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ON CERTAIN PROPERTIES OF THE SOLUTIONS OF A NON LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER

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The first part of this paper is a presentation of some results concerning the boundedness of solutions of a non-linear differential equation of the second order together with their first derivatives. A theorem is proved which is a generalization of a theorem of Kiguradze [1] and some results from [2], [3] and [4] are generalized and extended.

The second part deals with the oscillatory properties of solutions.

1. THE BOUNDEDNESS OF SOLUTIONS AND THEIR FIRST DERIVATIVES

Consider a non-linear differential equation of the second order o the form

$$a(t)\, u'' + b(t)\, g(u, u') + f(t, u) = 0 \tag{1}$$

Let $F(t, x) = \int_0^x f(t, s)\, ds$. In some theorems we shall postulate the following conditions:

α) $f(t, x)$ and $\dfrac{\partial f(t,x)}{\partial t}$ are continuous for $t \geqq t_0 \geqq 0$, $|x| < \infty$;

β) $g(x, y)$ is continuous for all x and y and there exists a non-negative constant k such that $yg(x, y) \geqq ky^2$ for all x and y;

γ) $a(t)$, $b(t)$ are continuous non-negative functions for $t \geqq t_0 \geqq 0$ and $2kb(t) \geqq a'(t)$.

Theorem 1: *Suppose that α), β) and γ) hold. Suppose further that for any continuously differentiable function $x(t)$ on $(t_0, \bar{t})$ where $t_0 < \bar{t} \leqq \infty$ which is unbounded for $t \to \bar{t}_-$, there exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that, for $t_i \to \bar{t}_-$,*

$$\frac{\partial F(t, x(t))}{\partial t} \leqq \frac{\partial F(t, x(t_i))}{\partial t}, \quad t_0 \leqq t \leqq t_i, \tag{2}$$

$$\lim_{i \to \infty} F(t_0, x(t_i)) = F, \tag{3}$$

with $F \leqq \infty$ independent of $x(t)$.

Then every solution $u(t)$ of (1) defined for $t \geqq t_0$, for which

$$F(t_0, u(t_0)) + \frac{1}{2} a(t_0)\, u'^2(t_0) < F, \text{ is bounded for all } t \geqq t_0. \tag{4}$$

Proof: Let the solution $u(t)$ exist on $\langle t_0, \bar{t})$ and suppose that it satisfies the condition (4). Suppose that $\limsup_{t \to \bar{t}_-} |u(t)| = +\infty$ for $t_0 < \bar{t} \leqq \infty$. By multiplying (1) by $u'(t)$ we get

$$a(t)\, u''u' + b(t)\, g(u, u')\, u' + f(t, u)\, u' = 0,$$

and by integrating

$$(5)\quad \frac{1}{2}\int_{t_0}^{t} a(s)\frac{\mathrm{d}}{\mathrm{d}s}u'^2(s)\,\mathrm{d}s + \int_{t_0}^{t} b(s)\,g(u,u')\,u'(s)\,\mathrm{d}s + F(t,u(t)) = F(t_0,u(t_0)) + \int_{t_0}^{t}\frac{\partial F(s,u(s))}{\partial s}\,\mathrm{d}s.$$

Since

$$\frac{1}{2}\int_{t_0}^{t} a(s)\frac{\mathrm{d}}{\mathrm{d}s}u'^2(s)\,\mathrm{d}s + \int_{t_0}^{t} b(s)\,g(u,u')\,u'(s)\,\mathrm{d}s = \frac{1}{2}a(t)\,u'^2(t) - \frac{1}{2}a(t_0)\,u'^2(t_0) + \int_{t_0}^{t}\left[b(s)\,g(u,u')\,u'(s) - \frac{1}{2}a'(s)u'^2(s)\right]\mathrm{d}s \geqq \frac{1}{2}a(t)\,u'^2(t) - \frac{1}{2}a(t_0)\,u'^2(t_0),$$

we obtain, using (5), the inequality

$$(6)\quad \frac{1}{2}a(t)\,u'^2(t) + F(t,u(t)) \leqq \frac{1}{2}a(t_0)\,u'^2(t_0) + Ft_0,(u(t_0)) + \int_{t_0}^{t}\frac{\partial F(s,u(s))}{\partial s}\,\mathrm{d}s.$$

Since $u(t)$ is unbounded in $(\bar{t}-\partial,\bar{t})$, there exists a sequence $\{t_i\}_{i=1}^{\infty}$, $t_i \to \bar{t}_-$, which satisfies the assumptions (2) and (3) (if we put $x(t) = u(t)$). The last inequality then yields

$$F(t_i,u(t_i)) \leqq \frac{1}{2}a(t_0)\,u'^2(t_0) + F(t_0,u(t_0)) + \int_{t_0}^{t_i}\frac{\partial F(s,u(t_i))}{\partial s}\,\mathrm{d}s = \frac{1}{2}a(t_0)\,u'^2(t_0) + F(t_0,u(t_0)) + F(t_i,u(t_i)) - F(t_0,u(t_i)),$$

or

$$F(t_0,u(t_i)) \leqq \frac{1}{2}a(t_0)\,u'^2(t_0) + F(t_0,u(t_0)).$$

For $i \to \infty$ we get

$$F \leqq \frac{1}{2}a(t_0)\,u'^2(t_0) + F(t_0,u(t_0)),$$

which is contradictory to (4).

Lemma: *Let $\varphi(t)$ and $\varphi'(t)$ be continuous on $\langle t_0,\bar{t})$ where $\bar{t} < \infty$ and suppose that $\lim_{t\to\bar{t}_-}\varphi(t)$ does not exist. Then*

$$\limsup_{t\to\bar{t}-}\varphi'(t) = +\infty$$

and

$$\liminf_{t \to \bar{t}_-} \varphi'(t) = -\infty.$$

Proof: Let $\limsup_{t \to \bar{t}_-} \varphi(t) = A$ and $\liminf_{t \to \bar{t}_-} \varphi(t) = B$. This means that there exist sequences $\{t_i\}_{i=1}^{\infty}$, $\{\tilde{t}_i\}_{i=1}^{\infty}$, such that for $i \to \infty$, $t_i \to \bar{t}_-$, $\tilde{t}_i \to \bar{t}_-$ and that $\lim_{i \to \infty} \varphi(t_i) = A$ and $\lim_{i \to \infty} \varphi(\tilde{t}_i) = B$. By the mean value theorem by Langrange there exists a point $\xi_i \in (t_i, \tilde{t}_i)$ such that

$$(*) \qquad \frac{\varphi(t_i) - \varphi(\tilde{t}_i)}{t_i - \tilde{t}_i} = \varphi'(\xi_i).$$

Now let $\{\tilde{t}_{i_k}\}_{k=1}^{\infty}$ and $\{t_{i_k}\}_{k=1}^{\infty}$ be subsequences of $\{t_i\}_{i=1}^{\infty}$ and $\{\tilde{t}_i\}_{i=1}^{\infty}$ respectively such that for great k $t_{i_k} > \tilde{t}_{i_k}$, $t_{i_k} \to \bar{t}_-$, $\tilde{t}_{i_k} \to \bar{t}_-$ for $k \to \infty$. Using (*) for $k \to \infty$, we have

$$\lim_{k \to \infty} \varphi'(\xi_{i_k}) = +\infty$$

where $\xi_{i_k} \in (t_{i_k}, \tilde{t}_{i_k})$.

Analogously we prove that $\liminf_{t \to \bar{t}_-} \varphi'(t) = -\infty$. Here the subsequences $\{t_{i_k}\}_{k=1}^{\infty}$, $\{\tilde{t}_{i_k}\}_{k=1}^{\infty}$ are chosen so that $t_{i_k} < \tilde{t}_{i_k}$, $t_{i_k} \to \bar{t}_-$ $\tilde{t}_{i_k} \to \bar{t}_-$ for $k \to \infty$. Again we use (*) to get

$$\frac{\varphi(t_{i_k}) - \varphi(\tilde{t}_{i_k})}{\tilde{t}_{i_k} - t_{i_k}} = -\varphi'(\xi_{i_k}),$$

where $\xi_{i_k} \in (\tilde{t}_{i_k}, t_{i_k})$. Therefore

$$-\lim_{k \to \infty} \varphi'(\xi_{i_k}) = +\infty,$$

which completes the proof.

Theorem 1a: *Suppose that, in addition to the hypotheses of Theorem* 1, $a(t) > 0$ *for* $t \geqq t_0$. *Then any solution of* (1) *satisfying* (4) *is defined and bounded on* $\langle t_0, \infty)$.

Proof: Let $u(t)$ be a solution of (1) satisfying (4). According to Theorem 1 $u(t)$ is bounded for $t \geqq t_0$. It is therefore sufficient to prove that the solution exists on $\langle t_0, \infty)$.

Let $u(t)$ exist on $\langle t_0, \bar{t})$, $\bar{t} < \infty$. We can distinguish two cases:

I. $\lim_{t \to \bar{t}_-} u(t) = Y$ and either $\lim_{t \to \bar{t}_-} |u'(t)| = +\infty$ or $\lim_{t \to \bar{t}_-} u'(t)$ does not exist. Let $\lim_{t \to \bar{t}_-} |u'(t)| = +\infty$. For any sequence $\{t_i\}_{i=1}^{\infty}$, such that for $i \to \infty$ $t_i \to \bar{t}_-$ we get, using (6):

$$(6a) \qquad \frac{1}{2} a(t_i) u'^2(t_i) + F(t_i, u(t_i)) \leqq \frac{1}{2} a(t_0) u'^2(t_0) + F(t_0, u(t_0)) + \int_{t_0}^{t_i} \frac{\partial F(s, u(s))}{\partial s} \, ds.$$

Owing to α) and the fact $u(t)$ is continuous for $t \in \langle t_0, \bar{t})$, $\int_{t_0}^{\bar{t}} \frac{\partial F(s, u(s))}{\partial s} ds$ is convergent. Then, since $a(t) > 0$, from (6a) we can see that $F(t_i, u(t_i)) \to -\infty$ for $i \to \infty$, which contradicts the continuity of $F(t, x)$ for $t \geqq t_0$, $|x| < \infty$.

Now suppose that $\lim_{t \to \bar{t}_-} u'(t)$ dies not exist. Let $\limsup_{t \to \bar{t}_-} u'(t) = A$ and $\liminf_{t \to \bar{t}_-} u'(t) = B$. The assumptions $A = +\infty$, or $B = -\infty$ again lead to contradiction (using (6a)). Suppose therefore that both A and B are finite. Since $u'(t)$ and $u''(t)$ are continuous for $t \in \langle t_0, \bar{t})$, we can use the lemma to show that $\limsup_{t \to \bar{t}_-} u''(t) = +\infty$ and $\liminf_{t \to \bar{t}_-} u''(t) = -\infty$. This gives us — for the numbers of a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \to \bar{t}_-$ for $i \to \infty$ and $\lim_{i \to \infty} u'(t_i) = +\infty$ — the following equation (by substituting into (1)): $a(t_i)\, u''(t_i) + b(t_i)\, g(u(t_i), u'(t_i)) + f(t_i, u(t_i)) = 0$. Since $a(\bar{t}) > 0$ and $b(t)$, $g(u, u')$ and $f(t, u)$ remain finite for $t_i \to \bar{t}_-$, this equation leads to contradiction if $i \to \infty$.

II. $\lim_{t \to t_-} u(t)$ does not exist. Since $u(t)$ is bounded and continuous for $t \in \langle t_0, \bar{t})$ and $u'(t)$ is likewise continuous for $t \in \langle t_0, \bar{t})$, it is a consequence of the lemma that $\limsup_{t \to t_-} u'(t) = +\infty$ and $\liminf_{t \to t_-} u'(t) = -\infty$. Analogously as in I., a contradiction can be deduced from (6a).

We have therefore proved that it is necessary that $\lim_{t \to \bar{t}_-} u(t) = Y$ and $\lim_{t \to \bar{t}_-} u'(t) = Y_1$ where both Y and Y_1 are finite. Therefore the solution $u(t)$ passing through the point $(t_0, u(t_0), u'(t_0))$ can be extended to pass through the point $(\bar{t}, Y, Y_1)$. By the appropriate existence theorem there exists a solution $u_1(t)$ of (1) which is defined for $t \in \langle \bar{t}, \bar{t}_1)$ and $u_1(\bar{t}) = Y$, $u_1'(\bar{t}) = Y_1$. Define a function $\tilde{u}(t)$ as follows:

$$\tilde{u}(t) = \begin{cases} u(t) & \text{for } t \in \langle t_0, \bar{t}) \\ u_1(t) & \text{for } t \in \langle \bar{t}, \bar{t}_1). \end{cases}$$

Now $\tilde{u}(t)$ is an extension of $u(t)$ to $(t_0, \bar{t}_1)$ which satisfies the condition (4) and is therefore bounded. If $\bar{t}_1 < \infty$, then the extension can be repeated. This proves the theorem.

Remark 1: *The assumption that* $a(t) > 0$ *for* $t \geqq t_0$ *is essential. The following example will demonstrate that for* $a(t) \geqq 0$ *not every solution can be extended.*

Example: The differential equation

$$(1 - t)^2 u'' + |1 - t|\, u' + \frac{3}{4} u = 0,$$

satisfies the assumptions of Theorem 1 with $F = +\infty$. Therefore every solution is bounded on its domain. It is easily demonstrated that $u(t) = \sqrt{1 - t}$ is a solution which cannot be extended beyond $t = 1$.

Theorem 2: *Suppose that* α), β) *and* γ) *hold and that there exists a sequence* $\{x_i\}_{i=1}^{\infty}$ *such that* $(-1)^i x_i > 0$ for $i = 1, 2, \ldots$, $\lim_{i \to \infty} |x_i| = \infty$ *and*

$$(7) \qquad \frac{\partial F(t, x)}{\partial t} \leqq \frac{\partial F(t, x_i)}{\partial t}, \quad |x| \leqq |x_i|, \; t \geqq t_0,$$

(8) $$\lim_{i\to\infty} F(t_0, x_i) = F \leqq +\infty.$$

Then every solution $u(t)$ *of* (1) which satisfies the condition (4) is bounded for all $t \geqq t_0$ *from its domain.*

Proof: Let $x(t)$ be a function which is continuous on $(t_0, \bar{t})$ and unbounded for $t \to \bar{t}_-$. Evidently there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that, for $k \to \infty$, $t_k \to \bar{t}_-$ and that $x(t_k) = x_{i_k}$, $|x(t)| \leqq |x_{i_k}|$ for $t_0 \leqq t \leqq t_k$, where $\{x_{i_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_i\}_{i=1}^{\infty}$.
Now (7) and (8) imply (2) and (3) with F independent of $x(t)$. Therefore under the assumptions of Thoerem 2, the hypotheses of Theorem 1 hold and so does its conclusion which is also a conclusion of Theorem 2.

Theorem 3: *Suppose that in the hypothesis of Theorem* 1 *the conditions* (2) *and* (3) *are replaced by*

(9) $$0 \leqq \frac{\partial f(t,x)}{\partial t} = -\frac{\partial f(t,-x)}{\partial t}, \quad x > 0, t \geqq t_0 \geqq 0$$

and

(10) $$\limsup_{|x|\to\infty} F(t_0, x) = +\infty.$$

Then all solutions of (1) *are bounded for* $t \geqq t_0$ *from their domain.*

Proof: Let $u(t)$ be defined on $\langle t_0, \bar{t})$. Using (9), we find that

$$\frac{\partial F(t,x)}{\partial t} \leqq \frac{\partial F(t,\tilde{x})}{\partial t} \text{ for } |x| \leqq |\tilde{x}|, t \geqq t_0.$$

Owing to (10), there exists a sequence $\{x_i\}_{i=1}^{\infty}$ such that $(-1)^i x_i > 0$, $|x_i| < |x_{i+1}|$ $(i = 1, 2, \ldots)$ and $\lim_{i\to\infty} F(t_0, x_i) = +\infty$. Thus the hypotheses of Theorem 2 are satisfied. Since in addition to that, $F = +\infty$, every solution of (1) is bounded on its domain.

Remark 2: *If* $a(t) \equiv 1, b(t)\, g(x,y) \equiv 0$ *for* $t \geqq t_0 \geqq 0$ *and* $|x| + |y| < \infty$, *then our Theorem* 1 *becomes Theorem* 1 *of* [1].

Theorem: 4 *Suppose that the assumptions* α), β) *and* γ) *hold. Moreover, suppose that for* $t \geqq t_0 \geqq 0$, $|x| < \infty$,

(11) $$\frac{\partial F(t,x)}{\partial t} \leqq 0$$

and that for any sequences $\{t_i\}_{i=1}^{\infty}$, $\{x_i\}_{i=1}^{\infty}$ *such that for* $i \to \infty$, $t_i \to \infty$ *and* $|x_i| \to \infty$

(12) $$\lim_{i\to\infty} F(t_i, x_i) = F \leqq \infty.$$

Then every solution of (1) *which satisfies the inequality*

(13) $$F(t_0, u(t_0)) + \frac{1}{2} a(t_0)\, u'^2(t_0) < F,$$

is bounded for $t \geqq t_0$ *from its domain.*

If in addition $a(t) \geqq a > 0$ *for* $t \geqq t_0$ *and*

(14) $$f(t,x) \operatorname{sgn} x > 0 \textit{ for } x \neq 0 \textit{ and } t \geqq t_0,$$

then the first derivative of any solution $u(t)$ of (1) *is bounded for $t \geqq t_0$ from the domain of $u(t)$ which, if $u(t)$ satisfies the inequality* (13), *is for $t \in \langle t_0, \infty)$.*

Proof: The method is analogous to that used in proving Theorem 1. Suppose that, although a solution $u(t)$ of (1) is defined on $\langle t_0, \bar{t})$ and satisfies the condition (13), $\limsup_{t \to \bar{t}_-} | u(t) | = + \infty$. By multiplying (1) by $u'(t)$ and integrating over (t_0, t), where $t < \bar{t}$, we get the following modification of (6):

$$(15) \qquad \frac{1}{2} a(t)\, u'^2(t) + F(t, u(t)) \leqq \frac{1}{2} a(t_0)\, u'^2(t_0) + F(t_0, u(t_0)) + + \int_{t_0}^{t} \frac{\partial F\,(s, u\,(s))}{\partial s}\, \mathrm{d}s,$$

from which, using (11), we get

$$(16) \qquad F(t, u(t)) \leqq \frac{1}{2} a(t_0)\, u'^2\,(t_0)\, u'^2(t_0) + F(t_0, u(t_0)).$$

Let $\bar{t} = + \infty$. Since $\limsup_{t \to \bar{t}_-} | u\,(t) | = + \infty$, there exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that for $i \to \infty$ both t_i and $|u\,(t_i)\,|$ tend to infinity. Thus (16) leads to a contradiction with (13).

If the domain of $u(t)$ is a finite interval, i.e. $\bar{t} < \infty$ and $u(t)$ is unbounded at $\bar{t}$, there exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that for $i \to \infty$ $t_i \to \bar{t}_-$ while $| u(t_i) | \to \infty$. Define a sequence $\{\tilde{t}_i\}_{i=1}^{\infty}$ such that for all i $t_i \leqq \tilde{t}_i$ and $\lim_{i \to \infty} \tilde{t}_i = \infty$. Using (11), we get from (16):

$$F(\tilde{t}_i, u(t_i)) \leqq F(t_i, u(t_i)) \leqq \frac{1}{2} a(t_0)\, u'^2(t_0) + F(t_0, u(t_0)),$$

so that, for $i \to \infty$, we have

$$F \leqq \frac{1}{2} a(t_0)\, u'^2\,(t_0) + F(t_0, u(t_0))$$

which contradicts the assumption (13).

Furthermore, using (11) and (14), from (15) we get

$$\frac{1}{2} a u'^2(t) \leqq \frac{1}{2} a(t_0)\, u'^2(t_0) + F(t_0, u(t_0)),$$

so that $u'(t)$ is bounded.

The proof that the domain of $u(t)$ is $\langle t_0, \infty)$ if $a(t) \geqq a > 0$ for $t \geqq t_0$ and provided (14) holds, follows from the proof of Theorem 1a.

Theorem 5: *Let the hypotheses of Theorem* 4 *hold and suppose that in* (12) $F = + \infty$. *If $c(t)$ is absolutely integrable, i.e.* $\int^{\infty} | c(t) | \, \mathrm{d}t \leqq K < \infty$, *then every solution $u(t)$ of the equation*

$$(17) \qquad a(t)\, u'' + b(t)\, g(u, u') + f(t, u) = c(t)$$

together with its first derivative is bounded for all $t \geqq t_0$ *from its domain.*

If in addition $c(t)$ *and* $c'(t)$ *are continuous, then the same holds for* $\langle t_0, \infty)$.

Proof: By multiplying (17) by $u'(t)$ and integrating over (t_0, t) (where $t_0 \leqq t < \bar{t}$, $\langle t_0 \bar{t})$ being the domain of $u(t)$), we get

$$(18) \qquad \frac{1}{2} a(t)\, u'^2(t) + F(t, u(t)) \leqq \frac{1}{2} a(t_0)\, u'^2(t_0) + F(t_0, u(t_0)) + $$
$$+ \int_{t_0}^{t} \frac{\partial F(s, u(s))}{\partial s}\, ds + \int_{t_0}^{t} c(s)\, u'(s)\, ds,$$

from which, using (11) and (14), we have

$$\frac{1}{2} a u'^2(t) \leqq K_0 + \int_{t_0}^{t} |c(s)|\, |u'(s)|\, ds$$

and therefore

$$a\, |u'(t)| \leqq \frac{a}{2}(|u'|^2 + 1) \leqq K_0 + \frac{a}{2} + \int_{t_0}^{t} |c(s)|\, |u'(s)|\, ds,$$

where

$$K_0 = \frac{1}{2} a(t_0)\, u'^2(t_0) + F(t_0, u(t)).$$

By Bellman's lemma in [5] we have

$$|u'(t)| \leqq K_1 \exp\left[\frac{1}{a} \int_{t_0}^{t} |c(s)|\, ds\right] \leqq K_2 < \infty,$$

where

$$K_1 = K_0 + \frac{a}{2}.$$

From (18) we have further

$$F(t, u(t)) \leqq K_0 + \int_{t_0}^{t} c(s)\, u'(s)\, ds,$$

and since $|u'(t)| \leqq K_2$ and $c(t)$ is absolutely integrable, we have

$$F(t, u(t)) \leqq K_0 + K_2 K < \infty \text{ for } t \geqq t_0.$$

Using (12) with $F = +\infty$, we see that $u(t)$ is also bounded.

The last assertion of the theorem may be proved by the fact that the functions

$\bar{f}(t, x) = f(t, x) - c(t)$ and $\dfrac{\partial \bar{f}(t, x)}{\partial t} = \dfrac{\partial f(t, x)}{\partial t} - c'(t)$ satisfy the condition α).

In connection with (1), let us consider the equation

$$a(t)\, u'' + b(t)\, g(u, u') + (1 + \psi(t))\, f(t, u) = 0. \tag{19}$$

Theorem 6: *Suppose that the hypotheses of Theorem 4 hold, with F in (12) equal to $+\infty$. If*

$$\lim_{t\to\infty} \varphi(t) = 0 \text{ and } \int_{t_0}^{\infty} |\, \psi'(t)\, |\, \mathrm{d}t < \infty, \tag{20}$$

then there exists $t_1 \geqq t_0$ such that every solution of (19) together with its first derivative is bounded for all $t \geqq t_1$ from its domain.

If in addition $\psi'(t)$ is also continuous, than this holds for $t \in \langle t_1, \infty)$.

Proof: (20) implies the existence of $t_1 \geqq t_0$ such that, for $t \geqq t_1$, $1 + \psi(t) \geqq k_1 > 0$. Now if $t_1 \leqq t < \bar{t}$, where $\langle t_1, \bar{t})$ is the domain of $u(t)$, the equation (19) yields the inequality

$$\frac{1}{2} a(t)\, u'^2(t) + (1 + \psi(t))\, F(t, u(t)) \leqq \frac{1}{2} a(t_1)\, u'^2(t_1) + (1 + \psi(t_1))\, F(t_1, u(t_1)) +$$

$$+ \int_{t_1}^{t} \psi'(s)\, F(s, u(s))\, \mathrm{d}s + \int_{t_1}^{t} (1 + \psi(s))\, \frac{\partial F\,(s, u(s))}{\partial s}\, \mathrm{d}s$$

and therefore

$$\frac{1}{2} au'^2(t) + k_1 F(t, u(t)) \leqq K_0 + \int_{t_1}^{t} \psi'(s)\, F(s, u(s))\, \mathrm{d}s, \tag{21}$$

where

$$K_0 = \frac{1}{2} a(t_1)\, u'^2(t_1) + F(t_1, u(t_1))\, (1 + \psi(t_1)).$$

From (14) we see that $F(t, u) \geqq 0$ for $t \in \langle t_1, \bar{t})$ and therefore if we omit the term $\frac{1}{2} au'^2(t)$ in (21) and use Bellman's lemma, we get

$$F(t, u(t)) \leqq K_0 \exp\left[\frac{1}{k_1} \int_{t_1}^{t} |\, \psi'(s)\, |\, \mathrm{d}s\right] \leqq K < \infty$$

and also

$$\frac{1}{2} au'^2(t) \leqq K_0 + K \int_{t_1}^{t} |\, \psi'(s)\, |\, \mathrm{d}s \leqq K_1 < \infty.$$

But this means, owing to (12), that, $u(t)$ and $u'(t)$ are bounded on $\langle t_1, \bar{t})$. The last part of the theorem again follows from the proof of Theorem 1a.

Theorem 7: *Suppose that the hypotheses of Theorem 4 hold with F in (12) equal to $+\infty$. Suppose further that $1 + \psi(t) \geqq k_1 > 0$ and $\psi'(t) \leqq 0$ for $t \geqq t_0 \geqq 0$. Then every solution $u(t)$ of (19) together with its first derivative is bounded for $t \geqq t_0$ from its domain. If in addition $\psi'(t)$ is continuous, then this holds for $\langle t_0, \infty)$.*

Proof: The proof is a direct consequence of the proof of Theorem 6. In fact (21) yields

$$F(t, u(t)) \leqq \frac{K_0}{k_1}$$

and also

$$\frac{1}{2} au'^2(t) \leqq K_0,$$

which completes the proof.

Theorem 8: *Suppose that all hypotheses of Theorem 6 except (20) hold and that instead of satisfying the condition (20), $\psi(t)$ is such that for $t \geqq t_0 \geqq 0$*

$$1 + \psi(t) \geqq k_1 > 0 \text{ and } \int_{t_0}^{\infty} | \psi'(t) | \, dt < \infty.$$

Then every solution of (19) together with its first derivative is bounded for $t \geqq t_0$ from its domain. If in addition $\psi'(t)$ is also continuous, then this is true for $\langle t_0, \infty)$.

The proof of this theorem realizes the condition (21) and the method is analogous to that used in proving Theorem 6.

The conditions of boundedness in these theorems will be considerably simplified if we put $a(t) \equiv 1$ in (1), (17) and (19). We have

Theorem 9: *Let the hypotheses (2) and (3) of Theorem 1 hold an suppose that*

α') $f(t, x)$ and $\frac{\partial F(t, x)}{\partial t}$ are continuous for $t \geqq t_0 \geqq 0$, $| x | < \infty$;

β') $g(x, y)$ is continuous and $g(x, y) \operatorname{sgn} y \geqq 0$ for all x and y;

γ') $a(t) \equiv 1$, $b(t) \geqq 0$ is a continuous function for $t \geqq t_0 \geqq 0$.

Then every solution $u(t)$ of (1) for which

$$F(t_0, u(t_0)) + \frac{1}{2} u'^2(t_0) < F,$$

is bounded on $\langle t_0, \infty)$.

Proof: Let a solution $u(t)$ of (1) be defined on $\langle t_0, t).$). By multiplying (1) by $u'(t)$ we get

$$u'u'' + b(t)\, g(u, u')\, u' + f(t, u)\, u' = 0,$$

from which by integrating we get the following form of (6):

$$\frac{1}{2} u'^2(t) + F(t, u(t)) \leqq \frac{1}{2} u'^2(t_0) + F(t_0, u(t_0)) + \int_{t_0}^{t} \frac{\partial F(s, u(s))}{\partial s} \, ds.$$

Now we proceed as we did in proving Theorem 1. From the proof of Theorem 1a it is obvious that any solution can be extended to $\langle t_0, \infty)$.

Remark 3: *If $a(t) \equiv 1$, then in Theroems 2—8 it suffices to postulate the conditions α'), β') and γ') instead of the ,,undashed" conditions.*

Theorem 10: *Suppose that $f(t, x, y)$ and $\frac{\partial f(t, x, y)}{\partial t}$ are continuous for $t \geqq t_0 \geqq 0$,*

$|x| + |y| < \infty$; *let that* $F(t, x, y) \int_0^x = f(t, s, y)\, ds$ *and suppose that* $\frac{\partial F}{\partial y} f(t, x, y) \geqq 0$,

for $t \geqq t_0 \geqq 0$, $|x| + |y| < \infty$. *If for any continuously differentiable function* $x(t)$ *on* $(t_0, \bar{t})$ *which is unbounded for* $t \to \bar{t}_-$, $t_0 < \bar{t} \leqq \infty$, *there exists a sequence* $\{t_i\}_{i=1}^{\infty}$ *such that if* $t_i \to \bar{t}_-$ *for* $i \to \infty$, *then*

$$\frac{\partial F(t, x(t), x'(t))}{\partial t} \leqq \frac{\partial F(t, x(t_i), x'(t_i))}{\partial t}, \; t_0 \leqq t \leqq t_i \tag{22}$$

and

$$\lim_{i \to \infty} [\inf_{|y| < \infty} F(t_0, x(t_i), y)] = \bar{F} \tag{23}$$

with $\bar{F} \leqq \infty$ *independent of* $x(t)$, *then every solution of the equation*

$$u'' + f(t, u, u') = 0, \tag{24}$$

for which

$$K_0 = \frac{1}{2} u'^2(t_0) + F(t_0, u(t_0), u'(t_0)) < \bar{F}, \tag{25}$$

is bounded for $t \geqq t_0$ *from its domain.*

Proof: Suppose that a solution $u(t)$ of (24) is defined on $\langle t_0, \bar{t})$ and satisfies(25) and that $\limsup_{t \to \bar{t}_-} |u(t)| = +\infty$. By multiplying (24) by $u'(t)$ and integrating over (t_0, t) with $t_0 \leqq t < \bar{t}$, we obtain the equation

$$\frac{1}{2} u'^2(t) + \int_{t_0}^{t} f(s, u(s), u'(s))\, u'(s)\, ds = \frac{1}{2} u'^2(t_0)$$

or

$$\frac{1}{2} u'^2(t) + \int_{t_0}^{t} \frac{d}{ds} F(s, u(s), u'(s))\, ds = \frac{1}{2} u'^2(t_0) +$$

$$+ \int_{t_0}^{t} \frac{\partial F(s, u(s), u'(s))}{\partial s} ds - \int_{t_0}^{t} \frac{\partial F}{\partial y} f(s, u(s), u'(s))\, ds$$

with $\frac{\partial F}{\partial y}$ taken in the point $(s, u(s), u'(s))$. Therefore

$$\frac{1}{2} u'^2(t) + F(t, u(t), u'(t)) \leqq \frac{1}{2} u'^2(t_0) + F(t_0, u(t_0), u'(t_0)) + \tag{26}$$

$$+ \int_{t_0}^{t} \frac{\partial F(s, u(s), u'(s))}{\partial s} ds.$$

Owing to (22), we have further

$$F(t_i, u(t_i), u'(t_i)) \leqq K_0 + \int_{t_0}^{t_i} \frac{\partial F(s, u(s), u'(s))}{\partial s} \mathrm{d}s \leqq$$
$$\leqq K_0 + F(t_i, u(t_i), u'(t_i)) - F(t_0, u(t_i), u'(t_i)),$$

so that

$$F(t_0, u(t_i), u'(t_i)) \leqq K_0.$$

Since $\inf\limits_{|y|<\infty} F(t_0, u(t_i), y) \leqq F(t_0, u(t_i), u'(t_i)) \leqq K_0$,
we get from (23)

$$F = \lim_{i\to\infty} [\inf_{|y|<\infty} F(t_0, u(t_i), y)] \leqq K_0,$$

which contradicts the fact that $u(t)$ satisfies (25).

This proof is the source of a further theorem.

Theorem 11: *Let $f(t, x, y)$ be continuous for $t \geqq t_0 \geqq 0$, $|x| + |y| < \infty$. Suppose further that $F(t, x, y) = \int_0^x f(t, s, y)\,\mathrm{d}s$ is such that $\frac{\partial F(t, x, y)}{\partial t} \leqq 0$, $\frac{\partial F}{\partial y} f(t, x, y) \geqq 0$ for $t \geqq t_0 \geqq 0$, $|x| + |y| < \infty$.*

If for all sequences $\{t_i\}_{i=1}^{\infty}$ such that $\lim\limits_{i\to\infty} t_i = \infty$ and all sequences $\{x_i\}_{i=1}^{\infty}$ such that $\lim\limits_{i\to\infty} |x_i| = \infty$ we have

(27) $$\lim_{i\to\infty} [\inf_{|y|<\infty} F(t_i, x_i, y)] = F,$$

with $F \leqq +\infty$, then every solution of (24) which satisfies (25) is bounded on its domain.

If in addition for $t \geqq t_0$, $|x| + |y| < \infty$

(28) $$f(t, x, y) \operatorname{sgn} x > 0,\ x \neq 0,$$

then the derivative $u'(t)$ of any solution $u(t)$ is also bounded.

Proof: Suppose that $u(t)$ is a solution of (24) defined on $\langle t_0, \bar{t})$, $\bar{t} \leqq +\infty$ which satisfies (25) and let $\limsup\limits_{t\to\bar{t}} |u(t)| = +\infty$. Using (26) and the asumption $\frac{\partial F}{\partial t} \leqq 0$, we get

$$F(t, u(t), u'(t)) \leqq K_0.$$

If $\bar{t} = +\infty$, then there must exist a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \to \infty$ for $i \to \infty$ and $|u(t_i)| \to \infty$ for $i \to \infty$. If we put $t = t_i$ in the last inequality, we get a contradiction with (25).

If $\bar{t} < \infty$, then there must exist a sequence $\{t_i\}_{i=1}^{\infty}$ such that $\lim\limits_{i\to\infty} |u(t_i)| = +\infty$. If $\{\tilde{t}_i\}_{i=1}^{\infty}$ is a sequence such that $\tilde{t}_i \to \infty$ for $i \to \infty$ and $t_i \leqq \tilde{t}_i$ for any i then, using (28), we have again

$$F(t_i, u(t_i), u'(t_i)) \leqq K_0$$

and therefore

$$F = \lim_{i\to\infty} [\inf_{|y|<\infty} F(\tilde{t}_i, u(t_i), y)] \leqq \lim_{i\to\infty} [\inf_{|y|<\infty} F(t_i, u(t_i), y)] \leqq K_0$$

which again contradicts (25).

From (26) and (28) we see that if $u(t)$ is defined on $< t_0, t)$, then for any t from this interval we have

$$\frac{1}{2} u'^2(t) \leqq K_0.$$

Thus $u'(t)$ is also bounded.

Considerations similar to those which have led to Theorems 10, 11 and same proceeding theorems could now be used to prove the following theorems:

Theorem 12: *Assume the validity of the hypotheses of Theorem* 10 *and the conditions* β' *and* γ'. *If*

$$\frac{\partial F}{\partial y} g(x, y) \geqq 0 \text{ for } t \geqq t_0 \geqq 0, |x| + |y| < \infty \tag{29}$$

then any solution of the equation

$$u'' + b(t)\, g(u, u') + f(t, u, u') = 0 \tag{30}$$

which satisfies (25) *is bounded for* $t \geqq t_0$ *from its domain.*

Theorem 13: *Assume the validity of the hypotheses of Theorem* 11 *and the conditions* β' *and* γ'. *If* (29) *holds, then any solution of* (30) *which satisfies* (25) *and the derivative of any solution are bounded for* $t \geqq t_0$ *from their domain.*

Theorem 14: *Make the same assumptions as in the proceeding theorem with* F *in* (27) *equal to* $+\infty$. *If* $c(t)$ *is absolutely integrable, then all solutions of the equation*

$$u'' + b(t)\, g(u, u') + f(t, u, u') = c(t)$$

are bounded, together with their first derivatives, for $t \geqq t_0$ *from their domain.*

Theorem 15: *Assume the validity of the hypotheses of Theorem* 11 *and the conditions* β' *and* γ' *with* F *in* (27) *equal to* $+\infty$. *If* $\psi(t)$ *satisfies* (20), *then there exists* $t_1 \geqq t_0$ *such that all solutions of the equation*

$$u'' + b(t)\, g(u, u') + (1 + \psi(t))\, f(t, u, u') = 0 \tag{31}$$

together with their first derivatives are bounded for $t \geqq t_1$ *from their domains.*

Theorem 16: *Replace in the assumptions of Theorem* 15 *the condition* (20) *by the following one:*
$1 + \psi(t) \geqq k_1 > 0$ *and* $\psi'(t) \leqq 0$ *for* $t \geqq t_0 \geqq 0$. *Then all solutions of* (31) *together with their first derivatives are bounded for* $t \geqq t_0$ *from their domains.*

Theorem 17: *Replace in the assumptions of Theorem* 15 *the condition* (20) *by the following one:*

$$1 + \psi(t) \geqq k_1 > 0 \text{ for any } t \geqq t_0 \geqq 0 \text{ and } \int_{t_0}^{\infty} |\psi'(t)|\, \mathrm{d}t < \infty.$$

Then all solutions of (31) *together with their first derivatives are bounded for* $t \geqq t_0$ *from their domains.*

In [3] we find sufficient conditions for the boundedness of all solutions of the equation

$$u'' + a(t)\, f(u)\, g(u') = 0, \tag{32}$$

together with their first derivatives.
Let us investigate the boundedness of a more general equation.

In the following theorems we shall use the following assumptions:

a) $f(x)$ *is continuous for all* x *and* $f(x)$ sgn $x > 0$ *for* $x \neq 0$;

b) $g(y)$ *is continuous and* $g(y) > 0$ *for all* y;

c) $\lim_{|x| \to \infty} F(x) = +\infty$, $\lim_{|y| \to \infty} G(y) = +\infty$, *where* $F(x) = \int_0^x f(s)\, \mathrm{d}s$,

$G(y) = \int_0^y \frac{s}{g(s)}\, \mathrm{d}s$.

Theorem 18: *Suppose that* $f(t, x)$ $\frac{\partial f(t\ x)}{\partial}$ *are continuous for* $t \geqq t_0 \geqq 0$, $|x| < \infty$ *and let* $F(t, x)$ *satisfy the conditions* (2) *and* (3) *of Theorem* 1. *If b) holds, then any solution of the equation*

$$u'' + f(t, u)\, g(u') = 0, \tag{33}$$

which satisfies the condition

$$K_0 = G(u'(t_0)) + F(t_0, u(t_0)) < F, \tag{34}$$

is bounded for $t \geqq t_0$ *from its domain.*

Proof: From (33) we obtain

$$\frac{u''u'}{g(u')} + f(t, u)\, u' = 0$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} G(u'(t)) + \frac{\mathrm{d}}{\mathrm{d}t} F(t, u(t)) - \frac{\partial F(t, u(t))}{\partial t} = 0.$$

By integrating over (t_0, t) with $t_0 \leqq t < \bar{t}$ where $\langle t_0, \bar{t})$ is the domain of $u(t)$ we obtain

$$G(u'(t)) + F(t, u(t)) = K_0 + \int_{t_0}^{t} \frac{\partial F(s, u(s))}{\partial s}\, \mathrm{d}s \tag{35}$$

and further

$$F(t, u(t)) \leqq K_0 + \int_{t_0}^{t} \frac{\partial F(s, u(s))}{\partial s}\, \mathrm{d}s.$$

Analogously as in the proof of Theorem 1, the boundedness of $u(t)$ is proved using (2) and (3).

Theorem 19: *Let* $f(t, x)$ *be continuous for* $t \geqq t_0 \geqq 0$, $|x| < \infty$ *and suppose that the conditions* (11), (12) *and* (14) *of Theorem* 4 *hold. If b) holds and if* $G(y)$ *satisfies* c), *then any solution of* (33) *which satisfies* (34) *is bounded together with its first derivative, for* $t \geqq t_0$ *from their domain.*

Proof: Let $\langle t_0, \bar{t})$ be the domain of $u(t)$. Owing to (11), we obtain from (35)

$$G(u'(t)) + F(t, u(t)) \leqq K_0 \tag{36}$$

and further, using (14), we see that $G(u'(t)) \leqq K_0$ so that $u'(t)$ is bounded for $t \in \langle t_0, \bar{t})$.

The boundedness of $u(t)$ is proved using (36) and (12). In fact $F((t, u(t)) \leqq K_0$ for $t \in \langle t_0, \bar{t})$ and, owing to (12), $F \leqq K_0$ which contradicts (34).

Theorem 20: *Let $f(t, x)$ be continuous for $t \geqq t_0 \geqq 0$, $|x| < \infty$ and suppose that all hypotheses of Theorem 4 are valid with the exception of* α), β), γ) *and with F in* (12) *equal to $+\infty$. Further assume the validity of* b) *and that part of* c) *which concerns $G(y)$. If $\psi(t)$ satisfies* (20), *then there exists $T \geqq T_0$ such that any solution of the equation*

$$u'' + (A + \psi(t))\, f(t, u)\, g(u') = 0, \tag{37}$$

with A a positive constant is bounded together with its first derivative for $t \geqq T$ from its domain.

Proof: From (37) we get

$$\frac{u''u'}{g(u')} + (A + \psi(t))\, f(t, u)\, u' = 0$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} G(u'(t)) + (A + \psi(t)) \frac{\mathrm{d}}{\mathrm{d}t} F(t, u(t)) = (A + \psi(t)) \frac{\partial F}{\partial t}.$$

By integrating and using (11) we get

$$G(u'(t)) + (A + \psi(t))\, F(t, u(t)) \leqq K_o + \int_{t_0}^{t} \psi'(s)\, F(s, u(s))\, \mathrm{d}s,$$

$$\text{with } K_o = G(u'(t_o)) + (A + \psi(t_0))\, F(t_0, u(t_0)).$$

From (20) we deduce the existence of $T \geqq t_0$ such that $A + \psi(t) \geqq k_1 > 0$ for $t \geqq T$ and therefore

$$G(u'(t)) + k_1 F(t, u(t)) \leqq K_0 + \int_{T}^{t} |\psi'(s)|\, F(s, u(s))\, \mathrm{d}s. \tag{38}$$

From this, using Bellman's Lemma, we get

$$F(t, u(t)) \leqq K_0 \exp\left[\frac{1}{k_1} \int_{T}^{t} |\psi'(s)|\, \mathrm{d}s\right] \leqq K_1 < \infty,$$

so that $u(t)$ is bounded.

A further consequence of (38) is

$$G(u'(t)) \leqq K_0 + K_1 \int_{T}^{t} |\psi'(s)|\, ds \leqq K_2 < \infty,$$

and therefore $u'(t)$ is also bounded for $t \in \langle T, t)$.

Analogously we prove

Theorem 21: *Assume the validity of all hypotheses of Theorem* 20 *with the exception of* (20) *instead of which we assume for $t \geqq t_0 \geqq 0$ the validity of the following condition:*

$$A + \psi(t) \geqq k_1 > 0 \text{ and } \int_{t_0}^{\infty} |\psi'(t)|\, \mathrm{d}t < \infty.$$

Then any solution of (37) *is bounded, together with its first derivative, for $t \geqq t_0$ from its domain.*

2. OSCILLATION OF THE SOLUTION

Theorem 22: *Assume the validity of the hypotheses a), b) and let $F(x)$ satisfy c). If $a(t) \geqq a > 0$ for $t \geqq t_0 \geqq 0$, then every solution of (32) which is defined on $\langle T, \infty)$ with $T \geqq t_0$ is oscillatory.*

Proof: Suppose that a solution $u(t)$ of (32) is defined on $\langle T, \infty)$ and does not oscillate. For example, let $u(t) > 0$ for $t \geqq T$. From (32) we see that in that case we have, for $t \geqq T$

$$u''(t) = -a(t) f(u(t))\, g(u'(t)) < 0$$

so that $u'(t)$ is a decreasing function for $t \geqq T$. Two cases may occur:

1. There exist $t_1 \geqq T$ such that $u'(t_1) \leqq 0$, or
2. $u'(t) > 0$ for all $t \geqq T$.

In the first case there must exist a number $\xi > t_1$ such that $u(\xi) = 0$ which contradicts the hypothesis. Therefore it is necessary that $u'(t) > 0$ for $t \geqq T$ and $u(t)$ must be a monotonous increasing function of t. For $t \geqq T$ we have further

$$0 \leqq \lim_{t \to \infty} u'(t) \leqq u'(t) \leqq u'(T).$$

This means that $u'(t)$ it bounded. It is now easy to prove that so is $u(t)$. In fact, since $g(y) > 0$, (32) yields

$$\frac{u''(t)\, u'(t)}{g(u'(t))} + a(t) f(u(t))\, u'(t) = 0$$

and therefore

$$\frac{d}{dt} G(u'(t)) + a(t) \frac{d}{dt} F(u(t)) = 0. \tag{39}$$

Since $u(t)$ is an increasing function of t for $t \geqq T$, we see from the condition c) that so is $F(u(t))$. From (39) we get

$$G(u'(t)) + aF(u(t)) \leqq G(u'(T)) + aF(u(T)) = K_0,$$

so that

$$F(u(t)) \leqq \frac{1}{a} K_0$$

and therefore, owing to c), $u(t)$ is bounded on $\langle T, \infty)$. From the boundedness of $u(t)$ and $u'(t)$ and the continuity of $f(x)$ and $g(y)$ we see that *lim* $u'(t) = 0$ and that there exist numbers u_1 and u'_1 such that for all $t \geqq T$ we have

$$0 < f(u_1) \leqq (fu(t)),\ 0 < g(u_1) \leqq g(u'(t)),$$

with $u_1 \in \langle u(T), \lim_{t\to\infty} u(t)\rangle$, $u_1 \in \langle 0, u'(T)\rangle$. Then

$$-u''(t) = a(t) f(u(t))\, g(u'(t)) \geqq af(u_1)\, g(u_1) = c > 0,$$

so that

$$u(t) \leqq -\frac{1}{2} ct^2 + c_1 t + c_2,$$

with c_1 and c_2 constants dependent on c, T, $u(T)$ and $u'(T)$. Therefore for sufficiently large t we have $u(t) \leqq 0$ which is a contradiction.

Now new problems arise if $u(t) < 0$ for $t \geqq T$. For in that case we have for $t \geqq T$

$$u''(t) = -a(t) f(u(t)) g(u'(t)) > 0,$$

so that $u'(t)$ as a monotonous increasing function of t. The existence of a number $t_1 \geqq T$ such that $u'(t_1) \geqq 0$ again leads to contradiction. It is therefore neccessary that for $t \geqq T$

$$u'(t) < 0, u'(T) \leqq u'(t) \leqq \lim_{t\to\infty} u'(t) \leqq 0$$

so that $u(t)$ is a monotonous decreasing function which makes $|u(t)|$ a monotonous increasing function. The conditions *c*) and (39) can be used again to prove that $u(t)$ is bounded. But in that case

$$-u''(t) = a(t) f(u(t)) g(u'(t)) \leqq a(t) f(u_2) g(u_2) < 0$$

with $u_2 \in \langle \lim_{t\to\infty} u(t), u(T)\rangle, u'_2 \in \langle u'(T), 0\rangle$. Therefore $u'(t) - u'(T) \to \infty$ for $t \to \infty$ so that $u'(t)$ is unbounded for $t \to \infty$, giving a contradiction. This completes the proof.

The iem 23: *Let* $f(t, x)$ *be continuous,* $f(t, x) \operatorname{sgn} x > 0$ *for* $x \neq 0$, $\dfrac{\partial f(t, x)}{\partial x} \geqq 0$ *for* $t \geqq t_0 \geqq 0$, $|x| < \infty$. *Suppose further that b) holds. Then any solution of* (33) *which is defined on* $< T, \infty)$, $T \geqq t_0$ *and for which*

$$\int_T^\infty f(s, u(T)) \operatorname{sgn} u(T) \,\mathrm{d}s = + \infty \tag{40}$$

has at least one zero on (T, ∞).

Proof: Suppose that a solution $u(t)$ defined on $< T, \infty)$ satisfies (40) and that $u(t) > 0$ for $t \geqq T$. From (33) we get

$$u''(t) = -f(t, u(t)) g(u'(t)) < 0$$

so that $u'(t)$ is decreasing function of t for $t \geqq T$. In the same way as before we prove that $u'(t)$ must be positive for $t \geqq T$. Therefore $u(t)$ is an increasing function and its values lie in the interval $J = < u(T), \lim_{t\to\infty} u(t))$. Since $u'(t) > 0$ and decreases monotonously, it is bounded for $t \geqq T$. Thus there exists a constant $u'_1 \in \langle \lim_{t\to\infty} u'(t), u'(T)\rangle$ such that for $t \geqq T$

$$0 < g(u'_1) \leqq g(u'(t)).$$

But in that case, since $f(t, x)$ is in J a non-decreasing function of x, we have

$$-u''(t) = f(t, u(t)) g(u'(t)) \geqq g(u'_1) f(t, u(T))$$

and thus (40) implies that $u'(t) - u'(T) \to -\infty$, which is a contradiction with the assumption that $u'(t)$ is bounded for $t \geqq T$.

The method is analogous if we assume that $u(t) < 0$ for $t \geqq T$. Again we prove that the necessity of $u'(t) < 0$ for $t \geqq T$ and the fact that $u'(t)$ is a monotonous increasing function of t. There exists thus a contant $u_2 \in \langle u'(T), \lim_{t\to\infty} u'(t)\rangle$ such that for all $t \geqq T$ we have $g(u'(t)) \geqq g(u'_2)$. Since $f(t, x)$ is an increasing function of x, we have

$$—u''(t) = f(t, u(t))\, g(u'(t)) \leqq g(u'_2) f(t, u(T)) < 0$$

and again, using (40), $u'(t) — u'(T) \to + \infty$ for $t \to \infty$. This completes the proof.

It is easy to prove the following

Theorem 24: *Suppose that the hypotheses of Theorem* 23 *are valid and that, for* $t \geqq t_0 \geqq 0$, $A + \psi(t) \geqq k_1 > 0$. *Then any solution of* $u(t)$ *of* (37) *which is defined on* $\langle T, \infty)$ *and satisfies* (40) *has at least one zero on* (T, ∞).

Remark 4: *Evidently if in Theorems* 23 *and* 24 *the relation* (40) *holds for* $u_1 = u(T)$, *then any solution of* (32) *or* (37) *defined on* $\langle T, \infty)$ *for which* $u_1 \leqq u(T)$ *for* $u_1 > 0$, *and* $u_1 \geqq u(T)$ *for* $u_1 < 0$ *has at least one zero on* (T, ∞).

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