Archivum Mathematicum

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*Archivum Mathematicum*, Vol. 6 (1970), No. 4, 221--228

Persistent URL: http://dml.cz/dmlcz/104727

# DOUBLE LEXICOGRAPHIC PRODUCTS

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(Received April 15, 1968)

## INTRODUCTION

Let $Q$ be a nonempty set with a binary reflexive relation $\leqq$ and let $N_q$ be nonempty sets with binary relations $\tilde{N}_q (q \in Q)$. The lexicographic product is the Cartesian product $N = \underset{q \in Q}{\times} N_q$ with the relation $\tilde{N}$ defined as follows: Let $n^i = (n_q^i)_{q \in Q} \in N (i = 1, 2)$ . $(n^1, n^2) \in \tilde{N}$ iff

$$\underset{q}{\forall} \left[ (n_q^1 \neq n_q^2) \Rightarrow \underset{u}{\exists} \left[ (u \leqq q) \ \& \ (n_u^1 \neq n_u^2) \ \& \ \left( (n_u^1, n_u^2) \in \tilde{N}_u \right) \right] \right] \tag{1}$$

where $q$ and u range over $Q$.

Now let (I, $\leqq$) and (K, $\leqq$) be nonempty sets with binary reflexive relations and let $Z_{ik}$ be nonempty sets with binary relations $\tilde{Z}_{ik} (i \in I, k \in K)$. We can form the double Cartesian products

$$X = \underset{i \in I}{\times} (\underset{k \in K}{\times} Z_{ik}), \qquad Y = \underset{k \in K}{\times} (\underset{i \in I}{\times} Z_{ik}) \tag{2}$$

and the Cartesian product

$$W = \underset{(i,k) \in H}{\times} Z_{ik}, \tag{3}$$

where $H = I \times K$. In an obvious way we can identify the sets $X$, $Y$, and $Z$ (namely by means of the mappings which for arbitrary $z_{ik} \in Z_{ik}$ carry $\left((z_{ik})_{k \in K}\right)_{i \in I}$ onto $\left((z_{ik})_{i \in I}\right)_{k \in K}$ and onto $\left((z_{ik})_{(i,\ k) \in H}\right)$. The purpose of this note is to study the inclusion relations between the corresponding $\tilde{X}$, $\tilde{Y}$, and $\tilde{W}$. In this connection the fact that $\tilde{Z}_{ik}$ are binary relations has no impact. For the case of general relations we must replace the inequality relation $\neq$ in (1) by some general relations on $N_q$ which will be denoted by $N_q^{\#}$. In such a way we obtain a direct generalization of the lexicographic product.

## 1.

Let $A$, $M$ be nonempty sets and let $M^A$ denote the set of all mappings from $A$ to M. The elements of $M^A$ will be denoted by $(m^a)_a$. Any subset $\varrho$ of $M^A$ will be called an $A$-relation on $M$ (if card $A = h < \aleph_0$ we agree to identify the $A$-relations on $M$ with the $h$-ary relations on $M$, i.e. with the subsets of $M^h = M \times \ldots \times M$). The complement $\varrho'$ of an $A$-relation $\varrho$ is the $A$-relation $M^A \setminus \varrho$. For any $m \in M$ let $\bar{m} = (m^a)_a$, where $m^a = m$ for every $a \in A$. Further the $A$-relation $M^=$ is defined by $M^= = \{\bar{m} | m \in M\}$. Let $\tilde{M}$ and $M^\#$ be two $A$-relations on $M$. For the sake of brevity we set $\tilde{M}^\# = \tilde{M} \cap M^\#$, $\tilde{M}'^\# = (\tilde{M}')^\# = \tilde{M}' \cap M^\#$, and $\tilde{M}^{\#\prime} = (\tilde{M}^\#)' = (\tilde{M} \cap M^\#)' = \tilde{M}' \cup M^{\#\prime}$.

**1.1. Definition.** *Let $Q$ be a nonempty set with a binary reflexive relation $\leqq$. Let $A$ be a nonempty set and let $N_q$ be nonempty sets with two $A$-relations $\tilde{N}_q$ and $N_q^\#$ such that $N_q^\# \cap N_q^= = \emptyset$. The l-product of $N_q$ over $Q$ is the Cartesian product $N = \bigtimes_{q \in Q} N_q$ with two $A$-relations $\tilde{N}$ and $N^\#$ defined as follows: Let $n^a = (n_q^a)_{q \in Q} \in N (a \in A)$.*

$$(n^a)_a \in \tilde{N} \text{ iff } \forall_q \big[\big((n_q^a)_a \in N_q^\#\big) \Rightarrow \exists_u [(u \leqq q) \ \& \ \big((n_u^a)_a \in \tilde{N}_u^\#\big)]\big], \tag{4}$$

$$(n^a)_a \in N^\# \text{ iff } \exists_q [(n_q^a)_a \in N_q^\#], \tag{5}$$

*where $q$ and $u$ range over $Q$.*

Obviously $N^\# \cap N^= = \emptyset$. It is a simple matter to check that if card $A = 2$ and $(a, b) \in N_q^\#$ iff $a \neq b$, then (4) is (1) and (5) means $n^1 \neq n^2$.

If $(n^a)_a \in N^{\#\prime}$, then by (5) $(n_q^a)_a \in N_q^{\#\prime}$ for every $q \in Q$ and the right hand side of (4) is vacuous, hence we have

**1.2. Lemma.** $N^= \subseteq N^{\#\prime} \subseteq \tilde{N}$.

Let $n^a = (n_q^a)_{q \in Q} \in N (a \in A)$. The following assertions, that may be verified directly, will be helpful in the sequel.

**1.3. Lemma.**

$$(n^a)_a \in \tilde{N}' \text{ iff } \exists_q \big[\big((n_q^a)_a \in \tilde{N}'^\#_q\big) \ \& \ \forall_u \big[(u < q) \Rightarrow \big((n_u^a)_a \in \tilde{N}_u^{\#\prime}\big)\big]\big]. \tag{6}$$

**1.4. Lemma.** *Let $(n^a)_a \in \tilde{N}'$ and let $q \in Q$ such that $(n_q^a)_a \in \tilde{N}_q^\#$. Then there exists $u \in Q$ such that $u \ngeqq q$, $(n_u^a)_a \in N'^\#_u$, and $(n_v^a)_a \in \tilde{N}_q^{\#\prime}$ for any $v \in Q$, $v < u$.*

**1.5. Lemma.** *If $(n^a)_a \in \tilde{N}^\#$ then there exists $q \in Q$ such that $(n_q^a)_a \in \tilde{N}_q^\#$.*

**1.6. Lemma.** *If $(n_q^a)_a \in \tilde{N}'_q \cup N_q^{\#\prime}$ for every $q \in Q$, and $(n^a)_a \in N^\#$ then $(n^a)_a \in \tilde{N}'$.*

## 2.

The purpose of this paper is to study the inclusions between $\tilde{X}$, $\tilde{Y}$, and $\tilde{W}$, where $X$, $Y$, and $W$ are defined in (2) and (3), $Z_{ik}$ are nonempty sets with two $A$-relations $\tilde{Z}_{ik}$ and $Z_{ik}^\#$ (where $Z_{ik}^\# \cap Z_{ik}^= = \emptyset$), and the binary relation on $H = I \times K$ is the relation of the cardinal product (i.e. $(i, k) \leqq (i^*, k^*)$ iff $i \leqq i^*$ and $k \leqq k^*$). For further convenience we put

$$X_i = \underset{k \in K}{\times} Z_{ik}, \qquad Y_k = \underset{i \in I}{\times} Z_{ik} \qquad (i \in I,\ k \in K). \tag{7}$$

We shall denote the elements of $X_i$, $Y_k$, $X$, and $Y$ by $(z_{ik})_k$, $(z_{ik})_i$, $\big((z_{ik})_k\big)_i$, and $\big((z_{ik})_i\big)_k$, respectively. As in the introduction we identify $X$, $Y$, and $W$. In this section we shall describe the inclusion $\tilde{Y} \leqq \tilde{W}$.

**2.1. Lemma.** *If $\tilde{Y} \subseteq \tilde{X}$ or $\tilde{Y} \subseteq \tilde{W}$ then the following condition* (a) *is satisfied:* (a) *If $k' < k''$, $\tilde{Z}^{\#}_{i'k'} \neq \emptyset$, and $\tilde{Z}'^{\#}_{i''k''} \neq \emptyset$, then $i' \leqq i''$ $(i', i'' \in I,\ k', k'' \in K)$.*

Proof: Let $i', i'' \in I$; $k', k'' \in K$, $k' < k''$, and

$$(z^a_{i'k'})_a \in \tilde{Z}^{\#}_{i'k'}; \qquad (z^a_{i''k''})_a \in \tilde{Z}'^{\#}_{i''k''}. \tag{8}$$

For any $(i, k) \in H \setminus \{(i', k'), (i'', k'')\}$ we choose an arbitrary $z_{ik} \in Z_{ik}$ and we put $z^a_{ik} = z_{ik}$ for every $a \in A$. It is not difficult to check that in view of $k' < k''$, (8), the definition of $z^a_{ik}$, and 1. 2 we have $\Big(\big((z^a_{ik})_i\big)_k\Big)_a \in \tilde{Y}$. If $\tilde{Y} \subseteq \tilde{X}$, then $\Big(\big((z^a_{ik})_k\big)_i\Big)_a \in X$. But $\big((z^a_{i''k})_k\big)_a \in \tilde{X}'^{\#}_{i''}$, $\big((z^a_{i'k})_k\big)_a \in \tilde{X}^{\#}_{i'}$, and $\big((z^a_{jk})_k\big)_a \in X^{=}_j$ for any $j \in I$, $j \neq i', i''$. Thus it follows that $i' \leqq i''$. If $\tilde{Y} \subseteq \tilde{W}$ the proof is similar.

**2.2. Proposition.** *$\tilde{Y} \subseteq \tilde{W}$ iff* (a) *holds.*

Proof: In view of 2.1 it is sufficient to prove that (a) implies $\tilde{Y} \subseteq \tilde{W}$. To prove this let $\Big(\big((z^a_{ik})_i\big)_k\Big)_a \in \tilde{Y}$ and assume that there exists $(i'', k'') \in H$ such that $(z^a_{i''k''})_a \in \tilde{Z}'^{\#}_{i''k''}$. Then $\big((z^a_{ik''})_i\big)_a \in Y^{\#}_{k''}$ and therefore there exists $k' \in K$, $k' \leqq k''$ such that $\big((z^a_{ik'})_i\big)_a \in \tilde{Y}^{\#}_{k'}$. We must distinguish two cases: 1. Let $k' = k''$. Then from $(z^a_{i''k''})_a \in Z^{\#}_{i''k''}$ and from $\big((z^a_{ik''})_i\big)_a \in \tilde{Y}^{\#}_{k''}$ it follows that there is $i' \in I$, $i' \leqq i''$ such that $(z^a_{i'k'})_a \in \tilde{Z}^{\#}_{i'k'}$. 2. Let $k' < k''$. In view of $\big((z^a_{ik'})_i\big)_a \in \tilde{Y}^{\#}_{k'}$ there exists $i' \in I$ such that $(z^a_{i'k'})_a \in \tilde{Z}^{\#}_{i'k'}$ (see 1.5) and using (a) we see that $i' \leqq i''$. In both cases $i' \leqq i''$, $k' \leqq k''$ and $(z^a_{i'k'})_a \in \tilde{Z}^{\#}_{i'k'}$, and this shows that $\tilde{Y} \subseteq \tilde{W}$.

In order to describe $\tilde{W} \subseteq \tilde{X}$ and $\tilde{Y} \subseteq \tilde{X}$ we need the following definition.

**3.1.** Definition. *The condition* (*b*) *is satisfied if there are no finite sequences*
*$i_0 > i_1 > \ldots > i_{\alpha-1}$ in $I$ with $i_0 \geqslant i_l (l = 1, 2, \ldots, \alpha - 2)$ and $i_0 \ngeqq i_{\alpha-1}$ $(\alpha > 2)$*

$$k^*_0 \geqslant k_1 \nleqq k^*_1 \geqslant k_2 \nleqq k^*_2 \geqslant \ldots \nleqq k^*_{\alpha-2} \geqslant k_{\alpha-1} \text{ in } K$$

*or infinite sequences*

$$i_0 > i_1 > \ldots \text{ in } I \text{ and } k^*_0 \geqslant k_1 \nleqq k^*_1 \geqslant k_2 \nleqq k^*_2 \geqslant \ldots \text{ in } K$$

*such that*

$$1.\ \tilde{Z}^{\#}_{i_m k_m} \neq \emptyset; \qquad \tilde{Z}'^{\#}_{i_n k_n^*} \neq \emptyset$$

$$2.\ i_n = i_m \Rightarrow k_n^* \ngeq k_m$$

*for every $m$, $n$ such that $i_n$, $k_m$, $i_m$, $k_n^*$ belong to the sequences.*

*We say that the condition (c) is satisfied if there do not exist such sequences satisfying 1 and 2 and the additional condition*

$$3.\ k_{n-1}^* = k_m \Rightarrow k_n = k_{n-1}^*$$

*for every $m$ and $n$ such that $k_{n-1}^*$ and $k_m$ belong to the sequence.*

Note that obviously (b) $\Rightarrow$ (c).

**3.2. Lemma.** (a) & (c) $\Rightarrow \tilde{Y} \subseteq \tilde{X}$.

Proof: Suppose that (a) holds and $\tilde{Y} \nsubseteq \tilde{X}$. Then there exist $z_{ik}^a \in \in Z_{ik} (i \in I,\ k \in K,\ a \in A)$ such that

$$\Big(\big(((z_{ik}^a)_i\big)_k\Big)_a \in \tilde{Y}, \qquad \Big(\big(((z_{ik}^a)_k\big)_i\Big)_a \in \tilde{X}'. \tag{9}$$

We shall proceed by induction. The induction hypothesis is $V_p (p \geqslant 1)$: There exist $i_0 > i_1 > \ldots > i_{p-1}$ in $I$ and $k_0^* \geqslant k_1 \leqq\!\!\!\!\!/\; k_1^* \geqslant \ldots \geqslant \geqslant k_{p-1} \leqq\!\!\!\!\!/\; k_{p-1}^*$ in $K$ such that

I. The condition 2 in 3.1 holds for any $0 < m < p$ and $0 \leqq n < p$.

II. For every $0 \leqq j < p$ and any $r \in K$, $r < k_j^*$

$$\big(z_{i_j k_j^*}^a\big)_a \in \tilde{Z}'^{\#}_{i_j k_j^*}; \qquad \big(z_{i_j r}^a\big)_a \in \tilde{Z}^{\#'}_{i_j r}. \tag{10}$$

III. For every $0 < l < p$

$$\big(z_{i_l k_l}^a\big)_a \in \tilde{Z}^{\#}_{i_l k_l}; \qquad \big((z_{i k_l}^a)_i\big)_a \in \tilde{Y}^{\#}_{k_l}, \tag{11}$$

$$\big((z_{i k_{l-1}^*}^a)_i\big)_a \in \tilde{Y}^{\#}_{k_{l-1}^*} \Rightarrow k_l = k_{l-1}^*. \tag{12}$$

We shall prove first that $V_1$ holds. In virtue of (9) and (6) there exists $i_0 \in I$ such that

$$\big((z_{i_0 k}^a)_k\big)_a \in \tilde{X}'^{\#}_{i_0}; \qquad \big((z_{sk}^a)_k\big)_a \in \tilde{X}^{\#'}_{s} \tag{13}$$

for every $s \in I$, $s < i_0$. From (13) and (6) it follows again that there exists $k_0^* \in K$ such that

$$\big(z_{i_0 k_0^*}^a\big)_a \in \tilde{Z}'^{\#}_{i_0 k_0^*}; \qquad \big(z_{i_0 t}^a\big)_a \in \tilde{Z}^{\#'}_{i_0 t} \tag{14}$$

for any $t \in K$, $t < k_0^*$. It is easy to check that I-III hold for $p = 1$.

Suppose now that $V_p$ holds. For $j = p - 1$ (10) yields $\big(z_{i_{p-1} k_{p-1}^*}^a\big)_a \in \in \tilde{Z}'^{\#}_{i_{p-1} k_{p-1}^*}$. We shall consider two cases: Case A. Let

$$\big((z_{i k_{p-1}^*}^a)_i\big)_a \in \tilde{Y}^{\#}_{k_{p-1}^*}. \tag{15}$$

In this case we set $k_p = k^*_{p-1}$ and $(z^a_{i_{p-1}k^*_{p-1}})_a \in \tilde{Z}'^{\#}_{i_{p-1}k^*_{p-1}}$ together with (15) shows that there exists $i_p \in I$, $i_p < i_{p-1}$ such that

$$(16) \qquad (z^a_{i_pk_p})_a \in \tilde{Z}^{\#}_{i_pk_p}.$$

Case B. Assume that (15) does not hold. Then from $(z^a_{i_{p-1}k^*_{p-1}})_a \in Z^{\#}_{i_{p-1}k^*_{p-1}}$ it follows that $((z^a_{ik^*_{p-1}})_i)_a \in \overset{\#}{Y}_{k^*_{p-1}}$ and 1.2 and (9) show that there exists $k_p \in K$, $k_p < k^*_{p-1}$ such that

$$(17) \qquad ((z^a_{ik_p})_i)_a \in \tilde{Y}^{\#}_{k_p}.$$

From 1.5 it follows that there is $i_p \in I$ such that (16) holds. From (16), $(z^a_{i_{p-1}k^*_{p-1}})_a \in \tilde{Z}'^{\#}_{i_{p-1}k^*_{p-1}}$, $k_p < k^*_{p-1}$ and (a) we obtain $i_p \lesseqq i_{p-1}$. But $i_p \neq i_{p-1}$, since otherwise for $r = k_p < k^*_{p-1}$ (16) would contradict (10). Thus in both cases $i_p < i_{p-1}$ and (16) holds.

As the relation $\lesseqq$ in $I$ generally is not transitive, we must consider two cases:

a) Let $i_0 \ngeqq i_p$. Then in view of $i_{p-1} > i_p$ necessarily $p - 1 > 0$. We set $\alpha = p + 1$.

Using the hypothesis and (16) we can easily check that the sequences $i_0 > i_1 > \ldots > i_p$ and $k^*_0 \geqslant k_1 \nleqq k^*_1 \geqslant \ldots \geqslant k_p$ satisfy 1 in **3.1**. Now by hypothesis, 2 holds for any $0 < m < p$ and $0 \leqq n < p$. As $i_0 \ngeqq i_p$ and $i_0 \geqslant i_n$ we see that $i_n \neq i_p$ for any $0 \leqq n < p$ so that 2 is satisfied. Further (11) in the hypothesis, (15) in case A or (17) in case B show that (11) holds for any $0 < l < p + 1$. To prove **3** assume that $0 < m$, $n < p + 1$ and $k^*_{n-1} = k_m$. Then by (11) $((z^a_{ik^*_{n-1}})_i)_a \in \tilde{Y}^{\#}_{k^*_{n-1}}$. If $n < p$, then from (12) it follows $k_n = k^*_{n-1}$. On the other hand if $n = p$, then we have the case A and therefore again $k_p = k^*_{p-1}$. Thus **3** holds and we have found two sequences $i_0 > i_1 > \ldots > i_p$ with $i_0 \ngeqq i_p$ and $k^*_0 \geqslant k_1 \nleqq k^*_1 \geqslant \ldots \geqslant k_p$ that 1—**3** in **3.1** hold so that (c) is not satisfied.
b) Let $i_0 \geqslant i_p$. From (16), (13) (for $s = i_p$), and 1.4 we derive that there exists $k^*_p \in K$, $k^*_p \ngeqq k_p$ such that

$$(18) \qquad (z^a_{i_pk^*_p})_a \in \tilde{Z}'^{\#}_{i_pk^*_p}; \qquad (z^a_{i_pv})_a \in \tilde{Z}^{\#\prime}_{i_pv}$$

for any $v \in K$, $v < k^*_p$. Referring to (18), and $V_p$ we see that II holds for any $0 < j < p + 1$. **By** assumption 2 holds for any $0 < m < p$ and $0 \leqq n < p$. Let $i_n = i_p$. Using (16) we have $(z^a_{i_nk_p})_a = (z^a_{i_pk_p})_a \in \tilde{Z}^{\#}_{i_pk_p} = \tilde{Z}^{\#}_{i_nk_p}$; hence from (10) (for $j = n$ and $r = k_p$) it follows $k^*_n \ngeqq k_p$. Similarly, let $i_p = i_m$. By (11) $(z^a_{i_pk_m})_a = (z^a_{i_mk_m})_a \in \tilde{Z}^{\#}_{i_mk_m} = \tilde{Z}^{\#}_{i_pk_m}$ and from (18) (for $v = k_m$) we deduce that again $k^*_p \ngeqq k_m$. Thus 2 holds for $0 < m < p + 1$ and $0 \leqq n < p + 1$, i.e. I holds.

It remains to prove III for $l = p$. But when $l = p$, then (11) becomes

(16) and (15) or (17) so that (11) holds for $0 < l < p + 1$. Similarly, if $l = p$ in (12), we have $((z^a_{ik^*_{p-1}})_i)_a \in \tilde{Y}^{\#}_{k^*_{p-1}}$. Thus (15) holds, i.e. we have the case A and $k_p = k^*_{p-1}$ as required. This completes the induction step.

Thus we see that there are two possible cases. In the first case the condition (c) is not satisfied because there are finite sequences with the properties referred to in 3.1. In the contrary case $V_p$ holds for any $p \geqslant 1$. It is easy to check that the infinite sequences obtained in such a manner satisfy 1—3 in 3.1. Thus (c) does not hold and the proof is completed.

**3.3. Lemma.** (b) $\Rightarrow \tilde{W} \subseteq \tilde{X}$.

The proof is similar to the proof of 3.2 (the part III is omitted in $V_p$ and it is not necessary to consider the two cases A and B).

## 4.

**4.1. Lemma.** $\tilde{Y} \subseteq \tilde{X} \Rightarrow$ (a) & (c).

Proof: From 2.1 we know that $\tilde{Y} \subseteq \tilde{X} \Rightarrow$ (a). Suppose that (c) does not hold. Then there exists an ordinal $2 < \alpha \leqq \omega$ (where $\omega$ is the least infinite ordinal), $i_m \in I (0 \leqq m < \alpha)$, $k_n \in K$, and $k^*_{n-1} \in K (0 < n < \alpha)$ such that the conditions in 3.1 are satisfied. Referring to 2 we see that $(i_m, k_m) \neq (i_{n-1}, k^*_{n-1})$ for any $0 < m, n < \alpha$. From 1 it follows that for every $0 < n < \alpha$ there exist

$$(19) \qquad (z^a_{i_n k_n})_a \in \tilde{Z}^{\#}_{i_n k_n}; \qquad (z^a_{i_{n-1} k^*_{n-1}})_a \in \tilde{Z}'^{\#}_{i_{n-1} k^*_{n-1}}.$$

If $(i, k) \in H$, $(i, k) \neq (i_n, k_n)$ and $(i, k) \neq (i_{n-1}, k^*_{n-1})$ $(0 < n < \alpha)$ we choose an arbitrary $z_{ik} \in Z_{ik}$ and put $z^a_{ik} = z_{ik}$ for any $a \in A$.

To prove that $(((z^a_{ik})_i)_k)_a \in \tilde{Y}$, let $l \in K$ and let $((z^a_{il})_i)_a \in Y^{\#}_l$. Then there exists $0 < q < \alpha$ such that $k_q \leqq l$. For, if $l \neq k_m$ for every $0 < m < \alpha$, then in view of $((z^a_{il})_i)_a \in Y^{\#}_l$ and of the definition of $z^a_{ik}$ there exists $0 < q < \alpha$ such that $l = k^*_{q-1}$ and $k_q \leqq k^*_{q-1} = l$. We shall prove that $((z^a_{ik_q})_i)_a \in \tilde{Y}^{\#}_{k_q}$. Assume that $(z^a_{jk_q})_a \in \tilde{Z}'^{\#}_{jk_q}$. Then obviously there exists $0 < r < \alpha$ such that $j = i_{r-1}$, $k_q = k^*_{r-1}$. Using 3 we have $k_r = k^*_{r-1} = k_q$ and $i_r < i_{r-1}$. Since $(z^a_{i_r k_q})_a = (z^a_{i_r k_r})_a \in \tilde{Z}^{\#}_{i_r k_r} = \tilde{Z}^{\#}_{i_r k_q}$, this proves that $((z^a_{ik_q})_i)_a \in \tilde{Y}^{\#}_{k_q}$ and this shows that $(((z^a_{ik})_i)_k)_a \in \tilde{Y}^{\#}$.

It remains to prove that $(((z^a_{ik})_k)_i)_a \in \tilde{X}'$. Let $0 < n < \alpha$. By (19) $(z^a_{i_{n-1} k^*_{n-1}})_a \in \tilde{Z}'^{\#}_{i_{n-1} k^*_{n-1}}$ and in view of 2 in 3.1 there is no $0 < m < \alpha$ such that $i_{n-1} = i_m$ and $k^*_{n-1} \geqslant k_m$. Thus from the definiton of $z^a_{ik}$ we see that $((z^a_{i_{n-1}k})_k)_a \in \tilde{X}'^{\#}_{i_{n-1}}$. From the definition of $z^a_{ik}$ it follows that for every $j \in I$ such that $j \neq i_m$ for all $0 < m < \alpha$ we have $((z^a_{jk})_k)_a \in X^{\#'}_j$ and therefore by 1.6 $(((z^a_{ik})_k)_i)_a \in \tilde{X}'$. Thus we have proved that $\tilde{Y} \nsubseteq \tilde{X}$ and this completes the proof.

**4.2. Lemma.** $\tilde{W} \subseteq \tilde{X} \Rightarrow$ (b)
The proof is similar to the proof of 4.1.

## 5.

Now we can sumarize our results.

**5.1. Definition.** *$(\bar{a})$, $(\bar{b})$, and $(\bar{c})$ will denote the conditions which are obtained from $(a)$, $(b)$, and $(c)$, respectively, when the roles of $I$ and $K$ are interchanged.*

**5.2. Theorem.** *The following assertions hold:*

1) $\tilde{Y} \subseteq \tilde{W} \Leftrightarrow$ (a); $\quad \tilde{X} \subseteq \tilde{W} \Leftrightarrow (\bar{a})$;
2) $\tilde{W} \subseteq \tilde{X} \Leftrightarrow (b)$; $\quad \tilde{W} \subseteq \tilde{Y} \Leftrightarrow (\bar{b})$;
3) $\tilde{Y} \subseteq \tilde{X} \Leftrightarrow$ (a) & (c); $\quad \tilde{X} \subseteq \tilde{Y} \Leftrightarrow (\bar{a})$ & $(\bar{c})$.

Proof: We have to prove only the first assertions (because the second assertions are obtained from the first by interchanging the roles of I and K). But the first assertions have been established in 2.2, 3.3 and 4.2, 3.2, and 4.1.

Combining 1,2, and 3 we obtain

**5.3. Theorem.** *The following assertions hold:*

4) $\tilde{X} = \tilde{W} \Leftrightarrow (\bar{a})$ & (b); $\quad \tilde{Y} = \tilde{W} \Leftrightarrow$ (a) & $(\bar{b})$;
5) $\tilde{X} = \tilde{Y} \Leftrightarrow (a)$ & $(\bar{a})$ & $(c)$ & $(\bar{c})$.

**5.4. Remark.** If the relation $\leqq$ on $I$ is transitive, then $i_0 \ngeq i_{\alpha-1}$ in 3.1 is satisfied for no finite $\alpha$, thus the sequences in 3.1 are infinite. Moreover, if there is no infinite descending chain in I, then (b) (and therefore (c)) holds trivially.

We shall consider a special case. Let $Z_{ik}^{\#} = \{(x, y) \mid x, y \in Z_{ik}, x \neq y\}$. When the relations $\tilde{Z}_{ik}$ are binary, antisymmetric, and reflexive relations and card $Z_{ik} > 1$ for every $(i, k) \in H$, then $\tilde{Z}_{ik}'^{\#} \neq \emptyset$. First we need the following definition.

**5.5. Definition.** *Let $Q$ be a set with a reflexive binary relation $\leq$. We say that $q \in Q$ is a maximal (least) element of $Q$ if $q^* \ngtr q (q^* \ngeq q)$ for any $q^* \in Q$.*

**5.6. Proposition.** *Let* card $I > 1$, card $K > 1$, *and let $\tilde{Z}_{ik}'^{\#} \neq \emptyset$ for any $(i, k) \in H$. Then the following conditions are equivalent:*

(A) $\quad \tilde{X} \subseteq \tilde{W}$ & $\tilde{Y} \subseteq \tilde{W}$;
(B) $\quad \tilde{X} = \tilde{W} = \tilde{Y}$;

(C) *If $\tilde{Z}_{ik}^{\#} \neq \emptyset$ then $i$ and $k$ are both maximal or both least elements.*

Proof: (A) $\Rightarrow$ (C). The conditions $(a)$ and $(\bar{a})$ are satisfied on basis of 5.2. Suppose $\tilde{Z}_{ik}^{\#} \neq \emptyset$ and let $k$ be an element of $K$ which is not maximal. Then $k < k^*$ for some $k^* \in K$. Making use of $(a)$ and $\tilde{Z}_{i^*k^*}'^{\#} \neq \emptyset$

we see that $i \leqq i^*$ for any $i^* \in I$. Thus $k$ is maximal or $i$ is least. By symmetry, $k$ is least or $i$ is maximal. But card $I > 1$ and card $K > 1$, thus $k$ cannot be simultaneously maximal and least and $i$ cannot be simultaneously maximal and least so that (C) holds.

(C) $\Rightarrow$ (B). It is easily checked that (C) implies ($a$) and ($\bar{a}$). Assume that ($b$) does not hold. Then there are finite or infinite sequences $i_0 > i_1 > \ldots$ and $k_0^* \geqslant k_1 \nleqq k_1^* \geqslant \ldots$ satisfying 1—3 in 3.1. From (C), $\tilde{Z}^{\#}_{i_1 k_1} \neq \emptyset$ and $i_0 > i_1$ it follows that both $i_1$ and $k_1$ are least elements; but this contradicts $k_1^* \ngeqq k_1$. Thus (b) holds and, by symmetry, ($\bar{b}$) holds as well. Now from 5.3 it results that $\tilde{X} = \tilde{W} = \tilde{Y}$.

(B) $\Rightarrow$ (A) is trivial.

The following corollary gives the conditions for the equality of double lexicographic products of partially ordered sets.

**5.7. Corollary.** *Let* card $I > 1$, card $K > 1$, *and* card $Z_{ik} > 1$ $(i \in I, k \in K)$. *Let* $\tilde{Z}_{ik}$ *be partial orderings* $(i \in I, k \in K)$. *Then for the lexicographic products the conditions* (A), (B) *are equivalent to the following condition* (D).

(D) *If* $\tilde{Z}_{ik}$ *is not an antichain, then i and k are both maximal or both least elements.*

I would like to thank the reviewer doc. Dr. V. Novák for his most valuable remarks.

## BIBLIOGRAPHY

[1] Birkhoff G., *Lattice Theory*, New York 1948.
[2] Day M. M., *Arithmetic of ordered systems*, Trans. Amer. Math. Soc. 58 (1945), 1—43.
[3] Novák V., *On the lexicographic product of ordered sets*, Czechoslovak Math. Journal 15 (90) (1965), p. 270—282.
[4] Rosenberg I., *Die transitive Hülle des lexikographischen Produktes*, Matematický časopis 17 (1967) No 2, p. 92—107.
[5] Sekanina M., *On a Power of ordered Sets*. Archivum Math., 1 (1965), 75—82.

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