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ON THE BEHAVIOUR OF THE SOLUTIONS AND THEIR FIRST (n — 1) DERIVATIVES OF THE n — th ORDER DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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§1 . INTRODUCTION

Consider the differential equation

(a) $$L[x] \equiv x^{(n)} + a_1(t)\, x^{(n-1)} + \ldots + a_n(t)\, x = f(t),$$

where the coefficients $a_\mu(t)$ ($\mu = 1, 2, \ldots, n$) and the function $f(t)$ are continuous periodic functions of the same period p

$$a_\mu(t+p) = a_\mu(t)\ (\mu = 1, \ldots, n), f(t+p) = f(t).$$

The homogeneous and adjoint differential equations corresponding to (a) are

(b) $$L[y] \equiv y^{(n)} + a_1(t)\, y^{(n-1)} + \ldots + a_n(t)\, y = 0$$

and

(c) $$L[z] \equiv (-1)^n z^{(n)} + (-1)^{n-1}\big(a_1(t) z\big)^{(n-1)} + \ldots + a_n(t)\, z = 0$$

respectively.

Definition 1: The inhomogeneous differential equation (a) is said to be in the resonance case, if the adjoint differential equation (c) possesses at least one periodic solution $z(t)$ of period p, for which

$$\int_0^p z(t)\, f(t)\, \mathrm{d}t \neq 0.$$

Equation (a) is said to be in the exceptional case, if for all periodic solutions $z(t)$ of (c) the relation

$$\int_0^p z(t)\, f(t)\, \mathrm{d}t = 0.$$

holds; and in the principal case, if (c) has no periodic solution of period p.

Referring to [1] and [2], § 2, we can obtain the following:

Theorem 1: For the inhomogeneous differential equation (a) there exist bounded solutions, if the adjoint equation (c) has either at least one periodic solution $z(t)$ of period p for which $\int_0^p z(t)\, f(t) dt = 0$ (exceptional case) or no periodic solution of period p at all (principal case). But if (c) has periodic solutions $z(t)$ of period p, such that for all these $z(t)$, $\int_0^p z(t) f(t)\, \mathrm{d}t \neq \neq 0$ (resonance case), then the modulus of each solution of (a) independent of the initial values tends to ∞.

It was shown in [2], that in the resonance case the so-called "normal solution" $x(t)$ of (a) (see definition 4)—independent) of the initial conditions—takes at least values of power order equal to a certain power t^m.

In this paper, we consider an arbitrary solution of (a) and we study its minimal power order in the resonance case. Further, we study the minimal power order of the first (n — 1) derivatives of any arbitrary solution.

§ 2. THE PARTIAL-RESONANCE CASE

It is proved in [3], § 2, that the homogeneous differential equation (b) possesses the fundamental system of solutions $Y(t)$, which can be obtained in the form

$$Y(t) = \Phi(t)\, \mathrm{e}^{Kt}, \quad \Phi^{(t)} = \left(\bar{\varphi}_1(t), \ldots, \bar{\varphi}_n{}^{(t)}\right), \tag{1}$$

where the matrix $\Phi(t)$ is of period p and the constant matrix K is in the Jordan canonical normal form with the submatrices

$$K_\nu = \begin{bmatrix} \alpha_\nu & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \alpha_\nu \end{bmatrix},$$

of order $m_\nu \geqslant 1$. For the eigenvalues α_ν, we can make the limitation (see [3], (18))

$$-\frac{\Pi}{p} < I(\alpha_\nu) = \gamma_\nu \lesseqgtr \frac{\Pi}{p}. \tag{2}$$

The corresponding fundamental system of (c) is (see [1], (17) or [4], § 1.4)

$$Z(t) = \left(Y^{-1}(t)\right)^T = \left(\Phi^{-1}(t)\right)^T . \mathrm{e}^{-K^T t},$$

where $(Y^{-1})^T$ denotes the transposed matrix of Y^{-1}.

Let

$$\alpha_\nu \begin{cases} = 0 & \text{for } \nu = 1, 2, \ldots, \varrho \\ \neq 0 & \text{for } \nu = \varrho + 1, \ldots, s. \end{cases} \tag{3}$$

By virtue of [3], § 2 or [1], § 3, it follows that the differential equation

(b) has ϱ linear independent with p periodic solutions $y_{(\nu)}(t) = {}_1y_{(\nu)}(t)$, $(\nu = 1, \ldots, \varrho)$, which are the 1st components of the vectors $\mathfrak{y}_{(\nu)}(t)$ in the fundamental matrix $Y(t)$. Further the equation (c) has also the same number of linear independent with p periodic solutions $z_{[\nu]}(t) = {}_nz_{[\nu]}(t)$, which are the n^{th} components of the vectors $\mathfrak{z}_{[\nu]}(t)$ of the fundamental matrix $Z(t)$. The stated indices are defined by

$$(\nu) = \sum_{1}^{\nu-1} m_\mu + 1, \quad [\nu] = \sum_{1}^{\nu} m_\mu. \tag{4}$$

Definition 2: The inhomogeneous differential equation (a) is said to be w.r.t. an index ν in the partial-resonance case, if the adjoint differential equation (c) possesses the periodic solution $z_{[\nu]}(t)$ of period p (i.e. the corresponding $\alpha_\nu = 0$), such that

$$\int_0^p z_{[\nu]}(t)\, f(t)\, \mathrm{d}t = a_{[\nu]} \neq 0.$$

Equation (a) is said to be w. r. t. an index ν in the partial-exceptional case, when $\alpha_\nu = 0$ and simultaneously the relation

$$\int_0^p z_{[\nu]}(t)\, f(t)\, \mathrm{d}t = 0$$

holds. But if $\alpha_\nu \neq 0$, then the equation (a) is said to be w.r.t. the index ν in the partial-principal case.

Let the matrix $Y(t)$ or $Z(t)$ be so arranged that

$$\int_0^p z_{[\nu]}(t)\, f(t)\, \mathrm{d}t = a_{[\nu]} \begin{cases} \neq 0 & \text{for } \nu = 1, \ldots, \sigma \\ = 0 & \text{for } \nu = \sigma + 1, \ldots, \varrho. \end{cases} \tag{5}$$

Then the equation (a) is w.r.t. the indices $\nu = 1, \ldots, \sigma$ in the partial-resonance case, while (a) is w.r.t the indices $\nu = \sigma + 1, \ldots, \varrho$ in the partial-exceptional case.

Definition 3: The index ν will be called a resonance, exceptional or principal index, if the differential equation (a) is w. r. t. this index in the partial-resonance, partial-exceptional or partial-principal case.

By using the method of variation of parameters, we obtain the general solution of (a) in the form

$$x(t) = \sum_{1}^{s} {}^{\nu}x(t) \tag{6}$$

with (see [2], (33), (35) and (37))

$$ {}^{\nu}x(t) = \sum_{\mu=0}^{m_\nu} e^{\alpha_\nu t} \frac{t^{m_\nu - \mu}}{(m_\nu - \mu)!}\, {}^{\nu}\Theta_\mu(t), \text{ with } \alpha_\nu \begin{cases} = 0 \text{ for } \nu = 1, \ldots, \varrho \\ \neq 0 \text{ for } \nu = \varrho + 1, \ldots, s, \end{cases} \tag{7}$$

where

$$(8) \quad \begin{cases} {}^{\nu}\Theta_{\mu}(t) = \sum_{\gamma=0}^{\mu} {}^{\nu}d_{\mu-\gamma}\, \varphi_{(\nu)+\gamma}(t) \text{ for } \mu = 0, 1, \ldots, m_{\nu} - 1, \\ {}^{\nu}\Theta_{m_{\nu}}(t) = \sum_{\gamma=0}^{m_{\nu}-1} ({}^{\nu}d_{m_{\nu}-\gamma} + v^{*}_{(\nu)+\varphi})\, \varphi_{(\gamma)+\varphi}(t). \end{cases}$$

The constants ${}^{\nu}d_{\mu-\gamma}$ with $\mu - \gamma > 0$ are arbitrary constants of integration, while (see (5))

$$(9) \quad {}^{\nu}d_0 \begin{cases} = \dfrac{1}{p}\, a_{[\nu]} \neq 0 & \text{for } \nu = 1, \ldots, \sigma \\ = 0 & \text{for } \nu = \sigma + 1, \ldots, s. \end{cases}$$

Referring to [3], theorem 5, there exist for the components ${}^{\nu}x(t)$ (for $\nu = \varrho + 1,, \ldots, s$) of the solution $x(t)$ of (a) unique periodic functions ${}^{\nu}x^{*}(t)$ of period p.

Definition 4: A solution (6) of (a) whose components for $\nu = 1, \ldots, \varrho$ are represented by (7) with $\alpha_{\nu} = 0$ and for $\nu = \varrho + 1,, \ldots, s$ are unique periodic functions ${}^{\nu}x^{*}(t)$ of period p, is called a "normal solution".

§ 3. FUNDAMENTAL LEMMAS

In this paragraph we prove the following two lemmas:

Lemma 1: Let $\eta_1(t), \eta_2(t), \ldots, \eta_q(t)$ be non-identic vanishing periodic functions of period p. Let $\gamma_1, \gamma_2, \ldots, \gamma_q$ be real numbers, such that the numbers $e^{i\gamma_v p}$ $(v = 1, 2, \ldots, q)$ are pairwise distinct. Then there exists a number $K_0 > 0$ such that the modulus of the summation

$$(10) \quad \mathscr{S}(t) = \sum_{v=1}^{q} \eta_v(t)\, e^{i\gamma_v t}$$

takes values greater than K_0 for arbitrary large values of t.

Proof: We have only to prove that $\mathscr{S}(t)$—with increasing t—converges to a value different from zero. For this purpose, we form the functions

$$\begin{cases} \mathscr{S}(t) = \sum_{v=1}^{q} \eta_v(t)\, e^{i\gamma_v . t} \\ \mathscr{S}(t+p) = \sum_{v=1}^{q} \eta_v(t)\, e^{i\gamma_v . t} \,.\, e^{i\gamma_v . p} \\ \mathscr{S}(t+2p) = \sum_{v=1}^{q} \eta_v(t)\, e^{i\gamma_v . t} \,.\, e^{2i\gamma_v p} \\ \quad \vdots \\ \mathscr{S}\big(t + (q-1)\,p\big) = \sum_{v=1}^{q} \eta_v(t)\, e^{i\gamma_v . t} \,.\, e^{(q-1)i\gamma_v . p}. \end{cases}$$

This is a system of equations of the form

$$
(11)\qquad
\begin{bmatrix}
\mathscr{S}(t) \\
\mathscr{S}(t+p) \\
\mathscr{S}(t+2p) \\
\vdots \\
\mathscr{S}\big(t+(q+1)\,p\big)
\end{bmatrix}
=
$$

$$
=
\begin{bmatrix}
1 & , 1 & , \dots, & 1 \\
e^{i\gamma_1 p} & , e^{i\gamma_2 p} & , \dots, & e^{i\gamma_q p} \\
e^{2i\gamma_1 p} & , e^{2i\gamma_2 p} & , \dots, & e^{2i\gamma_q p} \\
\vdots & \vdots & \vdots & \vdots \\
e^{(q-1)i\gamma_1 p}, & e^{(q-1)i\gamma_2 p}, & \dots, & e^{(q-1)i_q p}
\end{bmatrix}
\begin{bmatrix}
\eta_1(t)\ e^{i\gamma_1 t} \\
\eta_2(t)\ e^{i\gamma_2 t} \\
\vdots \\
\eta_q(t)\ e^{i\gamma_q t}
\end{bmatrix}.
$$

Suppose the contrary, i.e. $\lim\limits_{t \to \infty} \mathscr{S}(t) = 0$. Then the vector on the L.H.S. converges to zero as $t \to \infty$. But the determinant of the constant matrix in (11) is different from zero "Vandermonde's determinant" (see [5], § 4.21), since the numbers $e^{i\gamma_v p}$ $(v = 1, 2, \dots, q)$ are pairwise distinct. It follows that the vector on the R.H.S. converges to zero, which contradicts with our hypothesis on the functions $\eta_1(t), \dots, \eta_q(t)$. Thus the lemma is proved.

Lemma 2. Let $\eta(t)$ be a non-trivial periodic solution of period p of the homogeneous differential equation (b) under the condition

$$(12)\qquad a_n(t) \not\equiv 0.$$

Let β, γ be real numbers, such that $\big($see (2)$\big)$

$$(13)\qquad -\frac{\pi}{p} < \gamma \leqq \frac{\pi}{p}.$$

Then the differential expression

$$(D + \beta + i\gamma)^r\ \eta(t)$$

for every $r = 1, 2, \dots, n-1$ is not identic zero, where D denotes the differential operator $\dfrac{d}{dx}$.

Proof: Suppose the contrary, i.e. one of the expressions is identic zero. Then $\eta(t)$ will be a solution of the differential equation with constant coefficients

$$(D + \beta + i\gamma)^r\ \eta(t) = 0$$

i.e.

$$\eta(t) = e^{-(\beta+i\gamma)t} \cdot \sum_{\mu=0}^{r-1} c_\mu t^\mu,$$

where c_μ are arbitrary constants. Since $\eta(t)$ has the period p, then it must have the form

$$\eta(t) = c_0\, e^{-\beta t}\, e^{i\gamma t} \text{ with } \beta = 0 \text{ and } e^{i\gamma p} = 1.$$

It follows from (13), that γ is equal to zero and consequently

$$\eta(t) = c_0 = \text{const},$$

i.e. the periodic solution $\eta(t)$ of (b) is a constant $\neq 0$. This is a contradiction with the fact that the homogeneous differential equation (b) can not—under the condition (12)—possess a constant solution. Otherwise, by substituting in (b), we get

$$a_n(t)\, y = a_n(t) \,.\, \text{const}] \equiv 0 \Rightarrow a_n(t) \equiv 0.$$

§ 4. THE SOLUTION OF THE INHOMOGENEOUS DIFFERENTIAL EQUATION (a)

Consider an arbitrary solution $x(t)$ of (a). This can be written in the form

(14) $$x(t) = x^*(t) + y(t),$$

where $x^*(t)$ is a particular solution of (a) and $y(t)$ is the general solution of (b). It is comfortable to sum up the terms of $y(t)$ whose constant factors are equal. Referring to (6), (7), (8), we obtain

(15) $$x(t) = \sum_{\nu=1}^{\sigma} \frac{1}{p} a_{[\nu]} \sum_{\gamma=0}^{m_\nu-1} \left(\varphi_{(\gamma)+\gamma}(t) \frac{t^{m_\nu-\gamma}}{(m_\nu-\gamma)!} \right) + \pi(t) + \\ + \sum_{\nu=1}^{s} e^{\alpha_\nu t} \sum_{\mu=1}^{m_\nu} \left({}^\nu d_\mu \cdot \sum_{\gamma=0}^{m_\nu-\mu} \varphi_{(\nu)+\gamma}(t) \frac{t^{m_\nu-\gamma-\mu}}{(m_\nu-\gamma-\mu)!} \right).$$

Here $\pi(t)$ represents the periodic function of period p:

$$\sum_{\nu=1}^{\varrho} \left(\sum_{\gamma=0}^{m_\nu-1} v^*_{(\nu)+\gamma} \varphi_{(\nu)+\gamma}(t) \right) + \sum_{\varrho+1}^{s} {}^\nu x^*(t),$$

where ${}^\nu x^*(t)$ for $\nu = \varrho + 1, \ldots, s$ are unique periodic functions of period p (see [3], theorem 5).

It is possible that some constants ${}^\nu d_\mu$ vanish. For this purpose, we introduce the index $l\nu$ with the following property: In the sequence ${}^\nu d_1, {}^\nu d_2, \ldots, {}^\nu d_{m_\nu}$ let ${}^\nu d_{l\nu}$ be the first non-vanishing constant. Setting

$$(16)\qquad \omega_\nu = m_\nu - l_\nu,\; m = \max_{(\nu=1,\ldots,\sigma)} (m_\nu),$$

we obtain for $x(t)$ the following representation

$$(17)\qquad x(t) = \left(\sum_{\substack{\nu=1\\(m_\nu=m)}}^{\sigma} \frac{1}{p}\, a_{[\nu]}\varphi_{[\nu]}(t)\, \frac{t^{m_\nu}}{m_\nu!} + \text{smaller powers}\right) + (\mathrm{I})$$

$$+ \left(\sum_{\nu=\sigma+1}^{\varrho} {}^\nu d_{l_\nu} \varphi_{(\nu)}(t)\, \frac{t^{\omega_\nu}}{\omega_\nu!} + \text{smaller powers}\right) + (\mathrm{II})$$

$$+ \left(\sum_{\nu=\varrho+1}^{s} e^{\alpha_\nu t}\, {}^\nu d_{l_\nu} \varphi_{(\nu)}(t)\, \frac{t^{\omega_\nu}}{\omega_\nu!} + \text{smaller powers}\right). \;(\mathrm{III})$$

(In the case, that all constants ${}^\nu d_1, {}^\nu d_2, \ldots, {}^\nu d_{m_\nu}$ vanish for an index $\nu > \sigma$, we put ${}^\nu d_{l\nu} = 0$). The corresponding terms for $\nu = 1, \ldots, \sigma$ are not needed to be written, since they are included in the 1st bracket in (17) under the "smaller powers".

Consider now the 3rd bracket (III) in (17), which includes exponential functions. Here the summations with negative $R(\alpha_\nu)$ can remain out of consideration, since they tend to zero as $t \to \infty$. Set

$$\alpha_\nu = \beta_\nu + i\gamma_\nu$$

and let β_ν (for $\nu = \varrho + 1, \ldots, s$) form a decreasing sequence

$$(18)\qquad \beta_{\varrho_1} > \beta_{\varrho_2} > \cdots > \beta_{\varrho_k} = 0.$$

Let us subdivide (III) in partial sums according to equal exponents of β_ν. Then we obtain

$$(19)\qquad (\mathrm{III}) = (\mathrm{III})_{\varrho_1} + (\mathrm{III})_{\varrho_2} + \cdots + (\mathrm{III})_{\varrho_k} + (\mathrm{III})_-,$$

where $(\mathrm{III})_-$ denotes the terms with negative β_ν.

On the power order of every partial sum $(\mathrm{III})_{\varrho\lambda}$ in (19) (for $\lambda = 1, 2, \ldots, k-1$), we state the following:

Theorem 2: Every partial sum $(\mathrm{III})_{\varrho\lambda}$ in (19) (for $\lambda = 1, 2, \ldots, k-1$) either takes values of the power order $e^{\beta_{\varrho\lambda}\cdot t}\, t^{\omega(\lambda)}$ with $\omega(\lambda)$ from (25) or it vanishes identically.

Proof: Referring to (15), (17), (19), it can be easily obtained for $(\mathrm{III})_{\varrho\lambda}$ ($\lambda = 1, 2, \ldots, k$) the convenient representation

$$(20)\quad (\mathrm{III})_{\varrho\lambda} = e^{\beta_{\varrho\lambda} t} \sum_{\substack{(\nu)\\(\beta_\nu=\beta_{\varrho\lambda})}} e^{i\gamma_\nu t} \sum_{\mu=0}^{m_\nu-1} \left({}^\nu d_{\mu+1} \sum_{r=0}^{m_\nu-1} \varphi_{(\nu)+r}(t)\, \frac{t^{m_\nu-1-r-\mu}}{(m_\nu-1-r-\mu)!}\right).$$

If the constants ${}^{\nu}d_{u+1}$ vanish for all $\mu = 0, 1, \ldots, m_{\nu} - 1$ and all ν for which $\beta_{\nu} = \beta_{\varrho\lambda}$, then $(\mathrm{III})_{\varrho\lambda}$ vanishes certainly. If all the constants ${}^{\nu}d_{\mu+1}$ do not vanish, then it will be proved that $(\mathrm{III})_{\varrho\lambda}$ (for $\lambda = 1, 2, \ldots, k-1$) takes values of the power order $e^{\beta_{\varrho\lambda}t}\, t^{w(\lambda)}$. Here we define for every index ν an index $\bar{\mu} = \bar{\mu}(\nu)$, such that in the sequence of constants

$$(21) \qquad {}^{\nu}d_1, {}^{\nu}d_2, \ldots, {}^{\nu}d_{m_{\nu}}$$

${}^{\nu}d_{\bar{\mu}}$ is the first constant, which is different from zero. Then we can start the sum over μ in $(\mathrm{III})_{\varrho\lambda}$ with $\mu = \bar{\mu}(\nu)$. When all the constants in the sequence (21) are equal to zero, then the sum over μ in (20) (for $\mu = 0, 1, \ldots, m_{\nu} - 1$) is certainly empty. Now in the sum over r with fixed μ and ν, the term $\varphi_{(\nu)} \dfrac{t^{m_{\nu}-\mu-1}}{(m_{\nu} - \mu - 1)!}$ is the only one which has the highest power factor. Therefore there exists exactly one term with the highest power factor $t^{m_{\nu}-\bar{\mu}-1}$ in the sum over μ for $\bar{\mu}(\nu), \ldots, m_{\nu} - 1$ with fixed ν. Putting

$$(22) \qquad \omega_{\nu}^{*} = m_{\nu} - \bar{\mu}(\nu) - 1,$$

we get for the sum over μ in (20) the following representation

$$(23) \qquad \sum_{\mu=\bar{\mu}(\nu)}^{m_{\nu}-1} \left({}^{\nu}d_{u+1} \sum_{r=0}^{m_{\nu}-1} \varphi_{(\nu)+r} \frac{t^{m_{\nu}-1-r-\mu}}{(m_{\nu} - 1 - r - \mu)!} \right) =$$

$$= {}^{\nu}d_{\bar{\mu}} \frac{t^{\omega_{\nu}^{*}}}{\omega_{\nu}^{*}!}\, \varphi_{(\nu)} + \text{smaller powers}.$$

Hence we obtain for $(\mathrm{III})_{\varrho\lambda}$ the representation

$$(24) \quad (\mathrm{III})_{\varrho\lambda} = e^{\beta_{\varrho\lambda}\cdot t} \sum_{\substack{(\nu) \\ (\beta_{\nu}=\beta_{\varrho\lambda})}} e^{i\gamma_{\nu}t} \left({}^{\nu}d_{\bar{\mu}} \frac{t^{\omega_{\nu}^{*}}}{\omega_{\nu}^{*}!}\, \varphi_{(\nu)} + \text{smaller powers} \right).$$

If all the constants in (24) vanish, then $(\mathrm{III})_{\varrho\lambda}$ will vanish identically. If not, we can arrange the occuring powers $t^{\omega_{\nu}^{*}}$ in (24) in decreasing exponents and sum up the terms having the same maximal exponents

$$(25) \qquad \omega(\lambda) = \max_{\substack{(\nu) \\ ({}^{\nu}d_{\bar{\mu}} \neq 0)}} (\omega_{\nu}^{*}).$$

Then $(\mathrm{III})_{\varrho\lambda}$ takes the form

$$(26) \quad (\mathrm{III})_{\varrho\lambda} = e^{\beta_{\varrho\lambda}t} \left(\frac{t^{\omega(\lambda)}}{(\omega(\lambda))!} \sum_{\substack{(\nu) \\ (\beta_{\nu}=\beta_{\varrho\lambda}) \\ ({}^{\nu}d_{\bar{\mu}} \neq 0)}} e^{i\gamma_{\nu}t}\, {}^{\nu}d_{\bar{\mu}}\varphi_{(\nu)} + \text{smaller powers} \right).$$

We want to sum up the terms with the same exponents $i\gamma_\nu$ in the existing sum in (26). Let $\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_q$ be the pairwise distinct exponents γ, then we obtain the sum over ν in the abbreviated form

$$(27) \qquad \sum_{\substack{(\nu) \\ (\beta_\nu = \beta_{\varrho\lambda}) \\ ({}^\nu d_{\bar{\mu}} \neq 0)}} e^{i\gamma_\nu t}({}^\nu d_{\bar{\mu}} \varphi_{(\nu)} + \text{smaller powers}) = \sum_{v=1}^{q} e^{i\bar{\gamma}_v \cdot t}\, \eta_v(t).$$

Here $\eta_v(t)$ are certain linear combinations of the functions $\varphi_{(\nu)}(t)$ with indices ν for which $\beta_\nu = \beta_{\varrho\lambda}$ and ${}^\nu d_\mu \neq 0$. But from (1), the functions $\varphi_{(\nu)}(t)$ are identic with the corresponding with p periodic linear independent solutions $y_{(\nu)}(t)$ of the homogeneous differential equation (b). Consequently the functions $\eta_v(t)$ are also non-trivial with p periodic solutions of (b). Further the numbers $e^{i\bar{\gamma}_v p}$ (for $v = 1, 2, \ldots, q$) are pairwise distinct because of the normalization of the eigen values α_ν in (2). Applying theorem 1, it follows that the modulus of the sum (27) takes values greater than a certain positive constant K_0. Thus each sum $(\mathrm{III})_{\varrho\lambda}$ ($\lambda = 1, 2, \ldots, k-1$) either takes values of the power order $e^{\beta_{\varrho\lambda} \cdot t}\, t^{\omega(\lambda)}$, or it vanishes identically.

§ 5. THE POWER ORDER OF ANY ARBITRARY SOLUTION OF THE DIFFERENTIAL EQUATION (a)

We are going to prove the following:

Theorem 3: In the resonance case any arbitrary solution $x(t)$ of the differential equation (a) takes—independent of the initial conditions—values of the minimal power orders t^m, with

$$(28) \qquad m = \max_{(\nu=1,\ldots\sigma)} (m_\nu).$$

Proof: Any arbitrary solution $x(t)$ of (a) can be written in the form (17). Here we must distinguish between two cases:

Case a: ${}^\nu d_{l_\nu} = 0$ in (17) for $\nu = \varrho + 1, \ldots, s$.

Then the 3rd bracket in (17) will vanish identically.

If it happens at the same time, that the 2nd bracket also vanishes, when ${}^\nu d_{l_\nu} = 0$ for $\nu = \sigma + 1, \ldots, \varrho$ (consequently there will not be smaller powers in it), then the solution $x(t)$ will be reduced only to the 1st bracket (I) in (17), which is the particular solution $x^*(t)$ of (a) (see (14)).

We shall prove that the factor of the highest power t^m is a non-identic vanishing periodic function of period p. Evidently the power t^m can be multiplied only by linear combinations of the periodic functions $\varphi_{(1)}, \varphi_{(2)}, \ldots, \varphi_{(\sigma)}$. Thus we need only to show, that no linear combination of the form $\sum_{\nu=1}^{\sigma} \lambda_\nu \varphi_{(\nu)}(t)$ can be identic zero. Otherwise the corresponding linear combination $\sum_{\nu=1}^{\sigma} \lambda_\nu y_{(\nu)}(t)$ will be identic zero, because the functions $\varphi_{(\nu)}(t)$ (for $\nu = 1, \ldots, \sigma$) are coincident with the functions $y_{(\nu)}(t)$ (see (1)). Since the functions $y_{(1)}, y_{(2)}, \ldots, y_{(\sigma)}$ are linear independent solutions of (b), then all the constants $\lambda_1, \lambda_2, \ldots, \lambda_\sigma$ must be equal to zero, which is impossible for (see (9))

$$\lambda_\nu = \frac{1}{pm_\nu!} = a_{[\nu]} \neq 0 \quad (\text{for } \nu = 1, \ldots, \sigma).$$

Thus $x(t)$ takes values of the power order t^m.

Now it can be assumed that for $\nu = \sigma + 1, \ldots, \varrho$ at least one of the constants ${}^\nu d_{l_\nu}$ (in (II) (see (17)) is different from zero. Let

$$\omega_{\mathrm{II}} = \max_{(\nu=\sigma+1,\ldots,\varrho)} (\omega_\nu). \tag{29}$$

Then the 2nd bracket (II) in (17) can be written in the form

$$(\mathrm{II}) = \frac{t^{\omega_{\mathrm{II}}}}{\omega_{\mathrm{II}}!} \sum_{\substack{\nu=\sigma+1 \\ (\omega_\nu=\omega_{\mathrm{II}})}}^{\varrho} {}^\nu d_{l_\nu} \varphi_{(\nu)} + \text{smaller powers}, \tag{30}$$

where all the constants ${}^\nu d_{l_\nu}$ do not vanish simultaneously.
Referring to (17), (30), we have three subcases:

a (1): If $\omega_{\mathrm{II}} < m$, then the 2nd bracket (II) in (17) will be included under the "smaller powers" in the bracket (I).

a (2): If $\omega_{\mathrm{II}} > m$, then the 1st bracket (I) will be included under the "smaller powers" in the bracket (II). Since the linear combination

$$\sum_{\substack{\nu=\sigma+1 \\ (\omega_\nu=\omega_{\mathrm{II}})}}^{\varrho} {}^\nu d_{l_\nu} \varphi_{(\nu)}(t) = \sum_{\substack{\nu=\varrho+1 \\ (\omega_\nu=\omega_{\mathrm{II}})}}^{\varrho} {}^\nu d_{l_\nu} y_{(\nu)} = \eta^{*(t)}$$

represents a non-trivial with p periodic solution of (b), then the sum (II) or (I) + (II), i.e. $x(t)$ takes values of the power order $t^{\omega_{\mathrm{II}}}$, thus at least values of the power order t^m.

a (3): If $\omega_{\text{II}} = m$, then we sum up the principal parts in both brackets (I) and (II) in the form

$$\frac{t^m}{m!}\left(\sum_{\substack{\nu=1 \\ (m_\nu = m)}}^{\sigma} \frac{\text{I}}{p} a_{[\nu]}\varphi_{(\nu)} + \sum_{\nu=\sigma+1}^{\varrho} {}^{\nu}d_{l_\nu}\varphi_{(\nu)} \right) = \frac{t^m}{m!}\eta^{**}(t),$$

where $\eta^{**}(t)$ represents again a non-trivial with p periodic solution of (b). (Notice that the functions $\varphi_{(\nu)}(t) = y_{(\nu)}(t)$ for $\nu = 1, \ldots, \sigma$ and $\nu = \sigma + 1, \ldots, \varrho$ are linear independent)

In all cases, we get for the union of (I) and (II) in (17) a representation of the form

$$x(t) = \frac{t^M}{M!}\, \eta(t) + \text{smaller powers,} \tag{31}$$

where

$$M = \max\,(m, \omega_{\text{II}}) \geqslant m \tag{32}$$

and $\eta(t)$ is a non-trivial with p periodic solution of (b) and $M \geqslant m$. Thus the theorem is proved in this case.

Case b: ${}^{\nu}d_{l_\nu} \neq 0$ in (17) for at least one index ν, $\varrho + 1 \leqq \nu \leqq s$. Consider now the total sum (III) in (19).

If all the elements of the sequence

$$(\text{III})_{\varrho_1},\ (\text{III})_{\varrho_2},\ \ldots,\ (\text{III})_{\varrho_{k-1}} \tag{33}$$

are not identic zero, then by means of theorem 2, the sum (III) takes with increasing t, values of the power order $e^{\beta t} \,.\, t^{\omega}$ with $\beta > 0$ and $\omega \geqslant 0$. Hence $x(t)$ takes at least values of the power order t^m and the theorem is proved.

But if all the elements in (33) are identic zeros, we sum up $(\text{III})_{\varrho_k}$, (I), (II), and we prove analogously, that $x(t)$ takes at least values of the power order t^m. Here it can be assumed that $(\text{III})_{\varrho_k}$ is not identic zero, otherwise the problem is reduced immediately to case a. Considering $(\text{III})_{\varrho_k}$ only, we obtain by virtue of theorem 2, a power order

$$t^{\omega(k)}\, e^{\beta_{\varrho_k} \cdot t}, \text{ but with } \beta_{\varrho_k} = 0.$$

Here we have three subcases:

b (1): In the case, that $\omega(k) > M$ with M from (32), $x(t)$ takes values of the power order $t^{\omega(k)}$ with $\omega(k) > M$.

b (2): In the case, that $\omega(k) < M$, we can negelect the values of (III) w.r.t. the value of (I) and (II), since (I) + (II) takes in all cases values of the power order t^M.

b (3): Only in the case that $\omega(k) = M$, a particular consideration is necessary.

We sum up all like terms from (I), (II) and (III) with the same power t^M then we obtain for $x(t)$ a representation of the form

$$x(t) = \frac{t^M}{M!}\left(\sum_{v=0}^{v=q} e^{i\bar{\gamma}_v \cdot t}\eta_v(t) + \text{smaller powers}\right),$$

where the exponents $\bar{\gamma}_v$ are pairwise distinct. Particularly $\bar{\gamma}_0$ is equal to zero (this corresponds to $\alpha = 0$ in (I) and (II)). The periodic functions $\eta_1(t), \eta_2(t), \ldots, \eta_q(t)$ are defined similarly as in (27). $\eta_0(t)$ is also a non-identic with p periodic function, since in (I) there is at least one term whose factor is the power t^m. Thus in the union of (I) and (II), there is a term whose factor is the power t^M with $M \geqslant m$. Further the highest power factors are always multiplied by the functions $\varphi_{(\nu)}(t) = y_{(\nu)}(t)$. Using lemma 1, it follows also in this case, that, $x(t)$ takes at least values of the power order t^m. Thus the theorem is completely proved.

§ 6. THE POWER ORDER OF THE DERIVATIVES OF ANY ARBITRARY SOLUTION OF (a)

In this paragraph, we study the power order of the first $(n-1)$ derivatives of any arbitrary solution of (a) under the essential condition (12).

Theorem 4: The derivatives $x^{(r)}(t)$ $(r = 1, 2, \ldots, n-1)$ of any arbitrary solution $x(t)$ of (a) take under the condition (12), the same power order as well as $x(t)$.

Proof: We consider the previous two cases, which are stated in theorem 3.

In case 1, we differentiate (31) successively:

$$\begin{cases} x'(t) &= \dfrac{t^M}{M!} \cdot \eta'(t) + \text{smaller powers} \\ x''(t) &= \dfrac{t^M}{M!} \cdot \eta''(t) + \text{smaller powers} \\ \vdots & \\ x^{(r)}(t) &= \dfrac{t^M}{M!} \cdot \eta^{(r)}(t) + \text{smaller powers} \quad (1 \leqq r \leqq n-1) \\ \vdots & \\ x^{(n-1)}(t) &= \dfrac{t^M}{M!} \cdot \eta^{(n-1)}(t) + \text{smaller powers}. \end{cases}$$

Then all derivatives $\eta^{(r)}(t)$ $(1 \leqq r \leqq n-1)$ must be non-identic vanishing with p periodic functions. Otherwise $\eta(t)$ will be a polynomial in t of degree $r-1$. But since the solutions $y_{(\nu)}(t)$ of (b) are periodic functions of period p, then $\eta(t)$ is reduced to a constant, which leads to a contradiction with the condition (12). Hence all derivatives $x^{(r)}(t)$ $(1 \leqq r \leqq \leqq n - 1)$ take values of the power order t^M.

In case 2, differentiating (26), we obtain for $\lambda = 1, 2, \ldots, k$ the following formulas successively (notice also (27)):

$$
\begin{cases}
(\mathrm{III})_{\varrho_\lambda} = \dfrac{t^{\omega(\lambda)}}{(\omega(\lambda))!} \sum\limits_{v=1}^{q} e^{(\beta_{\varrho_\lambda} + i\bar{\gamma}_v).t} \,.\, \eta_v(t) + \text{smaller powers}, \\
(\mathrm{III})'_{\varrho\lambda} = \dfrac{t^{\omega(\lambda)}}{(\omega(\lambda))!} \sum\limits_{v=1}^{q} e^{(\beta_{\varrho_\gamma} + i\bar{\gamma}_v)t} \,.\, (D + \beta_{\varrho\lambda} + i\bar{\gamma}_v)\, \eta_v(t) + \\
\qquad\qquad + \text{smaller powers}, \\
(\mathrm{III})''_{\varrho_\lambda} = \dfrac{t^{\omega(\lambda)}}{(\omega(\lambda))!} \sum\limits_{v=1}^{q} e^{(\beta_{\varrho\lambda} + i\bar{\gamma}_v)t} \,.\, (D + \beta_{\varrho_\lambda} + i\bar{\gamma}_v)^2\, \eta_v(t) + \\
\qquad\qquad + \text{smaller powers}, \\
\vdots \\
(\mathrm{III})^{(n-1)}_{\varrho_\lambda} = \dfrac{t^{\omega(\lambda)}}{(\omega(\lambda))!} \sum\limits_{v=1}^{q} e^{(\beta_{\varrho\lambda} + i\bar{\gamma}_v)t} \,.\, (D + \beta_{\varrho_\lambda} + i\bar{\gamma}_v)^{n-1} \eta_v(t) + \\
\qquad\qquad + \text{smaller powers}.
\end{cases}
$$

By virtue of (2), it follows from lemma 2, that the differential expression $(D + \beta_{\varrho_\lambda} + i\, \bar{\gamma}_v)^r\, \eta_v(t)$ is not identic zero for every $r = 1, 2, \ldots, n - 1$ and every $v = 1, 2, \ldots, q$. Then the first $(n - 1)$ dervatives of $(\mathrm{III})_{\varrho_\lambda}$ have the same power order as $(\mathrm{III})_{\varrho_\lambda}$. Analogously, it can be shown by using lemma 1, as in theorem 3, that all derivatives $x'(t)$, $x''(t)$, $\ldots$, $x^{(n-1)}(t)$ take the same power order as well as $x(t)$.

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