subset P)$.

Problem 2. To find the topologies (if they exist), for which it holds:

a) $s_u < u = u^2 < \sigma_u < \sigma_u . u < \sigma_u^2 < \sigma_u^2 . u < \sigma_u^3 < \dots$

b) $s_u < u = u^2 < \sigma_u < \sigma_u . u < \sigma_u^2 < \sigma_u^2 . u < \dots < \sigma_u^n (= \sigma_u^n . u) =$
$= \sigma_u^{n+1}, n \geqq 3.$

c) $s_u < u = u^2 < \sigma_u < \sigma_r . u < \sigma_u^2 < \sigma_u^2 . u < \dots < \sigma_u^n . u = \sigma_u^{n+1}, n \geqq 1.$

4.10. In this section we are going to introduce an example of a such topology u, for which it is $s_u \| u$, $u < \sigma_u$ and we shall show what is the subsemigroup $\mathscr{P}(P)$ for this topology u.

Let P be the set of all ordinal numbers which are greater than 0 and less or equal to ω. Let $u\emptyset = \emptyset$, $uM = M \cup \{n, \omega\}$ if M is a nonempty subset of P such that

$n \notin M$, $n < m < \omega \Rightarrow m \in M$, $uM = M \bigcup \{\omega\}$ if $M \neq \emptyset$ and for each number n such that $n \notin M$ there exists a number m such that $n < m < \omega$, $m \notin M$.

It is easy to verify that u is a topology. A set 0 is a neighborhood of ω if and only if $0 = P$. Let $x \neq \omega$. Then 0 is a neighborhood of x if and only if $x \in 0$ and there exists $y \in 0 - \{\omega\}$ such that $y > x$. From this we obtain that the topologies s_u and σ_u can be described as follows:

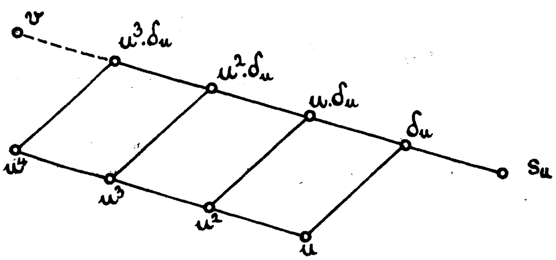
$$s_u\emptyset = \emptyset, \qquad \sigma_u\emptyset = \emptyset$$

$$s_u M = \begin{cases} P \text{ if } \omega \in M \\ M \text{ if } \omega \notin M \end{cases} (\emptyset \neq M \subset P) \quad \sigma_u M = \begin{cases} P \text{ if } \omega \in M \\ uM \text{ if } \omega \notin M \end{cases} (\emptyset \neq M \subset P)$$

The multiplication in $\mathscr{P}(P)$ is given by the table:

	s_u	u	σ_u
s_u	s_u	v	v
u	σ_u	u^2	$u \,.\, \sigma_u$
σ_u	σ_u	v	v

The letter v denotes the greatest topology in P, i.e. such a topology that $v\emptyset = \emptyset$, $vM = P$ if $\emptyset \neq M \subset P$. From this it is evident that $\mathscr{P}(P)$ consists of u^k (k is natural), s_u, $u^l \,.\, \sigma_u$ (l is a nonnegative integer) and v. The subsemigroup $\mathscr{P}(P)$ can be illustrated by the following diagram:



We shall prove the last statement:

First of all it is $u^{k+1} > u^k$ for each k natural because $u^{k+1} < k + 2, \omega) = < 1, \omega > = P \supsetneqq < 2, \omega > = u^k < k + 2, \omega)$.

Further it is $s_u < \sigma_u$ for $s_u(x) = \{x\} \subsetneqq \{x, \omega\} = \sigma_u(x)$ $(x \in P, x \neq \omega)$.

For $l \geqq 0$ it holds $u^l \,.\, \sigma_u < u^{l+1} \,.\, \sigma_u$ because $u^{l+1} [\sigma_u < l + 3, \omega)] = u^{l+2} < l + 3, \omega) = < 1, \omega > = P \supsetneqq < 2, \omega > = u^{l+1} < l + 3, \omega) = u^l [\sigma_u < l + 3, \omega)]$.
$u \parallel s_u$: $s_u(\omega) = P \supsetneqq \{\omega\} = u(\omega)$,

$s|_u(x) = \{x\} \subsetneqq \{x, \omega\} = u(x)$ $(x \neq \omega, x \in P)$.

$u < \sigma_u : u(\omega) = \{\omega\} \subsetneqq P = \sigma_u(\omega)$.

For $k \geqq 2$ it is $u^k \parallel s_u, \sigma_u, u^l \,.\, \sigma_u$ $(l \leqq k - 2)$:

$$u^k(\omega) = \{\omega\} \subsetneqq P = s_u(\omega) = \sigma_u(\omega) = u^l[\sigma_u(\omega)],$$

$$u^k < k + 1, \omega) = < 1, \omega > = P \supsetneqq s_u < k + 1, \omega) = < k + 1, \omega),$$

$$\sigma_u < k + 1, \omega) = u < k + 1, \omega) = < k, \omega >,$$

$$u^l[\sigma_u < k + 1, \omega)] = u^{l+1} < k + 1, \omega) = = < k - l, \omega >.$$

$$u^k < u^{k-1} \,.\, \sigma_u : \; u^k(\omega) = \{\omega\} \subsetneqq P = u^{k-1}[\sigma_u(\omega)].$$

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