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ON A CERTAIN DIFFERENTIAL EQUATION WITH TIME LAG

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This paper deals with the differential equation with time lag in the form

$$a_0u'(t) + b_0u(t) + b_1u(t-\omega) = f(t) + \int_{\alpha}^{\beta} b(t_1)u(t-t_1)\,dt_1,\ t > \gamma \tag{1}$$

where a_0, b_0, b_1, ω, α, β are constants; $a_0 \neq 0$, $\alpha > 0$, $\omega \geqq \alpha$, $\beta > \alpha$, $\gamma = \max(\beta, \omega)$. We are concerned with the existence and uniqueness of the solution, with relations between solutions of the equation (1) and finally with the representation of the solution of the equation (1) in the form of an integral over a simple regular curve.

Theorem 1. *Let $f(t) \in C^0 \langle 0, \infty)$, $b(t) \in C^0 \langle \alpha, \beta\rangle$ and $g(t) \in C^0 \langle 0, \gamma\rangle$. The equation (1) has one and only one solution $u(t)$ with $t > \gamma$, which satisfies the initial condition in the form*

$$u(t) = g(t) \text{ for } t \in \langle 0, \gamma \rangle. \tag{2}$$

Proof: First we shall prove this theorem for the interval $\langle\gamma, \gamma + \alpha\rangle$. Put

$$v(t) = f(t) + \int_{\alpha}^{\beta} b(t_1)\, u(t-t_1)\,dt_1 - b_1u(t-\omega).$$

Then the equation (1) can be written in the form

$$a_0u'(t) + b_0u(t) = v(t). \tag{3}$$

From the assumption of the theorem and from the representation of the function $v(t)$ it is evident that this function is continuous in $\langle\gamma, \gamma + \alpha\rangle$, i. e. $v(t) \in C^0 \langle\gamma, \gamma + \alpha\rangle$.

But the equation (3) is a linear differential equation. Since a_0, b_0 are constants, $v(t)$ is a continuous function in $\langle\gamma, \gamma + \alpha\rangle$, according to the Picard existential theorem there exists only one solution $u(t)$ of the equation (3), defined in the whole interval $\langle\gamma, \gamma + \alpha\rangle$ and satisfying initial condition $u(\gamma) = g(\gamma)$. The given solution is continuous and satisfies equation (1) in $(\gamma, \gamma + \alpha\rangle$. Since the solution $u(t)$ is continuous, it is easy to find out that the function $v(t)$ is continuous in the interval $\langle\gamma, \gamma + 2\alpha\rangle$. But it follows from the equation (3), that there exists only one continuous solution $u(t)$ which satisfies equation (1) in the whole interval $(\gamma, \gamma + 2\alpha\rangle$. Evidently we can repeat this process, extending thus the solution of the equation (1).

Thus the solution of the equation (1) with the initial condition (2) really exists and this solution is unique for any $t > \gamma$. This completes the proof.

Note 1. *Because $u(t) \in C^0 \langle 0, \infty)$, it follows from the equation (1), that $u'(t) \in C^0(\gamma, \infty)$ or that $u(t) \in C^1(\gamma, \infty)$.*

2. *If $g(t) \in C^1 < 0, \gamma >$, then $u'(t)$ is continuous in the point γ if and only if*

$$a_0 g'(\gamma - 0) + b_0 g(\gamma) + b_1 g(\gamma - \omega) = f(\gamma) + \int_\alpha^\beta b(t_1) \; g(\gamma - t_1) \, \mathrm{d}t_1.$$

The equation (1) can be written in the form

$$a_0 u'(t) + b_0 u(t) + b_1 u(t - \omega) - \int_\alpha^\beta b(t_1) \, u(t - t_1) \, \mathrm{d}t_1 = f(t)$$

and we define the linear operator $\mathbf{L}(u)$ as follows:

$$\mathbf{L}(u) = a_0 u'(t) + b_0 u(t) + b_1 u(t - \omega) - \int_\alpha^\beta b(t_1) \; u(t - t_1) \, \mathrm{d}t_1.$$

Then the following theorem holds:

Theorem 2. *Let $u_1(t)$, $u_2(t)$ — be any two solutions of the equation $\mathbf{L}(u) = 0$ and c_1, c_2 — arbitrary constants. Then $c_1 u_1(t) + c_2 u_2(t)$ is also a solution of the equation $\mathbf{L}(u) = 0$.*

To prove this theorem it is sufficient to note that $\mathbf{L}(c_1 u_1 + c_2 u_2) = c_1 \mathbf{L}(u_1) + c_2 \mathbf{L}(u_2) = 0$.

Theorem 3. *Let $w_1(t)$ be a solution of the equation $\mathbf{L}(u) = f(t)$ and let $w_2(t)$ be a solution of the equation $\mathbf{L}(u) = 0$. Then $w_1(t) + w_2(t)$ is solution of the equation $\mathbf{L}(u) = f(t)$.*

Again it is sufficient to show that $\mathbf{L}(w_1 + w_2) = \mathbf{L}(w_1) + \mathbf{L}(w_2) = f(t) + 0 = f(t)$.

Note: *The theorems we have proved for the equations $\mathbf{L}(u) = 0$ and $\mathbf{L}(u) = f(t)$ are analogues of those well known from the theory of ordinary differential equations.* Similarly, we shall try to find the solution of the equation $\mathbf{L}(u) = 0$ in the form of an exponential function. To do this, we shall investigate the conditions for a function $u(t) = \mathrm{e}^{st}$ to represent a solution of the equation $\mathbf{L}(u) = 0$.

$$\mathbf{L}(\mathrm{e}^{st}) = (a_0 s + b_0 + b_1 \mathrm{e}^{-\omega s} - \int_\alpha^\beta b(t_1) \, \mathrm{e}^{-st_1} \, \mathrm{d}t_1) \, \mathrm{e}^{st}.$$

Thus $u(t) = \mathrm{e}^{st}$ is a solution of the equation $\mathbf{L}(u) = 0$ if and only if s is a zero of the following transcendental function

$$h(s) = a_0 s + b_0 + b_1 \mathrm{e}^{-\omega s} - \int_\alpha^\beta b(t_1) \, \mathrm{e}^{-st_1} \, \mathrm{d}t_1. \tag{4}$$

The function $h(s)$ will be called the characteristic function and the equation $h(s) = 0$ the characteristic equation; this equation has in general infinitely many roots.

Theorem 4. *Let $\{s_r\}$ be an arbitrary sequence of roots of the characteristic equation $h(s) = 0$. Let $p_r(t)$ be a polynom whose order is lower than the multiplicity of root s_r. Then the function*

$$\sum_r p_r(t) \, \mathrm{e}^{s_r t}$$

satisfies the equation $\mathbf{L}(u) = 0$.

The sum is finite or infinite, and in the latter fall it is expected, that given sum uniformly converges.

Proof: The characteristic function of the equation $\mathbf{L}(u) = 0$ is

$$h(s) = a_0 s + b_0 + b_1 \mathrm{e}^{-\omega s} - \int_\alpha^\beta b(t_1)\, \mathrm{e}^{-st_1}\, \mathrm{d}t_1. \tag{4}$$

Then

$$h'(s) = a_0 - b_1 \omega \mathrm{e}^{-\omega s} + \int_\alpha^\beta b(t_1)\, t_1 \mathrm{e}^{-st_1}\, \mathrm{d}t_1,$$

$$h''(s) = b_1 \omega^2 \mathrm{e}^{-\omega s} - \int_\alpha^\beta b(t_1)\, t_1^2 \mathrm{e}^{-st_1}\, \mathrm{d}t_1,$$

(5) ....

$$h^{(k)}(s) = (-1)^k b_1 \omega^k \mathrm{e}^{-\omega s} - (-1)^k \int_\alpha^\beta b(t_1)\, t_1{}^k \mathrm{e}^{-st_1}\, \mathrm{d}t_1,\ k = 3, 4 \ldots$$

For any $n \geqq 1$ we have

$$\mathbf{L}(t^n \mathrm{e}^{st}) = a_0 (t^n \mathrm{e}^{st})' + b_0 t^n \mathrm{e}^{st} + b_1 (t - \omega)^n \mathrm{e}^{s\,(t-\omega)} -$$
$$- \int_\alpha^\beta b(t_1)\, (t - t_1)^n \,.\, \mathrm{e}^{s\,(t-t_1)}\, \mathrm{d}t_1.$$

Hence after modifications and using (5)

$$\mathbf{L}(t^n \mathrm{e}^{st}) = \mathrm{e}^{st} \sum_{k=0}^{n} \binom{n}{k} t^{n-k} h^{(k)}(s). \tag{6}$$

Now consider a root s with multiplicity m of the characteristic equation. Evidently $h(s)$, $h'(s)$, $\ldots$, $h^{(m-1)}(s)$ are all zero. Using the relation (6) and the note following the theorem 3 we see that $\mathbf{L}(t^n \mathrm{e}^{st}) = 0$ for $0 \leqq n \leqq m - 1$. Thus to a root s with multiplicity m of the characteristic equation there correspond m functions e^{st}, $t\mathrm{e}^{st}$, $\ldots$, $t^{m-1}\mathrm{e}^{st}$, which are soluutions of the equation $\mathbf{L}(u) = 0$ for all real values of the variable t. It is apparent that these functions are linearly independent.

According to theorem 2 can conclude that $p(t)\mathrm{e}^{st}$ will be also a solution of the equation $\mathbf{L}(u) = 0$, where $p(t)$ is an arbitrary polynom of the order lower than m. If, finally, we consider all roots s_r of the characteristic equation, we obtain the statement of the theorem 4. The proof is complete.

We shall require an upper estimate of the solution of the equation (1), and this assessment will be made using the following well known lemma.

Lemma 1. *Let $w(t)$ be positive nondecreasing continuous function, let $u(t) \geqq 0$ $v(t) \geqq 0$ be continuous and let*

$$u(t) \leqq w(t) + \int_a^t u(t_1)\, v(t_1)\, \mathrm{d}t_1,\ a \leqq t \leqq b,$$

then

$$u(t) \leqq w(t) \exp\left[\int_a^t v(t_1)\,\mathrm{d}t_1\right], \; a \leqq t \leqq b.$$

Theorem 5. *Let $u(t)$ be a solution of the equation*

$$a_0u'(t) + b_0u(t) + b_1u(t-\omega) = f(t) + \int_\alpha^\beta b(t_1)\,u(t-t_1)\,\mathrm{d}t_1 \tag{1}$$

in the interval $< 0, \infty)$ with the initial condition

$$u(t) = g(t), \; 0 \leqq t \leqq \gamma,$$

where $f(t) \in C^0 < 0, \infty)$, $b(t) \in C^0 \langle\alpha, \beta\rangle$, $g(t) \in C^0 < 0, \gamma >$.
Then let

a) $|f(t)| \leqq c_1 e^{c_2 t}, t \geqq 0$
where c_1 and c_2 are positive constants;

b) $m = \max\limits_{0 \leqq t \leqq \gamma} |g(t)|$;

c) $\boldsymbol{B} = \max\limits_{\alpha \leqq t \leqq \beta} |b(t)|$

Then there exist such positive constants c_3 and c_4, depending only on:
the coeficients of the equation (1),
c_1 and c_2,
the maximum of the function $g(t)$ in interval $\langle 0, \gamma\rangle$,
the maximum of the function $b(t)$ in interval $\langle\alpha, \beta\rangle$
and the length of the interval $\langle\alpha, \beta\rangle$,
that

$$|u(t)| \leqq c_3 e^{c_4 t}, t \geqq 0. \tag{7}$$

Proof: From the equation (1) we can see that

$$a_0u(t) = a_0u(\gamma) + \int_\gamma^t f(t_1)\,\mathrm{d}t_1 + \int_\gamma^t \int_\alpha^\beta b(t_2)\,u(t_1-t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1 -$$

$$-b_0 \int_\gamma^t u(t_1)\,\mathrm{d}t_1 - b_1 \int_\gamma^t u(t_1-\omega)\,\mathrm{d}t_1, \; t \geqq \gamma.$$

From the assumption of the theorem we obtain

$$|a_0|\,|u(t)| \leqq |a_0|\,m + c_1 \int_\gamma^t e^{c_2 t_1}\,\mathrm{d}t_1 + \boldsymbol{B} \int_\gamma^t \int_\alpha^\beta |u(t_1-t_2)|\,\mathrm{d}t_2\,\mathrm{d}t_1 + \tag{8}$$

$$+ |b_0| \int_\gamma^t |u(t_1)|\,\mathrm{d}t_1 + |b_1| \int_\gamma^t |u(t_1-\omega)|\,\mathrm{d}t_1, \; t \geqq \gamma.$$

First we shall modify the integral

$$\int_{\gamma}^{t}\int_{\alpha}^{\beta} | u(t_1 - t_2) | \, dt_2 \, dt_1.$$

Put $t_1 - t_2 = p$,
$t_2 = q$,
then $t_1 = p + q$,
$t_2 = q$,
where $\gamma - \beta \leqq p \leqq t - \alpha$,
$\alpha \leqq q \leqq \beta$ and $\mathbf{D} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$.

It is evident that the conditions for transforming the integral are satisfied, so we can write:

$$\int_{\gamma}^{t}\int_{\alpha}^{\beta} | u(t_1 - t_2) | \, dt_2 \, dt_1 \leqq \int_{\gamma-\beta}^{t-\alpha}\int_{\alpha}^{\beta} | u(p) | \, dq \, dp =$$

$$= (\beta - \alpha) \int_{\gamma-\beta}^{t-\alpha} | u(p) | \, dp \leqq \beta - \alpha) \int_{\gamma-\beta}^{t} | u(t_1) | \, dt_1.$$

After substituting this (8) we have:

$$(9) \qquad | a_0 | \, | u(t) | \leqq | a_0 | \, m + c_1 \int_{\gamma}^{t} e^{c_2 t_1} \, dt_1 + \boldsymbol{B}(\beta - \alpha) \int_{\gamma-\beta}^{t} | u(t_1) | \, dt_1 +$$

$$+ | b_0 | \int_{\gamma}^{t} | u(t_1) | \, dt_1 + | b_1 | \int_{\gamma}^{t} | u(t_1 - \omega) | \, dt_1, \; t \geqq \gamma.$$

Modifying the last term on the right hand side of the inequality (9) we find that the following holds:

$$| b_1 | \int_{\gamma}^{t} | u(t_1 - \omega) | \, dt_1 \leqq | b_1 | \int_{\gamma-\omega}^{t} | u(t_1) | \, dt_1.$$

So after further modifications of the inequality (9) we obtain:

$$| u(t) | \leqq m + \frac{c_1}{c_2 | a_0 |} e^{c_2 t} + \frac{\boldsymbol{B}(\beta - \alpha)}{| a_0 |} \int_{0}^{t} | u(t_1) | \, dt_1 + \frac{| b_0 |}{| a_0 |} \int_{0}^{t} | u(t_1) | \, dt_1 +$$

$$+ \frac{| b_1 |}{| a_0 |} \int_{0}^{t} | u(t_1) | \, dt_1, \; t \leqq \gamma$$

or

$$| u(t) | \leqq m + \frac{c_1}{c_2 | a_0 |} e^{c_2 t} + \frac{\boldsymbol{B}(\beta - \alpha) + | b_0 | + | b_1 |}{| a_0 |} \int_{0}^{t} | u(t_1) | \, dt_1, \; t \geqq \gamma.$$

From the last inequality and lemma 1 we obtain:

$$(10) \qquad | u(t) | \leqq (m + \frac{c_1}{c_2 \mid a_0 \mid} e^{c_2 t}) \cdot \exp \left(\frac{B(\beta - \alpha) + | b_0 | + | b_1 |}{| a_0 |} t \right) \leqq$$

$$\leqq \left(m + \frac{c_1}{c_2 | a_0 |} \right) \exp \left[\left(c_2 + \frac{B(\beta - \alpha) + | b_0 | + | b_1 |}{| a_0 |} t \right] , t \geqq \gamma .$$

As $| u(t) | \leqq m$ for $0 \leqq t \leqq \gamma$, we see that our estimate is valid for any $t \geqq 0$.

Finally designate

$$(11) \qquad c_3 = m + \frac{c_1}{c_2 \mid a_0 \mid}, \; c_4 = c_2 + \frac{B(\beta - \alpha) + | b_0 | + | b_1 |}{| a_0 |} .$$

Using (11) we can write the inequality (10) in the form

$$| u(t) | \leqq c_3 \, e^{c_4 t}, \quad t \geqq 0.$$

Thus the theorem is proved.

Prior to the formulation of the solution of equation (1) in form of an integral over simple regular curve, one more lemma will be stated.

Lemma 2. *Suppose that $b(t) \in C^0 < \alpha, \beta >$ and does not change its sign in $\langle \alpha, \beta \rangle$. Then all the zeroes of the characteristic function*

$$(4) \qquad h(s) = a_0 s + b_0 + b_1 e^{-\omega s} - \int_\alpha^\beta b(t_1) \, e^{-s t_1} \, dt_1$$

lie in the complex plane to the left of the line $x = M$, where the value of the constant M depends only on the coefficients of the equation (1) and on the value of the integral

$$\int_\alpha^\beta b(t_1) \, dt_1 .$$

Proof: As the function $h(s)$ can also have complex null points, put $s = x + iy$ and modify first the integral

$$\int_\alpha^\beta b(t_1) \, e^{-s t_1} \, dt_1 .$$

Then

$$\int_\alpha^\beta b(t_1) \, e^{-s t_1} \, dt_1 = \int_\alpha^\beta b(t_1) \, e^{-x t_1 - i y t_1} \, dt_1 =$$

$$= \int_\alpha^\beta b(t_1) \, e^{-x t_1} (\cos y t_1 - i \sin y t_1) \, dt_1 =$$

$$= \int_\alpha^\beta b(t_1) \, e^{-x t_1} \cos y t_1 \, dt_1 - i \int_\alpha^\beta b(t_1) \, e^{-x t_1} \sin y t_1 \, dt_1 .$$

With regard to the assumption for the function $b(t)$ we can use the mean value theorem obtaining:

$$\int_{\alpha}^{\beta} b(t_1)\, e^{-st_1}\, dt_1 = \int_{\alpha}^{\beta} b(t_1)\, e^{-xt_1} \cos yt_1\, dt_1 - i \int_{\alpha}^{\beta} b(t_1)\, e^{-xt_1} \sin yt_1\, dt_1 =$$

$$e^{-\mu x} \cos \mu y \int_{\alpha}^{\beta} b(t_1)\, dt_1 - ie^{-\nu x} \sin \nu y \int_{\alpha}^{\beta} b(t_1)\, dt_1,\ \mu, \nu \in \langle\alpha,\beta\rangle.$$

Put $\mathbf{K} = \int_{\alpha}^{\beta} b(t_1)\, dt_1$.

Then $\int_{\alpha}^{\beta} b(t_1)\, e^{-st_1}\, dt_1 = \mathbf{K}\,(e^{-\mu x} \cos \mu y - ie^{-\nu x} \sin \nu y)$

and the function $h(s)$ can be written in the form:

$$h(s) = h(x + iy) = a_0(x + iy) + b_0 + b_1 e^{-\omega x}(\cos \omega y - i \sin \omega y) -$$
$$- \mathbf{K}(e^{-\mu x} \cos \mu y - ie^{-\nu x} \sin \nu y).$$

Now the lemma can be proved by making an upper estimate of the absolute value for the exponential part of the function $h(s)$ and a lower estimate of the absolute value for its polynomial part. The lemma is proved by a comparison of both estimates.

Evidently it is sufficient to limit our considerations to the nonnegative values of the variable x.

Then

$$| b_1 e^{-\omega x}(\cos \omega y - i \sin \omega y) - \mathbf{K}e^{-\mu x} \cos \mu y + i\, \mathbf{K}e^{-\nu x} \sin \nu y | \leqq$$
$$\leqq | b_1 e^{-\omega x} | + | \mathbf{K}e^{-\mu x} | + | \mathbf{K}e^{-\nu x} | \leqq | b_1 | + 2 | \mathbf{K} |, \text{ for } x \geqq 0.$$

And then

$$| a_0(x + iy) + b_0 | = | a_0 x + b_0 + ia_0 y | = \sqrt{(a_0 x + b_0)^2 + a_0^2 y^2} \geqq | a_0 x + b_0 |.$$

We investigate now the conditions for the following inequality:

$$| a_0 x + b_0 | > | b_1 | + 2 | \mathbf{K} |, \text{ by } x \geqq 0.$$

Evidently the real component of no zero of the function $h(s)$ can lie in the interval which will be solution of the last inequality

1. Let $a_0 x + b_0 \geqq 0$,

a) let $a_0 > 0$, then $x > \dfrac{| b_1 | - b_0}{a_0} + \dfrac{2 | \mathbf{K} |}{a_0}$,

b) let $a_0 < 0$, then $x < \dfrac{| b_1 | - b_0}{a_0} + \dfrac{2 | \mathbf{K} |}{a_0}$.

But the case b) is impossible already for $x > -\dfrac{b_0}{a_0}$, since in that case $a_0 x + b_0 < 0$.

2. Let $a_0x + b_0 < 0$,

a) let $a_0 < 0$, then $x > \frac{|b_1| + b_0}{-a_0} + \frac{2|K|}{-a_0}$,

b) let $a_0 > 0$, then $x < \frac{|b_1| + b_0}{-a_0} + \frac{2|K|}{-a_0}$.

But again the case b) is impossible for $x > -\frac{b_0}{a_0}$, since in that case $a_0x + b_0 > 0$.

Put $M = \max\left\{\frac{|b_1| - b_0}{a_0} + \frac{2|K|}{a_0} + \varepsilon; \frac{|b_1| + b_0}{-a_0} + \frac{2|K|}{-a_0} + \varepsilon\right\}$,

where ε is an arbitrary positive number.

We see that all zeros of the function $h(s)$ lie left to the line $x = M$, thus for any zero s of the function $h(s)$ $\mathrm{Re}(s) < M$.

The proof is complete.

Theorem 6. *Let $u(t)$ be a solution of the equation*

$$(1) \qquad a_0u'(t) + b_0u(t) + b_1u(t-\omega) = f(t) + \int_\alpha^\beta b(t_1)\, u(t-t_1)\, dt_1,\ t > \gamma,\ a_0 \neq 0$$

with the initial condition $u(t) = g(t)$, $0 \leqq t \leqq \gamma$.
Let $g(t) \in C^1 < 0, \gamma >$; let $b(t) \in C^0 < \alpha, \beta >$ and let it not change sign in the interval $< \alpha, \beta >$;
let $f(t) \in C^0 < 0, \infty)$ and $|f(t)| \leqq c_1e^{c_2t}$, $t \geqq 0$, $c_1 > 0$, $c_2 > 0$.
Then for an arbitrary sufficiently large constant C

$$(12) \qquad u(t) = \int_{(C)} e^{ts}h^{-1}(s)\,[p(s) + q(s)]\, ds,\ t > \gamma,$$

where $p(s) = a_0g(\gamma)\, e^{-\gamma s} - b_1e^{-\omega s} \int_{\gamma-\omega}^{\gamma} g(t)e^{-st}\, dt + \int_\alpha^\beta b(t_1)\ e^{-st_1}\, dt_1 \cdot \int_{\gamma-t_1}^{\gamma} e^{-st}\, g(t)\, dt$,

$$q(s) = \int_\gamma^\infty f(t)\, e^{-st}\, dt.$$

Proof: Multiply the equation (1) by the term e^{-st} and integrate from γ to ∞. We note from the previous (theorem 5) the existence of the following integrals follows

$$(13) \qquad \int_\gamma^\infty u(t)e^{-st}\, dt,\ \int_\gamma^\infty u(t-\omega)\, e^{-st}\, dt,\ \int_\gamma^\infty f(t)\, e^{-st}\, dt$$

for an arbitrary complex s, for which $\mathrm{Re}(s) > c_4$.

We can modify individual terms.

Then

$$(14) \quad \int_\gamma^\infty u'(t)e^{-st}\, dt = \Big[u(t)e^{-st}\Big]_\gamma^\infty + s\int_\gamma^\infty u(t)e^{-st}\, dt = -g(\gamma)\, e^{-\gamma s} + s\int_\gamma^\infty u(t)e^{-st}\, dt$$

because $\lim\limits_{t\to\infty} u(t)e^{-st} = 0$, if $\mathrm{Re}(s) > c_4$.

Then by substitution $t - \omega = t_1$

(15) $$\int_{\gamma}^{\infty} u(t-\omega)\, e^{-st}\, dt = \int_{\gamma-\omega}^{\infty} u(t_1)\, e^{-s(t_1+\omega)}\, dt_1 = e^{-\omega s} \int_{\gamma-\omega}^{\infty} u(t) e^{-st}\, dt =$$

$$= e^{-\omega s} \left[\int_{\gamma-\omega}^{\infty} g(t)\, e^{-st}\, dt + \int_{\gamma} u(t) e^{-st}\, dt \right].$$

And finally using interchange of order of the integration, which can be done in this case we have

$$\int_{\gamma}^{\infty} e^{-st} \int_{\alpha}^{\beta} b(t_1)\, u(t-t_1)\, dt_1\, dt = \int_{\alpha}^{\beta} b(t_1) \int_{\gamma}^{\infty} u(t-t_1)\, e^{-st}\, dt\, dt_1.$$

Using the substitution $t - t_1 = t_2$, the right side of this equation can be written in the form

$$\int_{\alpha}^{\beta} b(t_1) \int_{\gamma-t_1}^{\infty} u(t_2)\, e^{-s(t_1+t_2)}\, dt_2\, dt_1 = \int_{\alpha}^{\beta} b(t_1)\, e^{-st_1} \int_{\gamma-t_1}^{\infty} u(t) e^{-st}\, dt\, dt_1 =$$

$$= \int_{\alpha}^{\beta} b(t_1)\, e^{-st_1}\, dt_1 \int_{\gamma-t_1}^{\gamma} g(t)\, e^{-st}\, dt + \int_{\alpha}^{\beta} b(t_1)\, e^{-st_1}\, dt_1 \int_{\gamma}^{\infty} u(t)\, e^{-st}\, dt.$$

Thus

(16) $$\int_{\gamma}^{\infty} e^{-st} \int_{\alpha}^{\beta} b(t_1)\, u(t-t_1)\, dt_1\, dt = \int_{\alpha}^{\beta} b(t_1)\, e^{-st_1}\, dt_1 \int_{\gamma-t_1}^{\gamma} g(t)\, e^{-st}\, dt +$$

$$+ \int_{\alpha}^{\beta} b(t_1)\, e^{-st_1}\, dt_1 \int_{\gamma}^{\infty} u(t)\, e^{-st}\, dt.$$

From (1) using (13), (14), (15) and (16) we obtain

$$-a_0 g(\gamma) e^{-\gamma s} + a_0 s \int_{\gamma}^{\infty} u(t) e^{-st}\, dt + b_0 \int_{\gamma}^{\infty} u(t)\, e^{-st}\, dt + b_1 e^{-\omega s} \cdot \int_{\gamma-\omega}^{\gamma} g(t) e^{-st}\, dt +$$

$$+ b_1 e^{-\omega s} \int_{\gamma}^{\infty} u(t) e^{-st}\, dt = \int_{\gamma}^{\infty} f(t) e^{-st}\, dt + \int_{\alpha}^{\beta} b(t_1)\, e^{-st_1}\, dt_1 \cdot \int_{\gamma-t_1}^{\gamma} g(t)\, e^{-st}\, dt +$$

$$+ \int_{\alpha}^{\beta} b(t_1)\, e^{-st_1}\, dt_1 \int_{\gamma}^{\infty} u(t)\, e^{-st}\, dt, \quad \mathrm{Re}(s) > c_4.$$

Hence

(17) $$\left[a_0 s + b_0 + b_1 e^{-\omega s} - \int_{\alpha}^{\beta} b(t_1) e^{-st_1}\, dt_1 \right] \int_{\gamma}^{\infty} u(t) e^{-st}\, dt = \int_{\gamma}^{\infty} f(t)\, e^{-st}\, dt +$$

$$+ a_0 g(\gamma)\, e^{-\gamma s} - b_1 e^{-\omega s} \int_{\gamma-\omega}^{\gamma} g(t)\, e^{-st}\, dt + \int_{\alpha}^{\beta} b(t_1)\, e^{-st_1}\, dt_1 \int_{\gamma-t_1}^{\gamma} g(t) e^{-st}\, dt, \quad \mathrm{Re}(s) > c_4.$$

According to the previously introduced designation, the left hand side of (17) is actually $h(s) \int_{\gamma}^{\infty} u(t)\, e^{-st}\, dt$.

Further put

$$p(s) = a_0 g(\gamma)\, e^{-\gamma s} - b_1 e^{-\omega s} \int_{\gamma-\omega}^{\gamma} g(t)\, e^{-st}\, dt + \int_{\alpha}^{\beta} b(t_1)\, e^{-st_1}\, dt_1 \cdot \int_{\gamma-t_1}^{\gamma} g(t)\, e^{-st}\, dt$$

and

$$q(s) = \int_{\gamma}^{\infty} f(t) e^{-st}\, dt.$$

Then (17) can be written in form

$$h(s) \int_{\gamma}^{\infty} u(t)\, e^{-st}\, dt = p(s) + q(s), \quad \operatorname{Re}(s) > c_4.$$

It is known from the lemma 2, that all the zeroes of the function $h(s)$ have the property that $\operatorname{Re}(s) < M$ for sufficiently large M. Therefore we can write

$$\int_{\gamma}^{\infty} u(t)\, e^{-st}\, dt = h^{-1}(s)\Big[p(s) + q(s)\Big], \quad \operatorname{Re}(s) > M, \quad \operatorname{Re}(s) > c_4.$$

Let $c = \max\,(M, c_4)$, then

$$\int_{\gamma}^{\infty} u(t)\, e^{-st}\, dt = h^{-1}(s)\Big[p(s) + q(s)\Big], \operatorname{Re}(s) > c. \tag{18}$$

According to the notes following theorem 1 $u(t) \in C^1 < \gamma, \infty)$; therefore it is continuous in that interval nad has a finite variation over any finite interval. The formula for determining a function from its known Laplace transform can therefore be used to represent the solution of (1).

From (18) then follows

$$u(t) = \int_{C} e^{ts} h^{-1}(s)\Big[p(s) + q(s)\Big]\, ds, \quad t > \gamma,$$

and proof is complete.

LITERATURE

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