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ON THE EXCEPTIONAL CASE IN SYSTEMS OF NON-LINEAR DIFFERENTIAL QUATIONS

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§ 1. INTRODUCTION

Consider the system of non-linear differential equations

(1) $$\mathbf{u}' = \mathbf{g}(\mathbf{u}, t) + \mathbf{h}(t),$$

where $\mathbf{g}$ and $\mathbf{h}$ are continuous vectors in t of n components and periodic of the same period p (not necessary the smallest one)

(2) $$\mathbf{g}(\mathbf{u}, t + p) = \mathbf{g}(\mathbf{u}, t),\ \mathbf{h}(t + p) = \mathbf{h}(t).$$

Let a periodic solution $\mathbf{u}_0(t)$ of period p of (1) be given:

(3) $$\mathbf{u}_0' = \mathbf{g}(\mathbf{u}_0, t) + \mathbf{h}(t),\ \mathbf{u}_0(t + p) = \mathbf{u}_0(t).$$

By varying the R.S. of (1), we consider the differential equation

(4) $$\mathbf{u}' = \mathbf{g}(\mathbf{u}, t) + \mathbf{h}(t) + \beta \mathbf{f}(t)$$

with a small parameter

(5) $$|\beta| \leqq \beta_0,$$

where the vector of n components $\mathbf{f}(t)$ is also continuous and periodic of period p

(6) $$\mathbf{f}(t + p) = \mathbf{f}(t).$$

We assume that the vector $\mathbf{g}(\mathbf{u}, t)$, for which

(7) $$|\mathbf{u} - \mathbf{u}_0(t)| \leqq \varepsilon,$$

possesses continuous partial derivatives w.r.t. the components of $\mathbf{u}$ till the second order. Let

(8) $$\mathbf{u}(t) = \mathbf{u}_0(t) + \mathbf{x}(t).$$

Setting

(9) $$\mathbf{u}(t, \lambda) = \mathbf{u}_0(t) + \lambda \mathbf{x}(t),$$

then

(10) $$\mathbf{u}_0(t) = \mathbf{u}(\mathrm{t}, 0),\ \mathbf{u}(t) = \mathbf{u}(t, 1).$$

Subtracting (4) from (1) and using Taylor's expansion for the vector function $\mathbf{g}$ w.r.t. λ, we obtain (see [1], §2)

$$\mathbf{x}'(t) = A(t)\,\mathbf{x} + \mathbf{f}(t) + \mathbf{r}(t, \mathbf{x}) \tag{11}$$

where

$$A(t) = \frac{\partial \mathbf{g}\big(\mathbf{u}_0(t), t\big)}{\partial \mathbf{u}} \tag{12}$$

or in components

$$\big(a_{ik}(t)\big) = \left(\frac{\partial g_i\big(\mathbf{u}_0(t), t\big)}{\partial u_k}\right) \qquad (i, k = 1, \ldots, n)$$

and

$$\mathbf{r}(t, \mathbf{x}) = \mathbf{g}\big(\mathbf{u}_0(t) + \mathbf{x}, t\big) - \mathbf{g}\big(\mathbf{u}_0(t), t\big) - \frac{\partial \mathbf{g}\big(\mathbf{u}_0(t), t\big)}{\partial \mathbf{u}}\,\mathbf{x} = \int_0^1 \mathbf{x}^T \,.\, T(\mathbf{u}_0(t) + \lambda \mathbf{x}, t)\,\mathbf{x}\cdot(1 - \lambda)\,\mathrm{d}\lambda \tag{13}$$

or in components for $i = 1, \ldots, n$ (see [1], (19))

$$r_i(t, \mathbf{x}) = g_i(\mathbf{u}_0(t) + \mathbf{x}, t) - g_i(\mathbf{u}_0(t), t) - \sum_{k=1}^{n} \frac{\partial g_i(\mathbf{u}_0(t), t)}{\partial u_k}\,x_k = \int_0^1 \sum_{k,l}^{1\ldots n} \frac{\partial^2 g_i(\mathbf{u}_0(t) + \lambda \mathbf{x}, t)}{\partial u_k \partial u_l}\,x_k x_l (1 - \lambda)\,\mathrm{d}\lambda. \tag{14}$$

We note that $\mathbf{x}^{T_1}$ means the transposed vector of $\mathbf{x}$ and T denotes the tensor

$$T(\mathbf{u}_0(t) + \lambda \mathbf{x}, t) = \begin{bmatrix} T_1(\mathbf{u}_0(t) + \lambda \mathbf{x}, t), \\ T_2(\mathbf{u}_0(t) + \lambda \mathbf{x}, t) \\ \cdot \\ \cdot \\ \cdot \\ T_n(\mathbf{u}_0(t) + \lambda \mathbf{x}, t) \end{bmatrix}, \text{ with } T_i(\mathbf{u}_0(t) + \lambda \mathbf{x}, \mathrm{t}) = \left(\frac{\partial^2 g_i(\mathbf{u}_0(t) + \lambda \mathbf{x}, t)}{\partial u_k \partial u_l}\right). \tag{15}$$

Here we shall seek solutions of (11) for sufficiently small ε and β_0 (see (7) and (5)). Evidently it is

$$A(t + p) = A(t),\ T(\mathbf{u}_0(t + p) + \lambda \mathbf{x}, t + p) = T(\mathbf{u}_0(t) + \lambda \mathbf{x}, t) \text{ for fixed } \mathbf{x} \text{ and } \lambda. \tag{16}$$

We set now for $\beta \neq 0$

$$\mathbf{x} = \beta \mathbf{x}^* \tag{17}$$

and we obtain first of all

$$\mathbf{x}^{*\prime} = A(t)\,\mathbf{x}^* + \mathbf{f}(t) + \frac{1}{\beta}\,\mathbf{r}(t, \beta\mathbf{x}^*). \tag{18}$$

Since the remainder term $\mathbf{r}$ $(t, \beta x^*)$ is in fact of order β^2 (see (13) and (14), then we can replace it by

$$\mathbf{r}(t, \beta\mathbf{x}^*) = \beta^2 \,.\, \mathbf{r}^*(t, x^*, \beta) \tag{19}$$

and we obtain for $\mathbf{x}^*$ the system of differential equations

$$\mathbf{x}^{*\prime} = A(t)\,\mathbf{x}^* + \mathbf{f}(t) + \beta\mathbf{r}^*(t, \mathbf{x}^*, \beta). \tag{20}$$

For $\beta \neq 0$, there exists by virtue of (17) a unique correspondence between the solutions of (11) and (20). For $\beta = 0$, (20) possesses still other solutions $\mathbf{x}^*$, for which (11) has no correspondents.

It is necessary to examine the continuity properties of $\mathbf{r}^*(t, \mathbf{x}^*, \beta)$, in particular as β tends to zero.

§ 2. STUDY OF THE CONTINUITY OF THE REMAINDER TERM $\mathbf{r}^*(t, \mathbf{x}^*, \beta)$ AND ITS DERIVATIVES

a) Continuity of $\mathbf{r}^*(t, \mathbf{x}^*, \beta)$.

Referring to (13) and (19), we obtain for $\mathbf{r}^*(t, \mathbf{x}^*, \beta)$ the representation

$$\mathbf{r}^*(t, \mathbf{x}^*, \beta) = \frac{\mathbf{g}(\mathbf{u}_0(t) + \beta\mathbf{x}^*, t) - \mathbf{g}(\mathbf{u}_0(t), t) - \dfrac{\partial \mathbf{g}(\mathbf{u}_0(t), t)}{\partial \mathbf{u}}\,\mathbf{x}^*}{\beta^2} = \tag{21}$$

$$= \int_0^1 \mathbf{x}^{*T} \,.\, T(\mathbf{u}_0(t) + \beta\lambda\mathbf{x}^*, t) \,.\, \mathbf{x}^* \,.\, (1 - \lambda)\,\mathrm{d}\lambda,$$

or in components

$$r_i^*(t, \mathbf{x}^*, \beta) = \frac{g_i(\mathbf{u}_0(t) + \beta\mathbf{x}^*, t) - g_i(\mathbf{u}_0(t), t) - \beta \sum\limits_k \dfrac{\partial g_i(\mathbf{u}_0(t), t)}{\partial u_k}\,x_k^*}{\beta^2} = \tag{22}$$

$$= \int_0^1 \sum_{k,\,l} \frac{\partial^2 g_i(\mathbf{u}_0(t) + \beta\lambda\mathbf{x}^*, t)}{\partial u_k \partial u_l}\,x_k^* x_l^* (1 - \lambda)\,\mathrm{d}\lambda.$$

Applying L'Hospital's rule, we get

$$\lim_{\beta\to 0} r_i^*(t, \mathbf{x}^*, \beta) = \lim_{\beta\to 0} \frac{\sum\limits_{k=1}^{n} \dfrac{\partial g_i(\mathbf{u}_0(t) + \beta\mathbf{x}^*, t)}{\partial u_k}\,x_k^* - \sum\limits_{k=1}^{n} \dfrac{\partial g_i(\mathbf{u}_0(t), t)}{\partial u_k}\,x_k^*}{2\beta} = \tag{23}$$

$$= \lim_{\beta\to 0} \left(\sum_{k,\,l}^{1\ldots n} {}^1\!/_2\, \frac{\partial^2 g_i(\mathbf{u}_0(t) + \beta\mathbf{x}^*, t)}{\partial u_k \partial u_l}\,x_k^* x_l^* \right) = \sum_{k,\,l} {}^1\!/_2\, \frac{\partial^2 g_i(\mathbf{u}_0(t), t)}{\partial u_k\, \partial u_l}\,x_k^* x_l^*\,.$$

From this, it follows immediately with

$$\mathrm{r}_i^*(t, \mathbf{x}^*, 0) = \sum_{k,l} {}^1/_2 \frac{\partial^2 g_i(\mathbf{u}_0(t), t)}{\partial u_k \partial u_l} x_k^* x_l^* \tag{24}$$

the continuity of $\mathrm{r}^*(t, \mathbf{x}^*, \beta)$ as β tends to zero. This result coincides with the integral representation in (22), namely

$$\int_0^1 \sum_{k,l} \frac{\partial^2 g_i(\mathbf{u}_0(t), t)}{\partial u_k \, \partial u_l} x_k^* x_l^* (1 - \lambda) \, \mathrm{d}\lambda.$$

b) The continuity of the first derivative w.r.t. $\mathbf{x}^*$.

We examine the continuity of the partial derivative $\dfrac{\partial r_i^*(t, \mathbf{x}^*, \beta)}{\partial x_\nu^*}$ $(i, \nu = 1, \ldots, n)$, in particular as $\beta \to 0$. For $\beta \neq 0$, we get from (21)

$$\frac{\partial r_i^*(t, \mathbf{x}^*, \beta)}{\partial x_\nu^*} = \frac{1}{\beta} \left(\frac{\partial g_i(\mathbf{u}_0(t) + \beta \mathbf{x}^*, t)}{\partial u_\nu} - \frac{\partial g_i(\mathbf{u}_0(t), t)}{\partial u_\nu} \right). \tag{25}$$

As $\beta \to 0$, we get

$$\lim_{\beta \to 0} \frac{\partial r_i^*(t, \mathbf{x}^*, \beta)}{\partial x_\nu^*} = \lim_{\beta \to 0} \left(\sum_l \frac{\partial^2 g_i(\mathbf{u}_0(t) + \beta x^*, t)}{\partial u_l \, \partial u_\nu} x_l^* \right) = \tag{26}$$

$$= \sum_l \frac{\partial^2 g_i(\mathbf{u}_0(t), t)}{\partial u_l \partial u_\nu} x_l^* .$$

Referring to (24), wo obtain at the same time for $\beta \to 0$

$$\frac{\partial r_i^*(t, \mathbf{x}^*, 0)}{\partial x_\nu^*} = {}^1/_2 \left(\sum_l \frac{\partial^2 g_i(\mathbf{u}_0(t), t)}{\partial u_\nu \, \partial u_l} x_l^* + \sum_k \frac{\partial^2 g_i(\mathbf{u}_0(t), t)}{\partial u_k \, \partial u_\nu} x_k^* \right) = \tag{27}$$

$$= \sum_k \frac{\partial^2 g_i(\mathbf{u}_0(t), t)}{\partial u_l \partial u_\nu} x_l^* .$$

The continuity of the first derivative follows from the concidence of (26) & (27).
Then $\mathbf{x}^*$ is uniquely determined from the parameter β and the initial vector $\mathbf{x}^*(0)$, and is differentiable w.r.t. components of the initial vector (see [2], § 17).

c) The continuity of the second derivative w.r.t. x^*.

Finally we examine the continuity of the second derivative $\dfrac{\partial^2 r_i^*(t, \mathbf{x}^*, \beta)}{\partial x_\lambda^* \partial x_\nu^*}$ $(i, \lambda, \nu = 1, \ldots, n)$. Differentiating (25) we get for $\beta \neq 0$

$$\frac{\partial^2 r_i^*(t, \mathbf{x}^*, \beta)}{\partial x_\lambda^* \partial x_\nu^*} = \frac{\partial^2 g_i(\mathbf{u}_0(t) + \beta \mathbf{x}^*, t)}{\partial u_\lambda \, \partial u_\nu}. \tag{28}$$

As $\beta \to 0$, we obtain

$$\lim_{\beta \to 0} \frac{\partial^2 r_i^*(t, \mathbf{x}^*, \beta)}{\partial x_\lambda^* \, \partial x_\nu^*} = \frac{\partial^2 g_i(\mathbf{u}_0(t), t)}{\partial u_\lambda \, \partial u_\nu}. \tag{29}$$

At the same sime we calculate by virtue of (27)

$$\frac{\partial^2 r_i(t, \mathbf{x}^*, 0)}{\partial x_\lambda^* \, \partial x_\nu^*} = \frac{\partial}{\partial x_\lambda^*} \left(\sum_l \frac{\partial^2 g_i(\mathbf{u}_0(t), t)}{\partial u_k \, \partial u_\nu} x_l^* \right) = \frac{\partial^2 g_i(\mathbf{u}_0(t), t)}{\partial u_k \, \partial u_\nu} . \tag{30}$$

From (29) and (30), it follows the continuity of the second derivative w.r.t. $\mathbf{x}^*$.

§ 3. THE CONDITIONS OF PERIODICITY IN THE EXCEPTIONAL CASE

The problem of the existence of periodic solutions $\mathbf{x}(t)$ with period p of the system of differential equations (11) of order $|\beta|$ leads immediately to the problem of the existence of periodic solutions $\mathbf{x}^*(t)$ with period p of the system of differential equations (20).

As in [1], we consider the homogeneous linear system of differential equations corresponding to (20)

$$\mathbf{y}'(t) = A(t)\mathbf{y} \tag{31}$$

and its adjoint (see e.g. [3], §3.2)

$$\mathbf{z}'(t) = A^T(t)\,\mathbf{z}, \tag{32}$$

where A^{T} denotes the transposed matrix of A.

The case, where the adjoint system (32) possesses at least one with p periodic solution $\mathbf{z}(t)$, for which

$$\int_0^p \mathbf{z}^T(t)\, \mathbf{f}(t)\, \mathrm{d}t = c \neq 0 \text{ (resonance case)} \tag{33}$$

is already treated in [1] and also the case where (32) possesses no with p periodic solutions at all (principal case).

We state

Theorem A (Resonance case): If the adjoint system of differential equations (32) possesses at least one periodic solution $\mathbf{z}(t)$ of period p, for which (33) is satisfied, then every solution of (11) — independent of the initial values — is such that $|\mathbf{x}(t)| = |\mathbf{u}(t)| - \mathbf{u}_0(t)$ is at least of order $\sqrt{|\beta|}$ with increasing t.

Theorem B (Principal case). If (32) has no periodic solution $\mathbf{z}(t)$ of period p, then (11) possesses — to every sufficiently small β — vector solutions $\mathbf{x}(t)$ for which $|\mathbf{x}(t)| \leqq \text{const.}\ |\beta|$ for all t, e.g. the unique existing small with p periodic solution $\mathbf{x}(t + p) = \mathbf{x}(t)$.
(For the proof, see [1]).

In this paper studied will be the so-called exceptional, case, where the adjoint system (32) possesses with p periodic solutions $\mathbf{z}(t)$, but for every such solution $\mathbf{z}(t)$ the exceptional condition.

$$\int_0^p \mathbf{z}^T(t)\mathbf{f}(t)\, \mathrm{d}t = 0 \tag{34}$$

is satisfied. In §4, it will be shown that under certain conditions, the system of differential equations (11) possesses in the exceptional case with p periodic solutions of order $|\beta|$.

It is well known from the study of the linear systems of differential equations (31) & (32), which has been already treated in details in [4], that a fundamental system of solutions $Y(t)$ of (31) or a fundamental system of solutions

$$Z(t) = (Y^{-1}(t))^T \tag{35}$$

of (32) (see e.g. [5], (17)) can be obtained such that the constant matrix (see [4], (10))

$$P = Y^{-1}(t)\, Y(t + p) \tag{36}$$

has the form

$$P = e^{Kp}. \tag{37}$$

Here the constant matrix K has the Jordan canonical normal form (see [4], (17))

$$K = \begin{bmatrix} K_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & H_s \end{bmatrix} \text{ with } K_\nu = \begin{bmatrix} \alpha_\nu & 1 & & \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ & & & \cdot\ 1 \\ & & & \alpha_\nu \end{bmatrix} \quad (\nu = 1, \ldots, s). \tag{38}$$

The submatrices K_ν have the order m_ν, where $n = \sum_1^s m_\nu$.

We consider now the vector solution of (20) in the form

$$\mathbf{x}^* = \mathbf{x}^*(t, \beta, \mathbf{c}), \tag{39}$$

where the vector $\mathbf{c}$ is related to the initial vector $\mathbf{x}^*(0, \beta, \mathbf{c})$ by the relation

$$\mathbf{x}^*(0, \beta, \mathbf{c}) = Y(0)\, \mathbf{c}. \tag{40}$$

We are going to prove the following.

Theorem 1. Necessary and sufficient condition, that the vector solution $\mathbf{x}^*(t, \beta, \mathbf{c})$ possesses the period p, is the satisfaction of the equation

$$(P^{-1} - I)\, \mathbf{c} - \int_0^p Z^T(\tau)\, \mathbf{f}(\tau)\, d\tau - \beta \int_0^p Z^T(\tau) \mathbf{r}^*(\tau, \mathbf{x}^*(\tau, \beta, \mathbf{c},), \beta)\, d\tau = \mathbf{o}. \tag{41}$$

Proof. By means of the method of variation of constants applied on the differential equation (20) and using (40), we obtain for $\mathbf{x}^*(t, \beta, \mathbf{c})$ the integral equation (see e.g. [4], (9) or [6]).

$$\mathbf{x}^*(t, \beta, \mathbf{c}) = Y(t) \left\{ \int_0^t Z^T(\tau)\, \mathbf{f}(\tau)\, d\tau + \beta \int_0^t Z^T(\tau)\, \mathbf{r}^*(\tau, \mathbf{x}^*(\tau, \beta, \mathbf{c}), \beta)\, d\tau + \mathbf{c} \right\}. \tag{42}$$

Referring to (36), (35) and (6), we get

$$(43)\qquad \mathbf{x}^*(t+p,\beta,\mathbf{c}) = Y(t)\,P\left\{\int_0^p Z^T(\tau)\,\mathbf{f}(\tau)\,\mathrm{d}\tau + \int_0^t Z^T(\tau+p)\,\mathbf{f}(\tau)\,\mathrm{d}\tau + \right.$$

$$+\beta\int_0^p Z^T(\tau)\,\mathbf{r}^*(\tau,\mathbf{x}^*(\tau,\beta,\mathbf{c}),\beta)\,\mathrm{d}\tau + \beta\int_0^t Z^T(\tau+p)\,\mathbf{r}^*(\tau+p,\mathbf{x}^*(\tau+$$

$$\left.+p,\beta,\mathbf{c}),\beta)\mathrm{d}\tau + \mathbf{c}\right\} = Y(t)\left\{P\int_0^p Z^T(\tau)\,\mathbf{f}(\tau)\,\mathrm{d}\tau + \int_0^t Z^T(\tau)\,\mathbf{f}(\tau)\,\mathrm{d}\tau + \right.$$

$$\left.+\beta P\int_0^p Z^T(\tau)\,\mathbf{r}^*(\tau,\mathbf{x}^*(\tau,\beta,\mathbf{c}),\beta)\,\mathrm{d}\tau + \beta\int_0^t Z^T(\tau)\,\mathbf{r}^*(\tau,\mathbf{x}^*(\tau+p,\beta,\mathbf{c}),\beta)\,\mathrm{d}\tau + P\mathbf{c}\right\}.$$

Subtracting (43) from (42), we obtain

$$(44)\qquad \mathbf{x}^*(t+p,\beta,\mathbf{c}) - \mathbf{x}^*(t,\beta,\mathbf{c}) = Y(t)\left\{(P-I)\,\mathbf{c} + P\int_0^p Z^T(\tau)\,\mathbf{f}(\tau)\,\mathrm{d}\tau + \right.$$

$$+\beta P\int_0^p Z^T(\tau)\,\mathbf{r}^*(\tau,\mathbf{x}^*(\tau,\beta,\mathbf{c}),\beta)\,\mathrm{d}\tau + \beta\int_0^t Z^T(\tau)\,[\mathbf{r}^*(\tau,\mathbf{x}^*(\tau+p,\beta,\mathbf{c}),\beta) -$$

$$\left.-\,\mathbf{r}^*(\tau,\mathbf{x}^*(\tau,\beta,\mathbf{c}),\beta)]\right\}\mathrm{d}\tau.$$

Setting $t = 0$ in (44), we get

$$(45)\qquad \mathbf{x}^*(p,\beta,\mathbf{c}) - \mathbf{x}^*(0,\beta,\mathbf{c}) = Y(0)P\left\{(I-P^{-1})\,\mathbf{c} + \int_0^p Z^T(\tau)\,\mathbf{f}(\tau)\,\mathrm{d}\tau + \right.$$

$$\left.+\beta\int_0^p Z^T(\tau)\,\mathbf{r}^*(\tau,\mathbf{x}^*(\tau,\beta,\mathbf{c}),\beta)\right\}\mathrm{d}\tau.$$

Referring to (45), we see that if $\mathbf{x}^*(t,\beta,\mathbf{c})$ has the period p, then (41) follows necessarily. Conversely, if (41) holds, then it follows from (45) that

$$\mathbf{x}^*(t+p,\beta,\mathbf{c}) = \mathbf{x}^*(t,\beta,\mathbf{c}),$$

and consequently, due to the periodicity of the differential equation (20), it follows in general that

$$(46)\qquad \mathbf{x}^*(t+p,\beta,\mathbf{c}) = \mathbf{x}^*(t,\beta,\mathbf{c}).$$

Thus the theorem is proved.

The problem of the existence of periodic solutions of period p of the differential equation (20) is therefore reduced to the existence of solutions $\mathbf{c} = \mathbf{c}(\beta)$ of the equation (41). If we suppose that to every sufficiently small β with

$$0 < |\beta| \leqq \beta_0$$

there corresponds a vector $\mathbf{c}(\beta)$ with

$$\lim_{\beta \to 0} \mathbf{c}(\beta) = \mathbf{c}(0), \tag{47}$$

so that the solution $\mathbf{x}^*(t, \beta, \mathbf{c}(\beta))$ has the period p, then there exists, because of the continuous dependence of the solution on β and $\mathbf{c}$, the limit

$$\lim_{\beta \to 0} \mathbf{x}^*(t, \beta, \mathbf{c}(\beta)) = \mathbf{x}^*(t, 0, \mathbf{c}(0)). \tag{48}$$

Here $\mathbf{x}^*(t, 0, \mathbf{c}(0))$ denotes a periodic solution of period p of (20) for $\beta = 0$. Then it follows necessarily from (41), that the relation

$$(P^{-1} - I)\, \mathbf{c}(0) - \int_0^p Z^T(\tau)\, \boldsymbol{f}(\tau)\, \mathrm{d}\tau = \boldsymbol{0} \tag{49}$$

holds for $\mathbf{c}(0)$. Now we know from [4], that to every submatrix (38) with $\alpha_\nu = 0$, there exists one and only one with β periodic solution $\mathbf{z}_{[\nu]}(t)$ of (32)

$$\mathbf{z}_{[\nu]}(t + p) = \mathbf{z}_{[\nu]}(t), \tag{50}$$

where the symbol $[\nu]$ denotes the index of the last column in (38), i.e.

$$[\nu] = \sum_1^\nu m_\mu . \tag{51}$$

Consequently, it follows in case $\beta_\nu = 0$ that the corresponding submatrix of the matrix $P^{-1} - I$ is singular. Hence the condition of solvability of (49) coincides with the exceptional condition (34).

We consider again (41) with arbitrary β. The matrix P in subdivided, as it follows easily from (38), in the elementary submatrices (see e.g. [7], (53))

$$P = \begin{bmatrix} P_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & P_s \end{bmatrix} \text{ with } P_\nu = \begin{bmatrix} 1, p, \dots, \dfrac{p^{m_\nu - 1}}{(m_\nu - 1)!} \\ 1, p, \dots, \dfrac{p^{m_\nu - 2}}{(m_\nu - 2)!} \\ \ddots \quad \vdots \\ p \\ 1 \end{bmatrix} e^{\alpha_\nu p}, \tag{52}$$

from which it follows that the matrix $P^{-1} - I$ is subdivided in the elementary submatrices ($\nu = 1, \dots, s$)

$$(53)\qquad (P^{-1}-I)_\nu = \begin{bmatrix} 1, -p, \ldots, \frac{(-p)^{m_\nu-1}}{(m_\nu-1)!} \\ 1, -p, \ldots, \frac{(-p)^{m_\nu-2}}{(m_\nu-2)!} \\ \ddots \qquad \vdots \\ \qquad -p \\ \qquad 1 \end{bmatrix} e^{-\alpha_\nu p} - \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}.$$

For $\alpha_\nu \neq 0$, the submatrices (53) are regular, while for $\alpha_\nu = 0$ they have the rank $m_\nu - 1$. We shall assume that the submatrices are arranged, such that

$$(54)\qquad e^{\alpha_\nu p}\begin{cases} = 1 & (\nu = 1, \ldots, \varrho) \\ \neq 1 & (\nu = \varrho = 1, \ldots, s). \end{cases}$$

Then for $\nu = 1, \ldots, \varrho$ the first column and the last row vanish in the submatrices (53).

The system of equations (41) consists of n equations of the form

$$(55)\qquad v_\mu(\mathbf{c}, \beta) = l_\mu(\mathbf{c}) - \beta b_\mu(\mathbf{c}, \beta) = 0 \qquad (\mu = 1, \ldots, n).$$

Here $l_\mu(\mathbf{c})$ denotes the linear part

$$(56)\qquad l_\mu(\mathbf{c}) = \mathbf{q}_\mu^{\mathrm{T}} \mathbf{c} + \int_0^p \mathbf{z}_\mu^{\mathrm{T}}(\tau) \boldsymbol{f}(\tau)\, \mathrm{d}\tau,$$

where $\mathbf{q}_\mu^T$ is the u-th row vector of the matrix $(P^{-1} - I)$ and $\mathbf{z}_\mu^T$ is the u-th row vector of the matrix Z^T. Further

$$(57)\qquad b_\mu(\mathbf{c}, \beta) = \int_0^p \mathbf{z}_\mu^T(\tau)\, \boldsymbol{r}^*(\tau, \mathbf{x}^*(\tau, \beta, \mathbf{c}), \beta)\, \mathrm{d}\tau.$$

Evidently the linear part (56) vanishes, if the u-th row of the matrix $P^{-1} - I$ vanishes, i.e. if $\mu = [\nu]$, $\nu = 1, \ldots, \varrho$ (see (34) and (50)). In this case (55) for $\mu = [\nu]$ becomes

$$(58)\qquad v_{[\nu]}(\mathbf{c}, \beta) \equiv -\beta b_{[\nu]}(\mathbf{c}, \beta) = 0 \qquad (\nu = 1, \ldots, \varrho).$$

Dividing by β, we obtain instead of (41) for $\beta \neq 0$ the system of equations

$$(59)\qquad v_\mu(\mathbf{c}, \beta) = \begin{cases} v_\mu(\mathbf{c}, \beta) = l_\mu(\mathbf{c}) - \beta b_\mu(\mathbf{c}, \beta) & (\mu = 1 \ldots, n \ \& \\ & \mu \neq [\nu], \nu = 1, \ldots \varrho) \\ -\dfrac{1}{\beta} v_\mu(\mathbf{c}, \beta) = b_\mu(\mathbf{c}, \beta) = 0 & (\mu = [\nu], \nu = 1 \ldots, \varrho.) \end{cases}$$

If we suppose again, that to every β there exists a vector $\mathbf{c}(\beta)$ such that (47) and (48) hold, then it follows necessarily from (57) and (58) that

$$(60)\qquad b_{[\nu]}(\mathbf{c}(0), 0) = \int_0^p \mathbf{z}_{[\nu]}^T(\tau) \boldsymbol{r}^*(\tau, \mathbf{x}^*(\tau, 0, \mathbf{c}(0)), 0)\, \mathrm{d}\tau = 0 \qquad (\nu = 1, \ldots, \varrho).$$

We use now (49) and (60) together as a system of equations to determine the vector $\mathbf{c}(0)$. By virtue of (49), the vector $\mathbf{c}(0)$ is uniquely determined up to the components $c_{(\nu)}(0)$ $(\nu = 1, \ldots, \varrho)$, which remain here arbitrary (see [4], §3 or [6]). The indices (ν) denote those indices of the first corresponding column in (52), i.e.

$$(\nu) = \sum_{1}^{\nu-1} m_\mu + 1. \tag{61}$$

Setting the just determined constants $c_\mu(0)$ in (60), then (60) will represent another system of equations in the ϱ constants $c_{(\nu)}(0)$. Let us assume now that (49) and (60) have a common vector solution $\mathbf{c}(0)$, that will be fixed. Thus the system of equations (59) for $\beta = 0$ is considered to be solved.

If we assume also that the functional determinant

$$\left| \frac{\partial b_{[\nu]}(\mathbf{c}, \beta)}{\partial c_{(\sigma)}} \right|_{\substack{\beta=0 \\ \mathbf{c}=\mathbf{c}(0)}} = \left| \frac{\partial b_{[\nu]}(\mathbf{c}(0), 0)}{\partial c_{(\sigma)}} \right| \neq 0 \qquad (\nu, \sigma = 1, \ldots, \varrho) \tag{62}$$

is different from zero, then it will be proved in § 4, that the system of equations (11) possesses, for sufficiently small values of β, a unique solution $\mathbf{c} = \mathbf{c}(\beta)$, which converges to $\mathbf{c}$ (0) as β approaches to zero.

The elements of this functional determinant (62) can be calculated as it follows: From the expressions (57) with $\mu = [\nu]$, $\beta = 0$, we obtain

$$\begin{aligned} b_{[\nu]}(\mathbf{c}(0),0) &= \int_0^p \mathbf{z}^T_{[\nu]}(t)\, \mathbf{r}^*(\tau, \mathbf{x}^*(\tau, 0, \mathbf{c}(0)), 0)\, d\tau = \\ &= {}^1/_2 \int_0^p \sum_{\gamma, l, \lambda}^{1\ldots n} z_{\lambda, [\nu]}(\tau) \frac{\partial^2 g_\lambda(\mathbf{u}_0(\tau), \tau)}{\partial u_\gamma\, \partial u_l} x^*_\gamma x^*_l \, d\tau. \end{aligned} \tag{63}$$

Further we obtain from (42)

$$x^*_l(t, 0, \mathbf{c}(0)) = \sum_{i, k}^{1\ldots n} y_{l, k}(t) \left(\int_0^t z_{i, k}(\tau) f_i(\tau)\, d\tau + c_k \right),$$

from which it follows that

$$\frac{\partial x^*_l(\tau, 0, \mathbf{c}(0))}{\partial c_{(\sigma)}} = y_{l, (\sigma)}(\tau).$$

Hence we obtain for the derivatives of (62)

$$\begin{aligned} \frac{\partial b_{[\nu]}(\mathbf{c}(0), 0)}{\partial c_{(\sigma)}} &= \int_0^p \sum_{8, l, \lambda} z_{l, [\nu]}(\tau) \frac{\partial^2 g_\lambda(\mathbf{u}_0(\tau), \tau)}{\partial u_\gamma \partial u_l} x^*_\gamma(\tau, 0, \mathbf{c}(0)) \cdot \\ &\cdot y_{l, (\sigma)}(\tau)\, d\tau \qquad (\nu, \sigma = 1,, \ldots, \varrho). \end{aligned} \tag{64}$$

Then it can be tested whether the functional determinant (62) vanishes or not.

§ 4. THE EXISTENCE OF PERIODIC SOLUTIONS $\mathbf{x}(t)$ OF (11) WITH PERIOD p OF ORDER $|\beta|$ IN THE EXCEPTIONAL CASE

Theorem 2. Under the hypothesis (34) and the existence of a vector $\mathbf{c}(0)$ such that (49), (60) and (62) hold, there exists to every sufficiently small values of β a vector solution $\mathbf{c}(\beta)$ of (41), which converges to $\mathbf{c}(0)$ as β tends to zero, such that the solution $\mathbf{x}^*(t, \beta, \mathbf{c}(\beta))$ has the period p.

To prove this theorem, we need the fundamental theorem of implicit functions, which will be formulated as the following

Lemma: Let the vector

$$\hat{\mathbf{v}}(c, \beta) = \hat{\mathbf{v}}(c, \ldots, c_n, \beta)$$

possess in a certain neighbourhood of the values

$$c_1 = c_1(0),\ c_2 = c_2(0),\ \ldots,\ c_n = c_n(0),\ \beta = 0$$

continuous first derivatives w.r.t. c_μ, and let it be continuous in all its $n + 1$ variables. Further let.

$$\hat{\mathbf{v}}(\mathbf{c}(0), 0) = \mathbf{0}; \tag{65}$$

and finally let the functional determinant

$$\left| \frac{\partial \hat{v}_u(c_1(0), c_2(0), \ldots, c_n(0), 0)}{\partial c_\lambda} \right| \neq 0 \ (\mu, \lambda = 1 \ \ldots, n). \tag{66}$$

Then there exists, to every sufficiently small β, a unique small vector solution $\mathbf{c} = \mathbf{c}(\beta)$ of the system of equations

$$\hat{\mathbf{v}}(\mathbf{c}, \beta) = \mathbf{0}, \tag{67}$$

which converges to $\mathbf{c}(0)$ as β tends to zero.

(For the proof see e.g. [8], § 7.)

Proof of theorem 2. Let the system of equations (41) be written in the form (59). We examine now the validity of the assumptions of the preceding lemma. Referring to (54), (49) and (60), we get for $\beta = 0$

$$\begin{cases} \hat{v}_\mu(\mathbf{c}(0), 0) = v_\mu(\mathbf{c}(0), 0) = l_\mu(\mathbf{c}(0)) = 0 \ (\mu = 1, \ldots, n \ \&\ \mu \neq [\nu] \text{ for } \nu = 1, \ldots, \varrho), \\ v_{[\nu]}(\mathbf{c}(0), 0) = b_{[\nu]}(\mathbf{c}(0), 0) = \displaystyle\int_0^p \mathbf{z}^T_{[\nu]}(\tau)\, \mathbf{r}^*(\tau, \mathbf{x}^*(\tau, 0, \mathbf{c}(0), 0)\, d\tau = 0 \ (\nu = 1, \ldots, \varrho). \end{cases}$$

Thus the condition (65) in satisfied.

Further we obtain from (59) for $\mu \neq [\nu]$ $(\nu = 1, \ldots, \varrho)$

$$\frac{\partial \hat{v}_\mu(\mathbf{c}, \beta)}{\partial c_\lambda} = \frac{\partial v_\mu(\mathbf{c}, \beta)}{\partial c}.$$

Since by virtue of (45) and (55), $\mathbf{v}(\mathbf{c}, \beta)$ satisfies the equation

$$\mathbf{x}^*(0, \beta, \mathbf{c}) - \mathbf{x}^*(p, \beta, c) = Y(0) P \mathbf{v}(\mathbf{c}, \beta).$$

then it can be written, by using (35) and (36) for $t = 0$, in the form

$$\mathbf{v}(\mathbf{c}, \beta) = Z^T(p)(\mathbf{x}^*(0, \beta, \mathbf{c}) - \mathbf{x}^*(p, \beta, \mathbf{c})).$$

Thus the assumptions of the preceding lemma of the continuity of the partial derivatives $\frac{\partial \hat{v}_\mu(\mathbf{c}, \beta)}{\partial c_\lambda}$ can be reduced to the continuity of the derivatives $\frac{\partial x^*_\mu(p, \beta, \mathbf{c})}{\partial c_\lambda}$, which follows from known theorem on the dependence of the solutions on the initial values and the parameters (see e.g. [2], § 17). For $\mu = [\nu]$ $(\nu = 1, \ldots, \varrho)$, it can be easily shown by virtue of (59), (57) and (27) that the derivatives $\frac{\partial \hat{v}_{[\nu]}(\mathbf{c},\beta)}{\partial c_\lambda}$ are also continuous.

It remains only to show, that the condition (66) is satisfied, i.e. it is required to prove that the system of equations (59) for $\beta = 0$ & $\mathbf{c} = \mathbf{c}(0)$ possesses a regular functional matrix

$$\left(\frac{\partial \hat{v}_\mu(\mathbf{c}, \beta)}{\partial c_\lambda}\right)_{\substack{\beta = 0 \\ \mathbf{c} = \mathbf{c}(0)}} = \left(\frac{\partial \hat{v}_\mu(\mathbf{c}(0), 0)}{\partial c_\lambda}\right) \qquad (\mu, \lambda = 1, \ldots, n). \tag{68}$$

Referring to (59) with $\beta = 0$, we obtain for (68):

$$\begin{cases} \dfrac{\partial \hat{v}_{[\nu]}(\mathbf{c}(0), 0)}{\partial c_\lambda} = \dfrac{\partial b_{[\nu]}(\mathbf{c}(0), 0)}{\partial c_\lambda} & (\mu = [\nu] \text{ for } \nu = 1, \ldots, \varrho). \\ \dfrac{\partial \hat{v}_\mu(\mathbf{c}(0), 0)}{\partial c_\lambda} = \dfrac{\partial l_\mu(\mathbf{c}(0))}{\partial c_\lambda} & (\text{for all other indices } \mu). \end{cases} \tag{69}$$

We see that the functional matrix (68), in all the rows with indices $\mu \neq [\nu]$ $(\nu = 1, \ldots, \varrho)$ coincide with the matrix $(P^{-1} - \mathrm{I})$. These rows are linear independent (see (53)). In the rows with indices $\mu = [\nu]$ $(\nu = 1, \ldots, \varrho)$, for which the elements of the matrix $(P^{-1} - \mathrm{I})$ are all zeros, there exists instead of them the partial derivatives

$$\frac{\partial b_{[\nu]}(\mathbf{c}(0), 0)}{\partial c_\lambda} \qquad (\lambda = 1, \ldots, s). \tag{70}$$

Thus we obtain the following representation:

Since the triangular submatrices of $(P^{-1} - \mathrm{I})$ for $\nu = 1, \ldots, \varrho$ of order $m_\nu - 1$, which are obtained by eliminating the (ν)-th column and the $[\nu]$-th row, are regular and since the other triangular submatrices for $\nu = \varrho + 1, \ldots, s$ of order m_ν are also regular, then by using row-combinations we can replace all derivatives (70) by zeros, with exception of the derivatives

$$\left(\frac{\partial b_{[\nu[}(\mathbf{c}(0),0)}{\partial c_{(\sigma)}}\right) \qquad (\nu, \sigma = 1, \ldots, \varrho). \tag{72}$$

It follows easily that the functional matrix (71) is regular if the submatrix (72) is regular, which is exactly the assumption (62). By virtue of the preceding lemma, there exists to every small sufficiently small β, a unique vector solution $\mathbf{c} = \mathbf{c}(\beta)$ of (59), which converges to $\mathbf{c}$ (0) as $\beta \to 0$. Since the system of equations (41) follows directly from (59), then the solution $\mathbf{x}^*(t, \beta, \mathbf{c}(\beta))$ of (20) is periodic with period p.

It follows also that $\mathbf{x} = \beta \mathbf{x}^*(t, \beta, \mathbf{c}(\beta))$ is a periodic solution with period p of the system of differential equations (11) of order $|\beta|$.

We summerise our results.

$$
(71)\qquad \left(\frac{\partial \hat{v}_u(\mathbf{c}(0),0)}{\partial c_\lambda}\right) =
\left|\begin{array}{c|c|c|c|c|c}
\begin{matrix} 0, -p, \dots, \dfrac{(-p)^{m_1-1}}{(m_1-1)!} \\ \ddots \qquad \vdots \\ 0, -p \end{matrix} & & & & & \\
\hline
\dfrac{\partial b_{[1]}(\mathbf{c}(0),0)}{\partial c_{(1)}}, \dots, \dfrac{\partial b_{[1]}(\mathbf{c}(0),0)}{\partial c_{[1]}} & \dots & \dfrac{\partial b_{[1]}}{\partial c_{(\varrho)}}, \dots, \dfrac{\partial b_{[1]}}{\partial c_{[\varrho]}} & \dfrac{\partial b_{[1]}}{\partial c_{(\varrho+1)}}, \dots, \dfrac{\partial b_{(1)}}{\partial c_{[\varrho+1]}} & \dots & \dfrac{\partial b_{[1]}}{\partial c_{(s)}}, \dots, \dfrac{\partial b_{[1]}(\mathbf{c}(0),0)}{\partial c_{[s]}} \\
\hline
 & \ddots & & & & \\
 & & \begin{matrix} 0, -p, \dots, \dfrac{(-p)^{m_\varrho-1}}{(m_\varrho-1)!} \\ \ddots \qquad \vdots \\ 0, -p \end{matrix} & & & \\
\hline
\dfrac{\partial b_{[\varrho]}(\mathbf{c}(0),0)}{\partial c_{(1)}}, \dots, \dfrac{\partial b_{[\varrho]}(\mathbf{c}(0),0)}{\partial c_{[1]}} & \dots & \dfrac{\partial b_{[\varrho]}}{\partial c_{(\varrho)}}, \dots, \dfrac{\partial b_{[\varrho]}}{\partial c_{[\varrho]}} & \dfrac{\partial b_{[\varrho]}}{\partial c_{(\varrho+1)}}, \dots, \dfrac{\partial b_{[\varrho]}}{\partial c_{[\varrho+1]}} & \dots & \dfrac{\partial b_{[\varrho]}}{\partial c_{(s)}}, \dots, \dfrac{\partial c_{[\varrho]}(\mathbf{c}(0),0)}{\partial c_{[s]}} \\
\hline
 & & & \begin{matrix} e^{-\alpha(\varrho+1)p}-1, \dots, \dfrac{(-p)^{m_{\varrho+1}-1}}{(m_{\varrho+1}-1)!} e^{-\alpha_{\varrho+1}p} \\ \ddots \qquad \vdots \\ e^{-\alpha_{\varrho+1}}-1 \end{matrix} & & \\
 & & & & \ddots & \\
 & & & & & \begin{matrix} e^{-\alpha_s p}-1, \dots, \dfrac{(-p)^{m_s-1}}{(m_s-1)!} e^{-\alpha_s p} \\ \ddots \qquad \vdots \\ e^{-\alpha_s p}-1 \end{matrix}
\end{array}\right|
$$

Let all with p periodic solutions $z_{[\nu]}(t)$ $(\nu = 1, \ldots, \varrho)$ of the adjoint system (32) satisfy the exceptional conditions (34), and let $\mathbf{c}(0)$ be a vector solution of (49), so that the corresponding vector solution $\mathbf{x}^*(t, 0, \mathbf{c}(0))$ of (20) has the period p. Further let all with p periodic solutions $z_{[\nu]}(t)$ of (32) satisfy also (60), and finally let the functional matrix (72) be regular. Then there exists, for sufficiently small values of β a periodic vector solution $\mathbf{x}^*(t, \beta, \mathbf{c}(\beta))$ with period p of the system of differential equations (20), which goes over in the vector solution $\mathbf{x}^*(t, 0, \mathbf{c}(0))$ as $\beta \to 0$ where at the same time $\mathbf{c}(\beta)$ converges to $\mathbf{c}(0)$. Consequently, it follows simultaneously the existence of a periodic vector solution $\mathbf{x}(t, \beta) = \mathbf{x}^*(t, \beta, \mathbf{c}(\beta))$ with period p of the system of differential equations (11) which is of order $|\beta|$.

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