SIMULTANEOUS NONDETERMINISTIC GAMES (II)*)

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§ 4. BASIC "GLOBAL" NOTIONS

a) The operations $^{-}$, $'$, $^{\sim}$, $+$, $-$ for collections

1. In this part a Q will be a fixedly chosen set, $Y \subset Q$ its subset, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ will be (Q)-collections, $(\mathfrak{A}_j)_{j \in J}$ will be a system of (Q)-collections.

2. On the set exp exp Q (of all (Q)-collections) we define three important unary operations $^{-}$, $'$, $^{\sim}$ in the following way:

$$\overline{\mathfrak{A}} := \{B \mid B \subset Q, \quad Q - B \in (\exp Q) - \mathfrak{A}\},$$

$$\mathfrak{A}' := (\overline{\mathfrak{A}})^{\#},$$

$$\tilde{\mathfrak{A}} := \overline{\mathfrak{A}'}.$$

There holds a trivial

3.1. Lemma. *For arbitrary (Q)-collections $\mathfrak{A}_1$, $\mathfrak{A}_2$ the following assertions are equivalent:*

(A)
$$A_1 \cup A_2 = Q \wedge A_1 \cap A_2 = \emptyset \Rightarrow$$
$$\Rightarrow (A_1 \in \mathfrak{A}_1 \wedge A_2 \notin \mathfrak{A}_2) \vee (A_1 \notin \mathfrak{A}_1 \wedge A_2 \in \mathfrak{A}_2)$$

(B)
$$\mathfrak{A}_1 = \overline{\mathfrak{A}_2}$$

(C)
$$\mathfrak{A}_2 = \overline{\mathfrak{A}_1}$$

3.2. Clearly there holds:

(1)
$$\overline{\mathfrak{A}} \text{ is regular} \Leftrightarrow Q \in \mathfrak{A},$$

(2)
$$\overline{\overline{\mathfrak{A}}} = \mathfrak{A},$$

(3)
$$\mathfrak{A} \subset \mathfrak{B} \Leftrightarrow \overline{\mathfrak{A}} \supset \overline{\mathfrak{B}},$$

(4)
$$\overline{\bigcup_{j \in J} \mathfrak{A}_j} = \bigcap_{j \in J} \overline{\mathfrak{A}_j}, \qquad \overline{\bigcap_{j \in J} \mathfrak{A}_j} = \bigcup_{j \in J} \overline{\mathfrak{A}_j}, \qquad (\text{for } J \neq \emptyset)$$

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*) The first part appeared in Spisy přírodov. fak. Univ. J. E. Purkyně (Brno), řada 2 (Archivum mathematicum), T 5 (1969), 29—64.

$$\overline{\emptyset} = \exp Q, \quad \overline{\exp Q} = \emptyset,$$
(5)
$$\overline{\exp Q - \{\emptyset\}} = \{Q\}, \ \overline{\{Q\}} = \exp Q - \{\emptyset\}.$$

3.3. Lemma. *Let* $\mathfrak{A}$ *be an* M*-collection. Then* $\overline{\mathfrak{A}}$ *is an* M*-collection, and if* $\mathfrak{A} \sqcap \{Y\} \subset \mathfrak{A}$, *then* $\overline{\mathfrak{A}} \sqcap \{Y\} \subset \overline{\mathfrak{A}}$.

Proof. If $B \subset C \subset Q$, $C \notin \overline{\mathfrak{A}}$, then $Q - C \in \mathfrak{A}$, $Q - C \subset Q - B$, hence $Q - B \in \mathfrak{A}$, $B \notin \overline{\mathfrak{A}}$; thus $\overline{\mathfrak{A}}$ is an M-collection. Further let $\mathfrak{A} \sqcap \{Y\} \subset \mathfrak{A}$. If $A \cup (Q - Y) \in \overline{\mathfrak{A}}$, then $(Q - A) \cap Y \notin \mathfrak{A}$, hence $Q - A \notin \mathfrak{A}$, $A \in \overline{\mathfrak{A}}$; therefore $\overline{\mathfrak{A}}$ satisfies the condition (v) of § 2.17.5, and hence $\overline{\mathfrak{A}} \sqcap \{Y\} \subset \overline{\mathfrak{A}}$. Q.E.D.

3.4. From Lemma 3.3 and (2) it follows that the assertions "$\mathfrak{A}$ is an M-collection (and $\mathfrak{A} \sqcap \{Y\} \subset \mathfrak{A}$)", "$\overline{\mathfrak{A}}$ is an M-collection (and $\overline{\mathfrak{A}} \sqcap \{Y\} \subset \overline{\mathfrak{A}}$)" are equivalent.

Let us note that $\mathfrak{A} \approx \mathfrak{B}$ does not imply $\overline{\mathfrak{A}} \approx \overline{\mathfrak{B}}$, as the following simple example shows: if $Q = \{0, 1\}$, $\mathfrak{A} = \{\{0\}\}$, $\mathfrak{B} = \{\{0\}, \{1\}, \{0, 1\}\}$, then $[\mathfrak{A}]_Q = [\mathfrak{B}]_Q = \mathfrak{B}$, $\mathfrak{A} \approx$ B, but $\overline{\mathfrak{A}} = \{\emptyset, \{0\}, \{0, 1\}\}$, $\overline{\mathfrak{B}} = \{\{0, 1\}\}$, $[\overline{\mathfrak{A}}]_Q = \exp Q \neq \{Q\} = [\overline{\mathfrak{B}}]_Q$, $\overline{\mathfrak{A}} \neq \overline{\mathfrak{B}}$.

The operation $-$ has a certain relation to conjugate systems of M-collections:

3.5. Theorem. *Let* $(\mathfrak{A}_j)_{j \in J}$ *be a system of* M*-collections, let* card $J \geqq 2$. *Then the following assertions are equivalent*:

(A) *For each* $j_0 \in J$ (*) *holds.*

(B) *There exists* $j_0 \in J$ *such that* (*) *holds.*

(C) $(\mathfrak{A}_j)_{j \in J}$ *is a conjugate system.*

where (*) *denotes the inclusion*

$$\overline{\mathfrak{A}_{j_0}} \supset \bigsqcap_{\substack{j \in J \\ j \neq j_0}} \mathfrak{A}_j.$$

Proof.

1. (A) implies (B).

2. Let (*) hold for some $j_0 \in J$ (i.e. let (B) be valid). Let $(A_j)_{j \in J} \in \bigtimes_{j \in J} \mathfrak{A}_j$. If $\bigcap_{j \in J} A_j = \emptyset$, then $A_{j_0} \subset Q - \bigcap_{\substack{j \in J \\ j \neq j_0}} A_j$, hence $Q - \bigcap_{\substack{j \in J \\ j \neq j_0}} A_j \in \mathfrak{A}_{j_0}$ ($\mathfrak{A}_{j_0}$ is an M-collection), but $\bigcap_{\substack{j \in J \\ j \neq j_0}} A_j \in \bigsqcap_{\substack{j \in J \\ j \neq j_0}} \mathfrak{A}_j \subset \overline{\mathfrak{A}_{j_0}}$ according to the supposition, and now $Q - \bigcap_{\substack{j \in J \\ j \neq j_0}} A_j \notin \mathfrak{A}_{j_0}$ gives a contradiction. Thus $\bigcap_{j \in J} A_j \neq \emptyset$. Therefore (C) holds.

3. Let (C) be valid. Let $j_0 \in J$, $A \in \bigsqcap_{\substack{j\in J\\ j\neq j_0}} \mathfrak{A}_j$. Then there exist sets $A_j \in \mathfrak{A}_j$ for $j \in J - \{j_0\}$ such that $A = \bigcap_{\substack{j\in J\\ j\neq j_0}} A_j$. If $Q - A \in \mathfrak{A}_{j_0}$, then we put $A_{j_0} := Q - A$, and now $\emptyset = \bigcap_{j\in J} A_j \in \bigsqcap_{j\in J} \mathfrak{A}_j$, which is a contradiction. Thus $Q - A \notin \mathfrak{A}_{j_0}$, $A \in \overline{\overline{\mathfrak{A}_{j_0}}}$. Therefore (A) holds.

Q.E.D.

3.6. Corollaries. If $\mathfrak{A}$, $\mathfrak{B}$ are M-collections, then the pair $(\mathfrak{A}, \mathfrak{B})$ is conjugate (i.e. $\mathfrak{A} \sqcap \mathfrak{B} \notin \emptyset$) iff $\overline{\mathfrak{A}} \supset \mathfrak{B}$ (or iff $\overline{\mathfrak{B}} \supset \mathfrak{A}$). This can be expressed in this form (see § 2 (20)):

If $\mathfrak{A}$ is an M-collection, then $\overline{\mathfrak{A}}$ is the greatest of the collections $\mathfrak{B}$ such that $(\mathfrak{A}, \mathfrak{B})$ is a conjugate pair.

From this there follows

(6) $\qquad \mathfrak{A}$ is an M-collection $\Rightarrow \overline{\mathfrak{A}} = \{B \mid B \subset Q,\ \{B\} \sqcap \mathfrak{A} \not\ni \emptyset\}.$

4.1. Now let us mention several properties of the operation $'$:

$$\text{(7)} \qquad \mathfrak{A}' = \overline{\mathfrak{A}} \Leftrightarrow Q \in \mathfrak{A},$$

$$\mathfrak{A}' = \emptyset \Leftrightarrow Q \notin \mathfrak{A},$$

$$\text{(8)} \qquad \emptyset' = \emptyset, \quad (\exp Q)' = \emptyset,$$

$$((\exp Q) - \{\emptyset\})' = \{Q\}^{\#},$$

$$\{Q\}' = (\exp Q) - \{\emptyset\},$$

(see (1), (5); § 2.10). By means of Lemma 3.3 we obtain

(9) $\qquad \mathfrak{A}$ is an M-collection $\Rightarrow \mathfrak{A}'$ is an RM-collection

4.2. Lemma. *Let $\mathfrak{A}$ be an RM-collection. Then*

$$\mathfrak{A}'' = \mathfrak{A}$$

and

$$\mathfrak{A}' = \begin{cases} \emptyset \\ \overline{\mathfrak{A}} = \emptyset \end{cases} \qquad \textit{if} \begin{cases} \mathfrak{A} = \emptyset. \\ \mathfrak{A} \neq \emptyset \end{cases}$$

Proof. If $\mathfrak{A} = \emptyset$, then $\mathfrak{A}' = \emptyset$ (see (7)), $\mathfrak{A}'' = \emptyset = \mathfrak{A}$. If $\mathfrak{A} \neq \emptyset$, then $Q \in \mathfrak{A}$, $\mathfrak{A}' = \overline{\mathfrak{A}}$ (see (7)), $\mathfrak{A}'' = (\overline{\mathfrak{A}})' = \overline{(\overline{\mathfrak{A}})^{\#}} = \mathfrak{A}^{\#} = \mathfrak{A}$, and $\overline{\mathfrak{A}} \neq \emptyset$ ($\overline{\mathfrak{A}} = \emptyset$ implies $\mathfrak{A} = \exp Q \ni \emptyset$, but $\mathfrak{A}$ is regular). Q.E.D.

4.3. Corollary. The following assertions are equivalent:

(A) $\qquad \mathfrak{A}_1, \mathfrak{A}_2$ are RM-collections $\wedge\ \mathfrak{A}_1' = \mathfrak{A}_2,$

(B) $\mathfrak{A}_1, \mathfrak{A}_2$ are M-collections $\wedge$ $[\mathfrak{A}_1 = \mathfrak{A}_2 = \emptyset \vee$
$\vee (\mathfrak{A}_1 \neq \emptyset \neq \mathfrak{A}_2 \wedge \overline{\mathfrak{A}}_1 = \mathfrak{A}_2)]$.

4.4. Let us note that from 4.2 and (9) it follows that if $\mathfrak{A}$ is an M-collection, then $\mathfrak{A}''' = \mathfrak{A}'$.

4.5. Lemma. *Let $\mathfrak{A}$ be a Y-generable RM-collection. Then $\mathfrak{A}'$ is a Y-generable RM-collection.*

Proof. $\mathfrak{A}'$ is an RM-collection (see (9)). If $\mathfrak{A}' = \emptyset$, then $\mathfrak{A} = \emptyset = \mathfrak{A}'$ (4.2), $Y = \emptyset$ (§ 2.17.2). Let $\mathfrak{A}' \neq \emptyset$, then $\mathfrak{A}' = \overline{\mathfrak{A}}$ (4.2), $\mathfrak{A} \sqcap \{Y\} \subset \mathfrak{A}$ (§ 2.17.2), $\overline{\mathfrak{A}} \sqcap \{Y\} \subset \overline{\mathfrak{A}}$ (Lemma 3.3). Hence always $\mathfrak{A}' - \{Y\} \subset \mathfrak{A}'$. Therefore $\mathfrak{A}'$ is Y-generable (§ 2.17.2). Q.E.D.

4.6. Lemma. *Let $(\mathfrak{A}_1, \mathfrak{A}_2)$ be a pair of M-collections. Then the following assertions are equivalent:*

(A) $(\mathfrak{A}_1)' \supset \mathfrak{A}_2 \wedge (\mathfrak{A}_2)' \supset \mathfrak{A}_1$,

(B) $(\mathfrak{A}_1, \mathfrak{A}_2)$ *is a regular pair.*

Proof. In both the cases (A), (B) there holds either $\mathfrak{A}_1 = \emptyset = \mathfrak{A}_2$, but then (A) and (B) hold, or $\mathfrak{A}_1 \neq \emptyset \neq \mathfrak{A}_2$, then $Q \in \mathfrak{A}_j$, $(\mathfrak{A}_j)' = \overline{\mathfrak{A}_j}$ for $j = 1, 2$ (see (7)), and now by means of 3.6 we conclude that (A), (B) are equivalent. Q.E.D.

4.7. If the operation $'$ is considered only as a mapping of the set of all RM-collections into itself, then it is an involution (i.e. it is inverse to itself). Let $\mathfrak{A}$, $\mathfrak{B}$ be *RM-collections*; we say that $\mathfrak{A}$ is *dual* to $\mathfrak{B}$ iff $\mathfrak{A}' = \mathfrak{B}$; thus the relation "to be dual to . . ." is symmetric, and we may say that $\mathfrak{A}$, $\mathfrak{B}$ are (mutually) *dual* iff $\mathfrak{A}' = \mathfrak{B}$. Further, we say that the pair $\mathfrak{A}$, $\mathfrak{B}$ (of RM-correspondences) is *complete* iff $\mathfrak{A}$, $\mathfrak{B}$ are dual. From Lemma 4.6 it follows that every complete pair is regular, moreover there holds (see 4.6, § 2 (46))

(10) $\mathfrak{A}$ is an RM-collection $\Rightarrow$ $\mathfrak{A}'$ is the greatest of all collections $\mathfrak{B}$ such that $(\mathfrak{A}, \mathfrak{B})$ is regular.

4.8. We say that a pair $(\mathfrak{A}, \mathfrak{B})$ of *R-collections* is *weakly complete* iff $([\mathfrak{A}]_Q, [\mathfrak{B}]_Q)$ is complete (i.e. iff the natural representatives of the classes of the M-decomposition — see § 2.14 — which contain $\mathfrak{A}$, $\mathfrak{B}$ form a complete pair). Clearly, every weakly complete pair is conjugate.

5. And now we characterize the operation $\tilde{}$: if $\mathfrak{A} \ni Q$, then $\mathfrak{A}' = \overline{\mathfrak{A}}$ (see (7)), hence $\tilde{\mathfrak{A}} = \overline{\mathfrak{A}'} = \overline{\overline{\mathfrak{A}}} = \mathfrak{A}$, if $\mathfrak{A} \not\ni Q$, then $\mathfrak{A}' = \emptyset$, $\tilde{\mathfrak{A}} = \bar{\emptyset} = \exp Q$:

$$(11) \qquad \tilde{\mathfrak{A}} = \begin{cases} \mathfrak{A} \\ \exp Q \end{cases} \quad \text{if} \begin{cases} Q \in \mathfrak{A}, \\ Q \notin \mathfrak{A}. \end{cases}$$

Thus

$$(12)\qquad \tilde{\tilde{\mathfrak{A}}} = \tilde{\mathfrak{A}},$$

$$(13)\qquad \mathfrak{A} \text{ is an } M\text{-collection} \Rightarrow \tilde{\mathfrak{A}} \text{ is an } M\text{-collection},$$

$$(14)\qquad \mathfrak{A} \text{ is an } RM\text{-collection} \Rightarrow \tilde{\mathfrak{A}} = \overline{\mathfrak{A}},$$

(see (11), (5), Lemma 4.2, etc.).

6.1. We define two mappings $+$, $-$: $\exp Q \to \exp\exp Q$ (for the fixedly chosen Q) in such a way:

$$Y^+ := \{A \mid A \subset Q, \quad \emptyset \neq Y \subset A\} \; (= \{A \mid Y \subset A \subset Q\}^{\#}),$$
$$Y^- := \{A \mid A \subset Q, \quad A \cap Y \neq \emptyset\}.$$

Thus

$$(15)\qquad Y^- = [\{\{y\} \mid y \in Y\}]_Q \supset [\{Y\}^{\#}]_Q = ([Y]_Q)^{\#} = Y^+,$$

hence

$$(16)\qquad Y^-,\ Y^+ \text{ are } Y\text{-generable } RM\text{-collections},$$

$$(17)\qquad \emptyset^+ = \emptyset^- = \emptyset, \qquad Q^+ = \{Q\}^{\#}, \qquad Q^- = (\exp Q) - \{\emptyset\}.$$

Clearly

$$(18)\qquad (Y^+)' = \overline{Y^+} = Y^-,$$

especially

$$(18')\qquad (Y^+,\ Y^-) \text{ is a complete pair}$$

Further, it can be easily proved (for $Y_1,\ Y_2 \subset Q$):

$$(19)\qquad Y_1 = Y_2 \Leftrightarrow Y_1^+ = Y_2^+ \Leftrightarrow Y_1^- = Y_2^-,$$

$$(20)\qquad Y_1^+ = Y_2^- \Leftrightarrow Y_1 = Y_2 \wedge \operatorname{card} Y_1 \leqq 1.$$

6.2. If $\mathfrak{A}$ is a Y-generable RM-collection, then clearly the pair $(\mathfrak{A}, Y^+)$ is regular ($\mathfrak{A} \sqcap Y^+ \supset (\mathfrak{A} \sqcap \{Y\}) \sqcap Y^+ = \mathfrak{A} \sqcap (\{Y\} \sqcap Y^+) = \mathfrak{A} \sqcap \{Y\} \supset \mathfrak{A} \not\ni \emptyset$ — see § 2.17.2; etc.), hence from Lemma 4.6 we get $\mathfrak{A} \subset Y^-$, $Y^+ \subset \mathfrak{A}'$; but $\mathfrak{A}'$ is a Y-generable RM-collection (Lemma 4.5), too, therefore also $\mathfrak{A}' \subset Y^-$, $Y^+ \subset \mathfrak{A}'' = \mathfrak{A}$:

$$(21)\qquad \mathfrak{A} \text{ is a } Y\text{-generable } RM\text{-collection} \Rightarrow Y^+ \subset \mathfrak{A} \subset Y^-.$$

6.3. If $\mathfrak{A}$ is an RM-collection, $Y^+ \subset \mathfrak{A} \subset Y^-$, then $\mathfrak{A}$ need not be Y-generable, as an example shows: let $Q = \{1, 2, 3\}$, $\mathfrak{A} = [\{\{1, 2\}, \{1, 3\}\}]_Q$, $Y = \{1, 2\}$, then $Y^+ = [\{\{1, 2\}\}]_Q \subset \mathfrak{A} \subset [\{\{1\}, \{2\}\}]_Q = Y^-$, but $\mathfrak{A} \sqcap \{Y\} = [\{\{1\}\}]_Q \not\subset \mathfrak{A}$, hence $\mathfrak{A}$ is not Y-generable (§ 2.17.2).

6.4. Let us put

$$Y^{(+)} := \{\{y\} \mid y \in Y\},$$

$$Y^{(-)} := \{Y\}^{\#}.$$

Thus for $\delta \in \{+, -\}$ $\| Y^{(\delta)} \| = Y$, and Y^{δ} is generated by $Y^{(\delta)}$ (see (15)); moreover, clearly $Y^{(\delta)}$ is the smallest (under $\subset$) collection generating Y^{δ}.

b) The corresponding operations for correspondences

7. In this part **b** (excluding Lemma 10) let P, Q be fixedly chosen sets, $A \subset Q$, u, v, w, u_1, $u_2 \in \mathrm{Corr}\,(P, \exp Q)$. $(u_j)_{j \in J}$ will be a system of elements of Corr $(P, \exp Q)$, $\Gamma \in \mathrm{Corr}\,(Q, P)$. Now the operations $^{-}$, $'$, $\sim$, and related notions defined by means of them can be considered as induced by the corresponding operations or notions of the part **a** in the sense given in § 2.7. Results of § 4a can be immediately transformed to correspondences, especially there hold 8.1—8.4:

8.1.

(22) $$\bar{\bar{u}} = u,$$

(23) $$u \subset v \Leftrightarrow \bar{u} \supset \bar{v},$$

(24) $$\overline{\bigcup_{j \in J} u_j} = \bigcap_{j \in J} \overline{u_j}, \quad \overline{\bigcap_{j \in J} u_j} = \bigcup_{j \in J} \overline{u_j} \qquad (J \neq \emptyset),$$

(25) u is an M-correspondence $\Rightarrow$ $\bar{u}$ is the greatest of all v such that (u, v) is conjugate,

(26) u is an M-correspondence $\Rightarrow$ $\bar{u}$, $\tilde{u}$ are M-correspondences, u' is an RM-correspondence,

(27) u is an RM-correspondence $\Rightarrow$ $u'' = u$

(28) u is an RM-correspondence $\Rightarrow$ u' is the greatest of all v such that (u,v) is regular,

(29) $$\tilde{\tilde{u}} = \tilde{u}.$$

8.2. Lemma. *The following assertions are equivalent:*

(A) $$A_1 \cup A_2 = Q \wedge A_1 \cap A_2 = \emptyset \Rightarrow$$
$$\Rightarrow u_1 A_1 \cup u_2 A_2 = P \wedge u_1 A_1 \cap u_2 A_2 = \emptyset,$$

(B) $$u_1 = \overline{u_2},$$

(C) $$u_2 = \bar{u}_1.$$

8.3. Lemma. *The following assertions are equivalent:*

(A) *u_1, u_2 are RM-correspondences and $(u_1)' = u_2$ (i.e. (u_1, u_2) is a complete pair of correspondences).*

(B) *u_1, u_2 are M-correspondences and there holds:*

$$A_1 \cup A_2 = Q \wedge A_1 \cap A_2 = \emptyset \Rightarrow$$
$$\Rightarrow u_1A_1 \cup u_2A_2 = u_1Q = u_2Q \wedge u_1A_1 \cap u_2A_2 = \emptyset.$$

8.4. Lemma. *Let u be a Γ-generable RM-correspondence.* Then *u' is a Γ-generable RM-correspondence.*

8.5. We shall introduce several phrases for shortness of expressions. Instead of "a partition of Q with $\{1, 2\}$ as the set of all indices" we shall use the term "a *complementary pair*". Let (A_1, A_2) denote a *complementary pair.*

Let $\mathscr{U} = (u_1, u_2)$ be an *RM-pair.* We say that $\mathscr{U}$ is *complete on* (A_1, A_2) iff $u_1A_1 \cup u_2A_2 = u_1Q = u_2Q$. We say that $\mathscr{U}$ is *complete in* x $(\in P)$ iff (xu_1, xu_2) is complete. (Thus $\mathscr{U}$ is complete iff it is complete on every complementary pair, or iff it is complete in all $x \in P$.) We say that $\mathscr{U}$ is *absolutely incomplete* iff there exists a complementary pair (A_1, A_2) such that $u_1A_1 \cup u_2A_2 = \emptyset \neq u_iQ$ (then evidently $\mathscr{U}$ is incomplete in all $x \in u_iQ$, $i = 1, 2$).

8.6. Lemma. *Let (u_1, u_2), (v_1, v_2) be RM-pairs of the same type, let (A_1, A_2) be a complementary pair. If (u_1, u_2) is complete on (A_1, A_2), and if $u_iA_i \subset v_iA_i$ for $i = 1, 2$, then (v_1, v_2) is complete on (A_1, A_2), and $u_iA_i = v_iA_i$ for $i = 1, 2$.*

Proof. In fact, $v_iQ \supset v_1A_1 \cup v_2A_2 \supset u_1A_1 \cup u_2A_2 = u_iQ = v_iQ$ for $i = 1, 2$ (§ 2 (15)), $u_1A_1 \cap u_2A_2 = \emptyset$, therefore $u_iA_i = v_iA_i$. Q.E.D.

9. From Lemma 8.2 there follows

(30) $$\bar{u}A = P - u(Q - A),$$

and hence $\bar{u}\emptyset = P - uQ$, $u'A = (\bar{u})^{\#}A = \bar{u}A - \bar{u}\emptyset = (P - u(Q - A)) - (P - uQ) = uQ - u(Q - A)$, $\tilde{u}A = \overline{u'A} = P - u'(Q - A) = P - (uQ - uA) = uA \cup (P - uQ)$ (see § 2,5.3, § 2 (9)):

(31) $$u'A = uQ - u(Q - A),$$

(32) $$\tilde{u} = u \cup \underline{P - uQ},$$

(where ____ is defined in § 2.11), especially

(33) $$\tilde{u}Q = P.$$

From (30), (31) we get

(34) $$u\emptyset = \emptyset \Rightarrow uQ = u'Q,$$

(35) $$uQ = P \Rightarrow \bar{u} = u',$$

(36) $$uA \subset uQ \quad \text{for all} \quad A \subset Q \Rightarrow \bar{u} = u' \cup \underline{P - uQ}.$$

The condition "$uA \subset uQ$ for all $A \subset Q$" can be written shortly as "$u \subset \underline{uQ}$". According to (34), (36), (35)

(37) $$u\emptyset = \emptyset \wedge u \subset \underline{uQ} \Rightarrow \bar{u} = \tilde{u}',$$

(38) $$u\emptyset = \emptyset \wedge uQ = P \Rightarrow \bar{u} = u' = \tilde{u}'.$$

10. Lemma. *Let E, G be sets, $u \in \operatorname{Corr}(E, \exp Q)$, $v \in \operatorname{Corr}(G, \exp E)$. Then*

$$\overline{v \,.\, u} = \bar{v} \,.\, \bar{u},$$

where the operation $\overline{}$ *is taken in* Corr $(G, \exp Q)$; Corr $(G, \exp E)$; Corr $(E, \exp Q)$ *in the expression* $\overline{v \,.\, u}$; $\bar{v}$; $\bar{u}$, *respectively.*

Proof. In fact, $(\bar{v} \,.\, \bar{u})\, A = \bar{v}(\bar{u}\, A) = \bar{v}(\overline{E - u(Q - A)}) = G - v(u(Q - A)) = G - (v \,.\, u)\,(Q - A) = (\overline{v \,.\, u})\, A$. Q.E.D.

11. Now to the operations ${}^{\delta}$, ${}^{(\delta)}$ introduced in § 4a ($\delta \in \{+, -\}$) which map $\exp Q$ into $\exp \exp Q$ we shall construct in a natural way the "induced" operations ${}^{\delta}$, ${}^{(\delta)}$, which will map Corr (Q, P) into Corr $(P, \exp Q)$: for $\Gamma \in \operatorname{Corr}(Q, P)$, $\delta \in \{+, -\}$ we define $\Gamma^{\delta}, \Gamma^{(\delta)} \in \operatorname{Corr}(P, \exp Q)$ by the conditions

$$x\Gamma^{\delta} = (\Gamma x)^{\delta}, \qquad x\Gamma^{(\delta)} = (\Gamma x)^{(\delta)},$$

(for all $x \in P$). The statements of § 2.6 can be easily transformed to the induced operations, especially

(39) (Γ^{+}, Γ^{-}) is a complete pair of Γ-generable correspondences,

(40) $\Gamma^{(\delta)}$ is the smallest (under $\subset$) correspondence generating Γ^{δ}; Γ is the graph of $\Gamma^{(\delta)}$,

(41) $(\Gamma^{(+)}, \Gamma^{(-)})$ is a weakly complete pair.

Clearly

(42)
$$\Gamma^{+}A = \{x \mid x \in P,\ \emptyset \neq \Gamma x \subset A\},$$
$$\Gamma^{-}A = \{x \mid x \in P,\ A \cap \Gamma x \neq \emptyset\},$$

therefore *the denotations* Γx, $\Gamma^{+}A$, $\Gamma^{-}A$ *have the same sense as in Berge's book* [1].

12. We say that u is a *simple correspondence* iff there exists Γ such that $xu \in \{x\Gamma^+, x\Gamma^-\}$ for all $x \in P$. Clearly every simple correspondence is an RM-correspondence. We shall understand under a *simple pair* such a pair (of correspondences) whose each member is simple. There holds

$$u \text{ is simple} \Rightarrow u' \text{ is simple,} \tag{43}$$

(see (18)), i.e. if a member of a complete pair of correspondences is simple, then the pair is simple.

c) The case $P = Q$.

13. In this part **c** we suppose the same as in part **b** (see **7**), and moreover $P = Q$. Let (P, P_0) be a type.

14. There holds

$$\bar{u} = (-\mathbf{1}) \,.\, u \,.\, (-\mathbf{1}), \tag{44}$$

$$\mathbf{1} \text{ is an } RM\text{-correspondence,} \tag{45}$$

$$\mathbf{1} \subset u \Rightarrow u' = \bar{u} \subset \mathbf{1}, \tag{46}$$

$$\bar{\mathbf{1}} = \mathbf{1}' = \mathbf{1}, \qquad \overline{(-\mathbf{1})} = (-\mathbf{1}), \qquad (-\mathbf{1})' = \underline{\varnothing}, \tag{47}$$

$$(\mathbf{1} \cup u)' = \overline{\mathbf{1} \cup u} = \mathbf{1} \cap \bar{u}, \qquad \overline{\mathbf{1} \cap u} = \mathbf{1} \cup \bar{u}, \tag{48}$$

(see (30), (31), (35), (23)).

15.1. In § 4.15 let $\mathscr{P} = (P_1, P_2)$ be a fixedly chosen *partition of the set* P. For (a given $\mathscr{P}$ and for) $i = 1, 2$ let $\mathsf{B}_i \in \operatorname{Corr}(P, \exp P)$ be defined by

$$\mathsf{B}_i := \mathbf{1} \cap \underline{P_i},$$

i.e. $\mathsf{B}_i A = A \cap P_i$ for $A \subset P$. Let us mention trivial properties:

$$\mathsf{B}_i \,.\, \mathsf{B}_i = \mathsf{B}_i, \qquad \mathsf{B}_i \,.\, \mathsf{B}_{3-i} = \underline{\varnothing}, \tag{49}$$

$$\begin{aligned} \mathsf{B}_i \bigcup_{j \in J} A_j &= \bigcup_{j \in J} \mathsf{B}_i A_j, \\ \mathsf{B}_i \bigcap_{j \in J} A_j &= \bigcap_{j \in J} \mathsf{B}_i A_j \quad (J \neq \emptyset), \end{aligned} \tag{50}$$

where $(A_j)_{j \in J}$ is a system of subsets of P.

15.2. For an arbitrary pair (u_1, u_2) (and for the given $\mathscr{P}$) we define $u \in \operatorname{Corr}(P, \exp P)$ by

$$(+) \qquad xu = \begin{cases} xu_1 \\ xu_2 \end{cases} \quad \text{if} \begin{cases} x \in P_1 \\ x \in P_2 \end{cases},$$

and we denote $u \leftarrow (u_1, u_2)$.

15.3. For $u \leftarrow (u_1, u_2)$, $v \leftarrow (u_2, u_1)$ there hold the following simple statements:

(51) $\quad u_1, u_2$ have a property $\mathsf{V} \Rightarrow u, v$ have the property V,

where V is some of the properties:
"to be an M-correspondence"
"to be an R-correspondence"
"to have the type (P, P_0)"
"to be a game correspondence"
"to be a Γ-generable correspondence"
"to be a simple correspondence"

(52) $\quad (u_1, u_2)$ has a property $\mathsf{V} \Rightarrow (u, v)$ has the property V,

where V is some of the properties:
"to be a conjugate pair"
"to be a regular pair"
"to be a weakly complete pair"
"to be a complete pair"

$$(53) \qquad u_1 \leftarrow (u, v), \qquad u_2 \leftarrow (v, u).$$

15.4. The condition (+) of 15.2 is equivalent to the condition $xu = x(B_1 . u_1) \cup x(B_2 . u_2)$ $(= x(B_1 . u_1 \cup B_2 . u_2))$ for all $x \in P$, i.e.

$$(54) \qquad u \leftarrow (u_1, u_2) \Leftrightarrow u = B_1 . u_1 \cup B_2 . u_2 .$$

15.5. In 15.5 let u be a *game correspondence*. Then

$$u = B_1 . u \cup B_2 . u \supset B_1 . \mathrm{u} . B_2 \cup B_2 . u . B_1 .$$

Let us introduce the condition

$$(*) \qquad u = B_1 . u . B_2 \cup B_2 . u . B_1 ,$$

(which is equivalent to $u \subset B_1 . u . B_2 \cup B_2 . u . B_1$). If (*) holds, then according to (49), (50) there holds

$$(**) \qquad B_i . u = B_i . u . B_{3-i} = u . B_{3-i} \qquad (\text{for } i = 1, 2),$$

and especially

$$(***) \qquad B_i . u \subset u . B_{3-i} \qquad (\text{for } i = 1, 2).$$

But (from (***) there follows $B_1 . u . B_2 \cup B_2 . u . B_1 \supset B_1 . B_1 . u \cup \cup B_2 . B_2 . u = u$, i.e. (*) holds iff (***) holds. Thus

(55) $$(*) \Leftrightarrow (**) \Leftrightarrow (***).$$

Let us note that especially

(56) $$(*) \Rightarrow u = uB_1 \cup uB_2,$$

but here the implication "$\Leftarrow$" is not valid, as there shows

15.6. Example. Let $P := \{1, 2\} \times \{1, 2\}$, $P_i := \{(1, i), (2, i)\}$, $\Gamma(k, i) := \{(3 - k, i)\}$ for $k, i \in \{1, 2\}$. Then $\Gamma^- = \Gamma^+$ (see 11; (20)), $\mathscr{P} := (P_1, P_2)$ is a partition of P. We put $u := \Gamma^- = \Gamma^+$, thus u is a Γ-generable game correspondence, $u = u'$ (see (39)). It is clear that $u = u . B_1 \cup u . B_2$, although $u . B_i \subset \underset{\smile}{P_i}$, $B_i . u . B_{3-i} \subset B_i . \underline{P_{3-i}} = \emptyset \neq \neq B_i . u$, i.e. (**) does not hold.

15.7. We say that Γ is *$\mathscr{P}$-alternative* iff

$$x \in P_i, \qquad i \in \{1, 2\} \Rightarrow \Gamma x \subset P_{3-i},$$

i.e. iff $P_i \cap \Gamma x = \emptyset$ whenever $x \in P_i$, $i \in \{1, 2\}$. Thus

(57) $$\Gamma \text{ is } \mathscr{P}\text{-alternative} \Leftrightarrow P = (P_1 - \Gamma^- P_1) \cup (P_2 - \Gamma^- P_2).$$

Let $\Gamma_{(\mathscr{P})}$ be the graph defined by

$$\Gamma_{(\mathscr{P})}x := \begin{cases} P_2 \\ \emptyset \\ P_1 \end{cases} \quad \text{if} \quad x \in \begin{cases} P_1 - P_0 \\ P_0 \\ P_2 - P_0 \end{cases};$$

clearly $\Gamma_{(\mathscr{P})}$ is the *greatest* (under $\subset$) *$\mathscr{P}$-alternative P_0-ended graph* (of Corr (P, P)).

Evidently (for $\Gamma_0 \in \operatorname{Corr}(P, P)$)

(58) $$\Gamma_0 \subset \Gamma,\ \Gamma \text{ is } \mathscr{P}\text{-alternative} \Rightarrow \Gamma_0 \text{ is } \mathscr{P}\text{-alternative}.$$

15.8. We say that $u \in \operatorname{Corr}(P, \exp P)$ is *$\mathscr{P}$-alternative* iff u is Γ-generable for some $\mathscr{P}$-alternative graph Γ. According to § 4.8.4

(59) u is a $\mathscr{P}$-alternative game correspondence $\Rightarrow u'$ is $\mathscr{P}$-alternative.
Further there holds for each $\delta \in \{+, -\}$

(60) Γ is $\mathscr{P}$-alternative $\Leftrightarrow \Gamma^\delta$ is $\mathscr{P}$-alternative ($\Rightarrow$: (39); $\Leftarrow$: (40), (58)).

15.9. Lemma. *Let u be a game correspondence of the type (P, P_0). Then the following statements are equivalent:*

(A) *u is $\mathscr{P}$-alternative.*

(B) *u is $\Gamma_{(\mathscr{P})}$-generable.*

(C) (*) *(of* 15.5) *holds.*

Proof. (B) and (A) are equivalent — see 15.7, § 2(27). The condition (*) can be written in the form

$$xuA \Rightarrow [xu(A \cap P_{3-i}) \text{ whenever } x \in P_i],$$

but $xuA \Rightarrow x \notin P_0 = P - uP$, and hence (*) holds iff u is $\Gamma_{\mathscr{P}}$-generable (§ 2: (15), (26)). Q.E.D.

15.10. Lemma. *Let u_1, u_2 be $\mathscr{P}$-alternative game correspondences, let $u \leftarrow (u_1, u_2)$. Then*

$$\begin{aligned} u &= B_1 . u_1 \cup B_2 . u_2 = u_1 . B_2 \cup u_2 . B_1 = \\ &= B_1 . u_1 . B_2 \cup B_2 . u_2 . B_1 = B_1 . u . B_2 \cup B_2 . u . B_1 = \\ &= u . B_1 \cup u . B_2, \end{aligned}$$

and u is a $\mathscr{P}$-alternative game correspondence.

Proof. See (54), 15.9, 15.5 (namely, $B_i . u . B_{3-i} = B_i . (B_1 . u_1 \cup B_2 . u_2) . B_{3-i} = B_i . u_i . B_{3-i}$ — see (49), (50) —, etc.), (51). Q.E.D.

d) Complete games. Games with perfect information

We suppose the same as in part **c** (13).

16.1. Let u, v be *game correspondences*. Then the assertion "(u, v) is a regular pair" is equivalent to the assertion "there exists an (R)SN-game such that u, v are the game correspondences of some two distinct players of this game" (see § 2.29—30); for an arbitrary fixed game correspondence u there is the greatest (under $\subset$) of v such that (u, v) is regular, namely $v := u'$ (see (28)), therefore, if u is the game correspondence of a player of a game, then the dual correspondence u' gives the greatest possibilities for another player.

We say that an (R)SN-game is *complete* iff it is a two-player game and the pair of the game correspondences of its players is complete.

According to the above, *every game correspondence u is the game correspondence of some player of a suitable complete game* (and, of course, *u' is the game correspondence of the other player of such a game*).

16.2. Now it is clear the meaning of regular weakly complete pairs of correspondences: if $\mathscr{G} = (\mathscr{R}, (f_j)_{j \in J})$ is a complete RSN-game, $J = \{j_1, j_2\}$, then $(u_{\mathscr{R}_{j_1}}, u_{\mathscr{R}_{j_2}})$ is a regular weakly complete pair (§ 2.28 etc.). It is easy to see that *every regular weakly complete pair can be obtainep* *in such a way. Further, if u is an R-correspondence, then there is v such that (u, v) is regular weakly complete* (it is sufficient to put $xv := ([xu]_P)' \sqcap \| xu \|$).

17.1. Now let us consider the formalizing of *game structures with perfect information* (and without chance moves) as certain (R)SN-game structures.

Let P be the set of all positions of a game (structure) with perfect information, let J be the set of all players ($P \neq \emptyset \neq J$); without loss of generality we can assume $0 \notin J$. Further let P_0 ($\subset P$) be the set of all final positions, Γ be the graph of this game (for $x \in P$ Γx is the set of all positions which may immediately follow after x); let $(P_j)_{j \in J}$ be the partition of P such that P_j is the set of all positions in which the player j "*nominatively*" *moves* (namely, at a final position it is irrelevant who moves). The corresponding SN-game structure will be a certain J-automaton $\mathscr{R} = (P, R, \varrho)$ with the pseudocomponents $\mathscr{R}_j = (P, R_j, \varrho_j)$ defined in the following way:

$$R_j(x) := \emptyset \qquad \text{for} \qquad x \in P_0, \qquad j \in J,$$

$$R_j(x) := \Gamma x \qquad \text{for} \qquad x \in P_j - P_0, \qquad j \in J;$$

for $x \in P_j - P_0$, $k \in J - \{j\}$ we put e.g. $R_k(x) := \{\emptyset\}$ (here we can choose an arbitrary non-empty set, too), further

$$\varrho(x, r) := \{\mathrm{pr}_j r\} \qquad \text{for} \qquad x \in P_j - P_0, \qquad r \in R(x) \; (:= \mathop{\times}_{j \in J} R_j(x)).$$

The interpretation is clear: *the player j moving at a nonfinal position $x (\in P_j - P_0)$ chooses some position $y \in \Gamma x$, and the position y will follow immediately after x*, since $y(x, r) = y$ for all $r \in R(x)$ such that $\mathrm{pr}_j r = y$; the other players "*play emptily*" at the position x. Thus

$$\varrho_k(x, y) = \begin{cases} \{y\} \\ \Gamma x \end{cases} \qquad \text{for} \quad x \in P_j - P_0, \quad y \in R_k(x), \begin{cases} k = j \\ k \in J - \{j\} \end{cases},$$

i.e.

$$xu_{\mathscr{R}_j} = \begin{cases} x\Gamma^{(-)} \\ x\Gamma^{(+)} \end{cases} \qquad \text{whenever} \qquad x \in \begin{cases} P_j \\ P - P_j \end{cases},$$

Hence, if u_j denotes the game correspondence of the player j (i.e. $u_j = [u_{\mathscr{R}_j}]_P$), then

$$xu_j = \begin{cases} x\Gamma^- \\ x\Gamma^+ \end{cases} \qquad \text{if} \qquad x \in \begin{cases} P_j \\ P - P_j \end{cases},$$

i.e.

$$u_j = B^j_+ \,.\, \Gamma^- \cup B^j_- \,.\, \Gamma^+,$$

where

$$B^j_+ := \underline{\mathbf{1} \cap P_j},$$
$$B^j_- := \underline{\mathbf{1} \cap P - P_j},$$

(see 15.2, 15.4). Further, if $(P_j)_{j\in J}$ satisfies the condition

$$(*) \qquad x \in P_j \Rightarrow P_j \cap \Gamma x = \emptyset,$$

(this is supposed in Berge's book [1]), then according to Lemma 15.10, (60), (39) there holds especially

$$u_j = \Gamma^- . B^j_- \cup \Gamma^+ . B^j_+ .$$

17.2. Passage 17.1 shows that *in an arbitrary game with perfect information each player's game correspondence is simple.* (Moreover, if card $J = 1$ in 17.1, then $u_j = \Gamma^-$, where $\{j\} = J$.)

On the other hand, *if v is a simple game correspondence and if J is a set,* card $J \geqslant 2$, $j \in J$, *then there exists a game with perfect information such that J is its set of players and v is equal to the game correspondence of j* (clearly, it is sufficient to choose some $k \in J - \{j\}$, and to put $P_j := \{x \mid x \in P,\ xv = x\Gamma^-\}$, $P_k := P - P_j$, $P_i := \emptyset$ for $i \in J - \{j, k\}$, where Γ is some graph such that $xu \in \{x\Gamma^+, x\Gamma^-\}$ for all $x \in P$, etc.).

17.3. Now let us consider the case of *two-player game with perfect information.* We may put $J := \{1, 2\}$ in 17.1; then for $i \in J$

$$B^i_+ = B_i, \qquad B^i_- = B_{3-i}.$$

Using the denotations of 15.2 we get

$$u_1 \leftarrow (\Gamma^-, \Gamma^+), \qquad u_2 \leftarrow (\Gamma^+, \Gamma^-),$$

i.e. (u_1, u_2) is a simple complete pair of game correspondences ((39), (52), 12).

On the other hand, if (v_1, v_2) is a simple complete pair of game correspondences, then there holds $v_1 \leftarrow (\Gamma^-, \Gamma^+)$, $v_2 \leftarrow (\Gamma^+, \Gamma^-)$ for a suitable graph Γ at a suitable partition (P_1, P_2) (see 12, 15.2).

Therefore *simple complete pairs of game correspondences and only they are the pairs of game correspondences of two-player games with perfect information.* We shall call these pairs *pairs with perfect information.*

Let us note that a pair with perfect information need not determine the partition (P_1, P_2) uniquely, although the graph Γ is determined by this pair (see (20), etc.): if card $\Gamma x \leqslant 1$, then it is irrelevant who moves at the position x.

Let us note that the Berge's condition (*) of 17.1 means the $\mathscr{P}$-alternativeness of Γ in the case $J = \{1, 2\}$.

e) On the "strategic completeness". Principal examples

We suppose the same as in part c.

18. In some sense, a game correspondence u determines the possibilities of a player at a game (see § 2.26.3 etc.); we may say that u describes the *local possibilities* of the player, and $\sim \boldsymbol{u}$ (or $\sim \mathfrak{U}$) describes his *global* (= *strategic*) *possibilities* if his memory is characterized by $\sim$ (see § 3a). In § 3a it was mentioned how some properties of u *induce* the corresponding properties of $\sim \boldsymbol{u}$ (or of $\sim \mathfrak{U}$), and conversely; that means, connections between certain global properties and corresponding local properties for isolated player were considered there. Analogous questions for properties related to two (sometimes many) players will be investigated in the following.

19. Lemma. *Let $(u_j)_{j\in J}$ be a regular system of the type (P, P_0). Let $(\tilde{j})_{j\in J}$ be a system of memory relations of the type (P, P_0). Then $(\tilde{j}\boldsymbol{u}_j)_{j\in J}$, $(\tilde{j}\,\mathcal{U}_j)_{j\in J}$ are RM-systems.*

Proof. The assertion immediately follows from § 3 (3), (5), (12), § 2.21.3—4. Q.E.D.

20.0 "Technical" note. Besides somewhat unusual denoting indexed memory relations, I had to modify other denotations: in (upper) index position, every expression of the type "$\langle\Xi\rangle$" (where Ξ may consist of several elementary symbols) is to be read "$\boldsymbol{\Xi}$". (The printing house has not some symbols which would have to occur in $\boldsymbol{\Xi}$ standing in index position.)

20.1. Let $\Gamma \in \mathrm{Corr}\,(P, P)$ be a graph of the type (P, P_0); to Γ we have defined the generalized graph $\boldsymbol{\Gamma} \in \mathrm{Corr}\,(\mathbf{P}, P)$ (of the type $(P, \emptyset)$, see § 3.0), and the correspondences $\boldsymbol{\Gamma}^+$, $\boldsymbol{\Gamma}^-$ (by § 4.11 with $Q := \mathbf{P}$, $\Gamma := \boldsymbol{\Gamma}$). We can define the $\sim$-strategic correspondences $\sim \boldsymbol{\Gamma}^{\langle(\delta)\rangle}$, $\sim \Gamma^{\langle\delta\rangle}$ (where $\delta \in \{+, -\}$, and $\sim$ is a memory relation of the type (P, P_0)) to the regular correspondences $\Gamma^{(\delta)}$ and to the game correspondences Γ^{δ}; clearly $\sim \boldsymbol{\Gamma}^{\langle(\delta)\rangle} = \sim \boldsymbol{\Gamma}^{\langle\delta\rangle}$ (see (39), (40), § 3 (9)). There is a near relationship between $\boldsymbol{\Gamma}^{\delta}$ and $\boldsymbol{\Gamma}^{\langle\delta\rangle}$: there holds

$$S(\Gamma^{(-)}) = \mathop{\times}_{\mathbf{z}\in\mathbf{Z}} \{\{z\} \mid z \in \Gamma\varkappa(\mathbf{z})\},$$
$$S(\Gamma^{(+)}) = \mathop{\times}_{\mathbf{z}\in\mathbf{Z}} \{\{\Gamma\varkappa(\mathbf{z})\}\},$$

(where $\mathbf{Z}$ is the set of all segments to $Z := P - P_0$), consequently

$$(61)\qquad x\boldsymbol{\Gamma}^{\langle(-)\rangle} = [\{\mathrm{s}(x, \boldsymbol{\sigma}) \mid \boldsymbol{\sigma} \in S(\Gamma^{(-)})\}]_{\mathbf{P}} = [\{\{\boldsymbol{x}\} \mid \boldsymbol{x} \in \boldsymbol{\Gamma}x\}]_{\mathbf{P}} = [x\boldsymbol{\Gamma}^{(-)}]_{\mathbf{P}} = x\boldsymbol{\Gamma}^-,$$
$$x\boldsymbol{\Gamma}^{\langle(+)\rangle} = [\{\mathrm{s}(x, \boldsymbol{\sigma}) \mid \boldsymbol{\sigma} \in S(\Gamma^{(+)})\}]_{\mathbf{P}} = [\{\boldsymbol{\Gamma}x\}]_{\mathbf{P}} = x\boldsymbol{\Gamma}^+,$$

i.e.

(62) $$\mathbf{\Gamma}^{\langle\delta\rangle} = \mathbf{\Gamma}^{\langle\langle\delta\rangle\rangle} = \mathbf{\Gamma}^{\delta} \qquad (\delta \in \{+, -\}).$$

Thus from (62), (39) we get

(63) $$(\mathbf{\Gamma}^{\langle+\rangle}, \mathbf{\Gamma}^{\langle-\rangle}) \text{ is a complete pair}$$

Further $\boldsymbol{S}(\Gamma^{(+)}) = \boldsymbol{S}(\Gamma^{(-)})$ (card $\boldsymbol{S}(\Gamma^{(+)}) = 1$), hence

(64) $$\sim \mathbf{\Gamma}^{\langle+\rangle} = \mathbf{\Gamma}^{\langle+\rangle}.$$

But it may happen $\mathbf{\Gamma}^{\langle-\rangle} \neq \mathring{\mathbf{\Gamma}}^{\langle-\rangle}$, e.g.: if $P := \{1, 2\}$, $\Gamma 1 := \Gamma 2 := P$ ($P_0 = \emptyset$), then $\{(1, 2, 1, 1, 1, \ldots)\} \in 1\mathbf{\Gamma}^{\langle-\rangle} - 1\mathring{\mathbf{\Gamma}}^{\langle-\rangle}$.

Let us note that there holds

(65) $$u = \Gamma^{-}, \quad v = \Gamma^{+} \Rightarrow \quad \mathfrak{A} = \mathbf{X}^{-}, \qquad \mathscr{V} = \mathbf{X}^{+},$$

where $\mathbf{X} := \mathbf{X}_{\Gamma}$.

21.1. Lemma. *Let (u_1, u_2) be a regular pair of the type (P, P_0), let $\tilde{\mathbf{1}}, \tilde{\mathbf{2}}$ be memory relations of this type. Then there holds:*

(1) $(\tilde{\mathbf{1}}\,\mathfrak{U}_1, \tilde{\mathbf{2}}\,\mathfrak{U}_2)$ *is complete* $\Rightarrow (\mathfrak{U}_1, \mathfrak{U}_2)$, $(\tilde{\mathbf{1}}\,\boldsymbol{u}_1, \tilde{\mathbf{2}}\,\boldsymbol{u}_2)$ *are complete,*

(2) $(\tilde{\mathbf{1}}\,\boldsymbol{u}_1, \tilde{\mathbf{2}}\,\boldsymbol{u}_2)$ *is complete* $\Rightarrow (\boldsymbol{u}_1, \boldsymbol{u}_2)$ *is complete,* (u_1, u_2) *is weakly complete.*

Proof.

1. Let $(\tilde{\mathbf{1}}\,\mathfrak{U}_1, \tilde{\mathbf{2}}\,\mathfrak{U}_2)$ be complete. Then $(\mathfrak{U}_1, \mathfrak{U}_2)$ is complete: $\overline{\tilde{\mathbf{2}}\,\mathfrak{U}_2} = \tilde{\mathbf{1}}\,\mathfrak{U}_1 \subset \mathfrak{U}_1 \subset \overline{\mathfrak{U}_2} \subset \overline{\tilde{\mathbf{2}}\,\mathfrak{U}_2}$ (see § 3 (12), § 4: (7), 19, 4.6, (3)), $\mathfrak{U}_1 = \overline{\mathfrak{U}_2}$. Let $(\mathbf{A}_1, \mathbf{A}_2)$ be a complementary pair ($Q := \mathbf{P}$), $x \in P$, then $\mathbf{A}_i \in \tilde{\imath}\,\mathfrak{U}_i$ for some $i \in \{1, 2\}$, but $\tilde{\imath}\,\mathfrak{U}_i \subset x(\tilde{\imath}\,\boldsymbol{u}_i)$, hence $x(\tilde{\imath}\,\boldsymbol{u}_i)\,\mathbf{A}_i$; therefore, $(\tilde{\mathbf{1}}\,\boldsymbol{u}_1, \tilde{\mathbf{2}}\,\boldsymbol{u}_2)$ is complete.

2. Let $(\tilde{\mathbf{1}}\,\boldsymbol{u}_1, \tilde{\mathbf{2}}\,\boldsymbol{u}_2)$ be complete. Then $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ is complete (Lemma 8.6). Let (A_1, A_2) be a complementary pair ($Q := P$), let $\mathbf{A}_i := \{\mathbf{x} \mid \mathbf{x} \in \mathbf{P}, l(\mathbf{x}) \geqslant 1, x_1 \in A_i\}$ for $i = 1, 2$; then $x\boldsymbol{u}_i\mathbf{A}_i$ iff $x[u_i]_P\mathbf{A}_i$ (compare § 3.2), hence $([u_1]_P, [u_2]_P)$ is complete, (u_1, u_2) is weakly complete. Q.E.D.

21.2. Let us note that if (in the situation of Lemma 21.1) $(\tilde{\mathbf{1}}\,\boldsymbol{u}_1, \tilde{\mathbf{2}}\,\boldsymbol{u}_2)$ $(\tilde{\mathbf{1}}\,\mathfrak{U}_1, \tilde{\mathbf{2}}\,\mathfrak{U}_2)$ is complete, then $\tilde{\imath}\,\boldsymbol{u}_i = \boldsymbol{u}_i$ ($\tilde{\imath}\,\mathfrak{U}_i = \mathfrak{U}_i$) for $i = 1, 2$, as there follows from the above.

21.3. Example. $P := \{1, 2\} \times \{0, 1\}$, $P_0 := \{1, 2\} \times \{0\}$, $P_i := \{i\} \times \{0, 1\}$, $\Gamma \in \mathrm{Corr}\,(P, P)$, $\Gamma(1, 1) := \Gamma(2, 1) := P_0$, $\Gamma(1, 0) := \Gamma(2, 0) = \emptyset$, $u_1 \leftarrow (\Gamma^{(-)}, \Gamma^{(+)})$, $u_2 \leftarrow (\Gamma^{(+)}\Gamma^{(-)})$ (at $\mathscr{P} := (P_1, P_2)$). Thus (u_1, u_2) is a weakly complete regular pair ((41), (52)). Let $\mathbf{A}_1 := \{((1, 1), (2, 0)), ((2, 1), (1, 0))\}$, $\mathbf{A}_2 := \mathbf{P} - \mathbf{A}_1$. Then $\mathbf{A}_1 \notin \mathfrak{U}_1$, $\mathbf{A}_2 \notin \mathfrak{U}_2$. Nevertheless, it is clear that even $(\mathring{\boldsymbol{u}}_1, \mathring{\boldsymbol{u}}_2)$ is complete.

21.4. It may happen (for a suitable weakly complete regular pair (u_1, u_2)), that $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ is complete, but $\mathring{\boldsymbol{u}}_i \neq \boldsymbol{u}_i$ for some i, and thus $(\mathring{\boldsymbol{u}}_1, \mathring{\boldsymbol{u}}_2)$ is not complete (see 20.0, 21.1).

21.5. After 21.1—4 there is a natural question: *if* (u_1, u_2) *is a regular weakly complete pair, is* $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ *complete?* (The converse implication holds, see 21.1.) It is known that in general the *answer is negative*, namely, this follows from results of the theory of two-player games with perfect information: Gale and Stewart [14] have proved this in a non-constructive way, using the possibility of well ordering of the continuum. In the following we give several examples based on the same principle; as we shall show they are only special cases of a certain more general example. First of all we prove three lemmata (22.1—3).

22.0. We say that an aim $\boldsymbol{A}$ has the *property* (t) iff $\boldsymbol{A}$ is a non-empty set of infinite variants such that for each $\mathbf{x} \in \boldsymbol{A}$ and for each $k \geqslant 0$ there exists $\mathbf{y} \in \boldsymbol{A}$ with the properties $(x_0, \ldots, x_k) = (y_0, \ldots, y_k)$, $\mathbf{y} \neq \mathbf{x}$.

Let us note that the following lemma is clear, especially if one uses the concepts of § 7a; nevertheless we shall present the proof in the usual style. $\aleph_0$ ($\aleph$) denotes the cardinality of the set of all natural numbers (of the continuum).

22.1. Lemma. *Let an aim* $\boldsymbol{A}$ *have the property* (t). *Then* card $\boldsymbol{A} \geqslant \aleph$.
Proof. To an arbitrary segment $\mathbf{z} = (z_0, \ldots, z_n)$ (at given P, Z) we put $A(\mathbf{z}) := \{x \mid x \in P$, there exists $\mathbf{x} \in \boldsymbol{A}$ such that $(x_0, \ldots, x_n) = (z_0, \ldots, z_n)$, $x_{n+1} = x\}$.
We say that $\mathbf{z}$ is essential iff card $A(\mathbf{z}) \geqslant 2$. Let some mapping $\psi_{\mathbf{z}}$ of $A(\mathbf{z})$ onto $\{1, 2\}$ be given for every essential segment $\mathbf{z}$. Now we define the mapping φ of $\boldsymbol{A}$ into $\{1, 2\}^N$ (where $N := \{1, 2, \ldots\}$) in the following way: let $\mathbf{x} \in \boldsymbol{A}$; then there exists (exactly one) increasing sequence $(k_n)_{n \in N}$, $0 \leqslant k_1 < k_2 < \ldots$, such that $\{k_n \mid n \in N\} = \{k \mid k \in N,\ (x_0, \ldots, x_k)$ is essential $\}$(this is implied by the property (t) of $\boldsymbol{A}$); and we put

$$\varphi(\mathbf{x}) := (\psi_{(x_0, \ldots, x_{k_1})}(x_{k_1+1}),\ \psi_{(x_0, \ldots, x_{k_2})}(x_{k_2+1}) \ldots).$$

It can be easily proved that φ is a mapping of $\boldsymbol{A}$ onto $\{1, 2\}^N$. Hence card $\boldsymbol{A} \geqslant$ card $\{1, 2\}^N = 2^{\aleph_0} = \aleph$. Q.E.D.

22.2. Lemma. *Let* card $P \leqslant \aleph_0$. *Then* card $S \leqslant \aleph$.

Proof. In fact, here card $\boldsymbol{Z}_k \leqslant \aleph_0$ for the set $\boldsymbol{Z}_k$ of all segments of the length k, hence card $\boldsymbol{Z} \leqslant \aleph_0$. Thus card $S =$ card $(\exp P - \{\emptyset\})_{\mathbf{z}} \leqslant$ $\leqslant (2^{\aleph_0})^{\aleph_0} =$ Q.E.D.

22.3. Lemma. *Let* m *be an infinite cardinal number, let* J *be a non-empty set such that* card $J <$ m. *Let* $\mathfrak{U}$ *be a collection in a set* M, *let* card $\mathfrak{U} \leqslant$ m, card $U \geqslant$ m *for all* $U \in \mathfrak{U}$.

Then there exists a partition $(M_j)_{j\in J}$ *of* M *such that* $U \cap M_j \neq \emptyset$ *for all* $U \in \mathfrak{U}$, $j \in J$.

Proof. Let ω° be the smallest ordinal number having the cardinality card $\mathfrak{U}$. There exists a transfinite sequence $(U^\xi)_{0\leqq\xi<\omega^\circ}$ such that $\{U^\xi \mid 0 \leqslant \leqslant \xi < \omega^\circ\} = \mathfrak{U}$. Let us define a system $(m_j^\xi)_{0\leqq\xi<\omega^\circ,\ j\in J}$ of elements of M by the transfinite induction in the following way:

let $0 \leqslant \xi < \omega^\circ$, let m_j^η be defined for all $\eta < \xi, j \in J$, then evidently card $\{m_j^\eta \mid 0 \leqslant \eta < \xi,\ j \in J\} <$ card $\mathfrak{U}$. card $J \leqslant \mathrm{m}^2 = \mathrm{m} \leqslant$ card U^ξ, and thus we (may) define a system $(\mathrm{m}_j^\xi)_{j\in J}$ of mutually distinct elements of the set $U^\xi - \{m_j^\eta \mid 0 \leqslant \eta < \xi,\ j \in J\}$ (hence now the elements m_j^η are defined for $\eta < \xi + 1$, $j \in J$).

Clearly there holds (for $0 \leqslant \xi, \xi_1, \xi_2 < \omega^\circ, j, j_1, j_2 \in J$):

(i) $$m_{j_1}^{\xi_1} = m_{j_2}^{\xi_2} \Rightarrow \xi_1 = \xi_2 \wedge j_1 = j_2,$$

(ii) $$m_j^\xi \in U_\xi.$$

Thus there exists a partition $(M_j)_{j\in J}$ of M such that

(iii) $$m_j^\xi \in M_j,$$

holds for $0 \leqslant \xi < \omega^\circ$, $j \in J$. From (ii), (iii) we conclude that if $U \in \mathfrak{U}$, $j \in J$, then $U \cap M_j \in m_j^\xi$, where ξ is such an ordinal number that $U = = U^\xi$. Q.E.D.

23.0. Now we shall give two simple but interesting examples of the "*strategic absolute incompleteness*". Let us note that the absolute incompleteness of $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ in them follows immediately from the (rather more complicated) example 24.1, too.

23.1. Example. Let $P := \{1, 2, 3\}$, $\mathfrak{A} := \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. It is easy to see that $(\mathfrak{A}, \mathfrak{A})$ is a regular weakly complete pair (*with equal members*!). Let $u = u_1 = u_2 \in \operatorname{Corr}(P, \exp P)$ be such that $xu = \mathfrak{A}$ for each $x \in P$. Then (u_1, u_2) is a regular weakly complete pair of correspondences of the type $(P, \emptyset)$. Clearly, $\mathrm{s}(x, \boldsymbol{\sigma})$ has the property (**t**) for all $\boldsymbol{\sigma} \in \boldsymbol{S}(u)$, $x \in P$. Thus, if we put $\mathfrak{U} := \{\mathrm{s}(x, \boldsymbol{\sigma}) \mid x \in P,\ \boldsymbol{\sigma} \in \boldsymbol{S}(u)\}$, $M := \mathbf{P}$, $J := \{1, 2\}$, then according to 22.1—3 there exists a partition $(\mathbf{A}_1, \mathbf{A}_2)$ of $M(= \mathbf{P})$ such that $U \cap \mathbf{A}_i \neq \emptyset$ for all $U \in \mathfrak{U}$, $i = \{1, 2\}$, i.e. $\mathrm{s}(x, \boldsymbol{\sigma}) \subset \mathbf{A}_i$ for each $x \in P$, $i \in \{1, 2\}$, $\boldsymbol{\sigma} \in \boldsymbol{S}(u) = \boldsymbol{S}(u_i)$. Therefore, $\boldsymbol{u}_1\mathbf{A}_1 \cup \boldsymbol{u}_2\mathbf{A}_2 = \emptyset$, hence $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ is *absolutely incomplete*.

23.2. Example. Let P be a set, $0 <$ card $P \leqslant \aleph_0$, $\Gamma \in \operatorname{Corr}(P, P)$, card $\Gamma x \geqslant 2$ for all $x \in P$. Let $\mathscr{P} = (P_1, P_2)$ be a partition of P such that Γ is $\mathscr{P}$-alternative. We define $u_1 \leftarrow (\Gamma^{(-)}, \Gamma^{(+)})$, $u_2 \leftarrow (\Gamma^{(+)}, \Gamma^{(-)})$ (at $\mathscr{P}$), then (u_1, u_2) is a weakly complete regular pair, and $([u_1]_P, [u_2]_P)$ is the pair of game correspondences of some *Bergean two-player game* *with perfect information* (see 17.3); we can prove that $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ *is absolutely incomplete* in a similar way as in example 23.1, namely, we put $\mathfrak{U} := \{\mathrm{s}(x, \boldsymbol{\sigma}) \mid x \in P, i \in \{1, 2\}, \boldsymbol{\sigma} \in S(u_i)\}$ etc. The case $\operatorname{card} P_1 = \operatorname{card} P_2 = 2$ can be considered as a certain *classic example* (compare [14] and [16]).

24.0. On the other hand, if $\operatorname{card} P = 1$ and (u_1, u_2) is a regular pair of correspondences, then clearly $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ is complete. Thus there is a natural question: does there exist a regular weakly complete pair (u_1, u_2) at $\operatorname{card} P = 2$ such that $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ is absolutely incomplete? — Example 24.2, based on Lemma 24.1 (which can be rather generalized) gives the positive answer to that question.

24.1. Lemma. *Let* (u_1, u_2) *be a regular pair of the type* (P, P_0), *let* $P^i := \{x \mid x \in P, xu_i$ *contains a one-element set*$\}$ *for* $i = 1, 2$. *If there holds*

(i) $$P_0 = \emptyset,$$

(ii) $$0 < \operatorname{card} P \leqslant \aleph_0,$$

(iii) $$P^1 \cap P^2 = \emptyset,$$

(iv) $i \in \{1, 2\}$, $x \in P^i \Rightarrow$ *there exists* $A \in xu_i$ *such that* $A \cap P^i = \emptyset$,

then the pair $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ *is absolutely incomplete.*

Proof. Under the suppositions we put

$$\mathsf{P}^i := \{\mathsf{x} \mid \mathsf{x} \in \mathsf{P}, \text{ there exists } k \text{ such that } x_k, x_{k+1}, \ldots, \in P^i\}$$

for $i = 1, 2$; clearly $\mathsf{P}^1 \cap \mathsf{P}^2 = \emptyset$. Let $\mathsf{P}^0 := \mathsf{P} - (\mathsf{P}^1 \cup \mathsf{P}^2)$. From the suppositions it follows clearly that (for $x \in P$, $i \in \{1, 2\}$)

$$\boldsymbol{\sigma} \in S(u_i), \quad \mathrm{s}(x, \boldsymbol{\sigma}) \subset \mathsf{P} - \mathsf{P}^i \Rightarrow \mathrm{s}(x, \boldsymbol{\sigma}) \cap \mathsf{P}^0 \text{ has the property (t)}$$

hence, the cardinality of each set of the collection

$$\mathfrak{U} := \{\mathrm{s}(x, \boldsymbol{\sigma}) \cap \mathsf{P}^0 \mid x \in P, \quad i \in \{1, 2\}, \quad \boldsymbol{\sigma} \in S(u_i), \quad \mathrm{s}(x, \boldsymbol{\sigma}) \subset \mathsf{P} - \mathsf{P}^i\}$$

is at least $\aleph$ (Lemma 22.1); clearly, $\operatorname{card} \mathfrak{U} \leqslant \operatorname{card} P \cdot \operatorname{card} S(u) \leqslant \aleph_0 \cdot \aleph = \aleph$ (Lemma 22.2). Therefore, if we put $M := \mathsf{P}^0$, $J := \{1, 2\}$, then according to 22.3 there exists a partition (M_1, M_2) of M such that $U \cap M_1 \neq \emptyset \neq U \cap M_2$ for all $U \in \mathfrak{U}$. Let $\boldsymbol{A}_i := M_i \cup \mathsf{P}^{3-i}$ for $i \in \{1, 2\}$. Then $(\boldsymbol{A}_1, \boldsymbol{A}_2)$ is a complementary pair of aims. If $x \in \boldsymbol{u}_1\boldsymbol{A}_1 \cup \boldsymbol{u}_2\boldsymbol{A}_2$, then there exist $i \in J$, $\boldsymbol{\sigma} \in S(u_i)$ such that $\mathrm{s}(x, \boldsymbol{\sigma}) \subset \boldsymbol{A}_i \subset \mathsf{P} - \mathsf{P}^i$, hence $\mathrm{s}(x, \boldsymbol{\sigma}) \cap \mathsf{P}^0 \in \mathfrak{U}$, and $\emptyset \neq (\mathrm{s}(x, \boldsymbol{\sigma}) \cap \mathsf{P}^0) \cap M_{3-i} \subset \mathrm{s}(x, \boldsymbol{\sigma}) \cap \boldsymbol{A}_{3-i}$, thus $\mathrm{s}(x, \boldsymbol{\sigma}) \not\subset \boldsymbol{A}_i$, which is a contradiction. Therefore $\boldsymbol{u}_1\boldsymbol{A}_1 \cup \boldsymbol{u}_2\boldsymbol{A}_2 = \emptyset$. Q.E.D.

24.2. Let P be a set, $0 < \operatorname{card} P \leqslant \aleph_0$, $\Gamma \in \operatorname{Corr}(P, P)$, $\operatorname{card} \Gamma x \geqslant 2$ for all $x \in P$. Let $\mathscr{P} = (P_1, P_2)$ be a partition of P such that $\Gamma x \not\subset P_i$

whenever $x \in P_t$. We define $u_1 \leftarrow (\Gamma^{(-)}, \Gamma^{(+)})$, $u_2 \leftarrow (\Gamma^{(+)}, \Gamma^{(-)})$ (at $\mathscr{P}$); therefore, 24.2 is a special case of 23.2. (u_1, u_2) is a regular weakly complete pair (and $([u_1]_P, [u_2]_P)$ is a pair *with perfect information*). Evidently, the suppositions of Lemma 24.1 are satisfied. Therefore $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ is *absolutely incomplete*. Let us note that here it is possible to choose P, Γ, $\mathscr{P}$ with card $P = 2$.

§ 5. THE FUNDAMENTAL PROBLEMS. ON IDEAS AND METHODS

a) The fundamental problems. Notes on methods

0. In this part **a** we shall formulate the *fundamental problems* investigated in this work. We suppose: (P, P_0) is a type, (u_1, u_2) denotes a *regular weakly complete pair* of elements of Corr $(P, \exp P)$, u denotes an *R-correspondence* of Corr $(P, \exp P)$, $\sim$, $\tilde{1}$, $\tilde{2}$ are memory relations, $u, u_1, u_2, \sim, \tilde{1}, \tilde{2}$ have the type (P, P_0); $(\mathbf{A}_1, \mathbf{A}_2)$ denotes a *complementary pair of aims*, $\mathbf{A}$ denotes an *aim* $(\mathbf{P} := \mathbf{P}_{(P,P_0)})$. Let us denote two equalities:

$$(*) \qquad \tilde{1}\ \boldsymbol{u}_1\mathbf{A}_1 \cup \tilde{2}\ \boldsymbol{u}_2\mathbf{A}_2 = P,$$

(which means that $(\tilde{1}\ \boldsymbol{u}_1, \tilde{2}\ \boldsymbol{u}_2)$ is complete on $(\mathbf{A}_1, \mathbf{A}_2)$),

$$(**) \qquad \sim \boldsymbol{u}\mathbf{A} = \boldsymbol{u}\mathbf{A}.$$

1. The equality (*) need not hold (as the principal examples show, it may happen even $\boldsymbol{u}_1\mathbf{A}_1 \cup \boldsymbol{u}_2\mathbf{A}_2 = \emptyset$). Therefore, we can consider the following *two-player problems*:

(I, $\tilde{1}$, $\tilde{2}$) *To give suitable sufficient conditions on* (u_1, u_2) *for the satisfaction of* (*) *at all* $(\mathbf{A}_1, \mathbf{A}_2)$.

(II, $\tilde{1}$, $\tilde{2}$) *To give suitable sufficient conditions on* $(\mathbf{A}_1, \mathbf{A}_2)$ *for the satisfaction of* (*) *at all* (u_1, u_2).

2. Similarly, the equality (**) need not hold (see e.g. § 4.20). Therefore, we can consider the following *one-player problems*:

(III, $\sim$) *To give suitable sufficient conditions on u for the satisfaction of* (**) *at all* $\mathbf{A}$.

(IV, $\sim$) *To give suitable sufficient conditions on* $\mathbf{A}$ *for the satisfaction of* (**) *at all u.*

3. Further, a $(\sim, \{\mathbf{A}\})$-absolute u-strategy need not exist (see § 3.8.4). Therefore, we can consider the following *problems*:

(V, $\sim$) *To give suitable sufficient conditions on u for the existence of a* $(\sim, \{\mathbf{A}\})$*-absolute u-strategy at all* $\mathbf{A}$.

(VI, $\sim$) *To give suitable sufficient conditions on* $\mathbf{A}$ *for the existence of a* $(\sim, \{\mathbf{A}\})$*-absolute u-strategy at all u.*

4. We shall write (K) $((\overset{\circ}{K}))$ instead of $(K, =, =)$ $((K, \overset{\circ}{\sim}, \overset{\circ}{\sim}))$ for $K \in \{\mathrm{I}, \mathrm{II}\}$, and instead of $(K, =)$ $((K, \overset{\circ}{\sim}))$ for $K \in \{\mathrm{III}, \mathrm{IV}, \mathrm{V}, \mathrm{VI}\}$. (The meaning of $=$, $\overset{\circ}{\sim}$ is the same as in § 1.2.)

In this work only the cases with $\tilde{1}$, $\tilde{2}$, $\sim \in \{=, \overset{\circ}{\sim}\}$ are investigated. Especially, then for $K \in \{\mathrm{III}, \mathrm{IV}, \mathrm{V}, \mathrm{VI}\}$ only the problems $(\overset{\circ}{K})$ are considered ((K) are trivial for these K).

5.0. Let us note that the "odd" problems are little interesting, as they need considerably strong sufficient conditions; we present only two results concerning them, namely very simple Lemma 5.1 (for (III)), and in § 7 (a theorem about the "*locally finite case*", for (I)).

5.1. Lemma. *Let the graph* Γ *of (the R-correspondence) u have the following property:*

If $(x_0, \ldots, x_m)$, $(y_0, \ldots, y_n)$ $(m, n \geqslant 0)$ *are (finite) sequences of elements of P such that* $x_r \in \Gamma x_{r-1}$, $y_s \in \Gamma y_{s-1}$ $(r = 1, \ldots, m;\ s = 1, \ldots, n)$, $x_0 = y_0$, $x_m = y_m$, *then* $m = n$, $x_k = y_k$ $(k = 0, \ldots, m)$. (*I.e. for two arbitrary positions x, y there is at most one path from x to y at the graph* Γ.)
Then

$$\mathring{u} = u.$$

(The *proof* is simple.)

6. The most important role play the problems (II, $\tilde{1}$, $\tilde{2}$); namely, (*) implies $\tilde{i}\ u_i A_i = u_i A_i$ for $i = 1, 2$ (§ 4.8.6 etc.), therefore, $(\overset{\circ}{\mathrm{IV}})$ will be considered in a near coherence with (II, ...).

The often used theorem 11 of § 3 is the most important result for $(\overset{\circ}{\mathrm{VI}})$; some more special results for this problem are obtained at the results for (II, ...) derived in § 6. (Compare part c.)

7. As we have mentioned above, the problems (II, ...) can be considered as the most important ones. At investigations of them there are two kinds of methods: to investigate either a *given game structure*, or an *auxiliary game structure* constructed in a suitable way to the given game structure. The latter alternative is used in § 7, namely a certain "extensive" form is introduced there in connection with the fact that the set of all variants can be in a natural way *topologized* (even metrized, namely, similarly as the Baire's space); the results of § 7 concern only the problem (II). The former alternative is used in § 6, and § 5c describes the so called *idea of active and passive aims* used for these purposes. This idea must be considered as main (for these problems) in this paper: namely, also results of § 7 are obtained by applications of that idea to the auxiliary game structures. Moreover, the use of the idea of active and passive aims together with the method of *successive approximations* (see part **d**) gives, without any applications of the theorem 11 of § 3, results concerning the problem (VI).

b) Aim-functions and aim-correspondences. Aim-modifications

Let (P, P_0) be a fixed type, $\boldsymbol{P} := \boldsymbol{P}_{(P,P_0)}$, let u, $v \in \operatorname{Corr}(P, \exp P)$.

8. The problems (II, ...) concern some complementary pairs of aims. Aims of such pairs can be given e.g. as (arbitrary) aims having certain properties, or as (arbitrary) values of suitable mappings, i.e. as aims depending on some "parameters". Such a mapping (i.e. a mapping into $\exp \boldsymbol{P}$) will be called an *aim-function.* Instead of mappings of $\exp P$ into $\exp \boldsymbol{P}$ we can consider elements of $\operatorname{Corr}(\boldsymbol{P}, \exp P)$ (compare § 2.7) which will be called *aim-correspondences.* We introduce a special symbolization for them: an aim-correspondence will be denoted by symbol $\boldsymbol{p}^{\varepsilon}$, where $^{\varepsilon}$ is a symbol with the distinction whether it is written above or below (therefore, only the symbols $^{\varepsilon}$ distinguish distinct aim-correspondences), e.g. if we substitute $^{\varepsilon} := {}_{\varepsilon_0}$, then $\boldsymbol{p}^{\varepsilon} = \boldsymbol{p}_{\varepsilon_0}$; further, we shall always consider no single aim-correspondences but pairs of them, namely $\boldsymbol{p}^{\varepsilon}$, $\boldsymbol{p}_{\varepsilon}$, and our denotations will always satisfy the condition

$$\boldsymbol{p}_{\varepsilon} = \overline{\boldsymbol{p}^{\varepsilon}},$$

i.e.

$$\boldsymbol{p}_{\varepsilon} A = \boldsymbol{P} - \boldsymbol{p}^{\varepsilon}(P - A)$$

(see § 4 (30)). Thus it is always sufficient to define only one of aim-correspondences $\boldsymbol{p}^{\varepsilon}$, $\boldsymbol{p}_{\varepsilon}$.

9. Let u be a regular correspondence of the type (P, P_0), let $\sim$ be a memory relation of this type, $\boldsymbol{p}^{\varepsilon} \in \operatorname{Corr}(P, \exp P)$. Then we can define

$$\sim u^{\varepsilon} := (\sim \boldsymbol{u} \,.\, \boldsymbol{p}^{\varepsilon}),$$

hence $\sim u^{\varepsilon} \in \operatorname{Corr}(P, \exp P)$ (§ 2.6.1), and if $\boldsymbol{p}^{\varepsilon}$ is an R-correspondence (M-correspondence), then $\sim u^{\varepsilon}$ has this property, too (§ 2 (33), (34)). Instead of $\mathring{\sim} u^{\varepsilon}$ ($= u^{\varepsilon}$) we shall write $\mathring{u}^{\varepsilon}$ (u^{ε}), too (compare § 3.6). The correspondence $\sim u^{\varepsilon}$ (u^{ε}, $\overset{\circ}{u}{}^{\varepsilon}$) will be called the ($\sim$, $^{\varepsilon}$)-*modification* ($^{\varepsilon}$-modification, plain $^{\varepsilon}$-modification) of u. Correspondences obtained in such a way will be sometimes called $\sim$-*strategic* (strategic, plain strategic) *modifications*, or only *aim-modifications.*

10.1. Let us note that if (u, v) is a regular pair fo the type (P, P_0), and $(\tilde{1}, \tilde{2})$ is a pair of memory relations of this type, then (for an arbitrary $\boldsymbol{p}^\varepsilon \in \mathrm{Corr}\,(\mathrm{P}, \exp P))$ the assertion

$(\tilde{1}\ \boldsymbol{u}, \tilde{2}\ \boldsymbol{v})$ is complete on all pairs $(\boldsymbol{p}^\varepsilon A, \boldsymbol{p}_\varepsilon(P - A))$ $(A \subset P)$

is equivalent to the equality

$$\tilde{1}\ u^\varepsilon = \overline{\tilde{2}\ v_\varepsilon},$$

(in fact, $\tilde{1}\ u^\varepsilon A = \tilde{1}\ \boldsymbol{u} \,.\, \boldsymbol{p}^\varepsilon A$, $\overline{\tilde{2}\ v_\varepsilon} A = P - (\tilde{2}\ v_\varepsilon(P - A)) = P - \tilde{2}\ \boldsymbol{v} \,.\, \boldsymbol{p}_\varepsilon(P - A) = P - \tilde{2}\ \boldsymbol{v}(\boldsymbol{P} - \boldsymbol{p}^\varepsilon A))$, and therefore this equality implies (see § 4.8.6)

$$\tilde{1}\ u^\varepsilon = u^\varepsilon, \qquad \tilde{2}\ v_\varepsilon = v_\varepsilon.$$

10.2. Therefore, the assertion
for every weakly complete regular pair (u, v) of the type (P, P_0)
$(\tilde{1}\ \boldsymbol{u}, \tilde{2}\ \boldsymbol{v})$ is complete on all $(\boldsymbol{p}^\varepsilon A, \boldsymbol{p}_\varepsilon(P - A))$ $(A \subset P)$

(which concerns the problem $(\mathrm{II}, \tilde{1}, \tilde{2})$) is equivalent to the assertion

$\tilde{1}\ u^\varepsilon = \overline{\tilde{2}\,(u')_\varepsilon}$ for all game correspondences u having the type (P, P_0)

(see § 4.4.8).

c) The idea of active and passive aims

11. Let the suppositions of § 5.0 be satisfied, let $A, B, X \subset P$. The *idea of active and passive aims* consists in such choices of (P)-collections $\mathfrak{A}_1$, $\mathfrak{A}_2$ and of $j \in \{1, 2\}$ that for $i := 3 - j$ there holds

(C) $$X \in \mathfrak{A}_j \Leftrightarrow P - X \in \mathfrak{A}_i,$$

(i.e. $\mathfrak{A}_j = (\exp P) - \overline{\mathfrak{A}_i} = \overline{(\exp P) - \mathfrak{A}_i}$),

(P) $$B \in \mathfrak{A}_j \Rightarrow B \subset \tilde{j}\ \boldsymbol{u}_j\boldsymbol{A}_j,$$

(A) there exists $A \in \mathfrak{A}_i$ such that $A \subset \tilde{i}\ \boldsymbol{u}_i\boldsymbol{A}_i$.

If these suppositions are satisfied, $B := P - A$, then $B \in \mathfrak{A}_j$, $B \subset \tilde{j}\ \boldsymbol{u}_j\boldsymbol{A}_j \subset P - \tilde{i}\ \boldsymbol{u}_i\boldsymbol{A}_i \subset P - A = B$, hence

$$A = \tilde{i}\ \boldsymbol{u}_i\boldsymbol{A}_i = \boldsymbol{u}_i\boldsymbol{A}_i,$$

$$B = \tilde{j}\ \boldsymbol{u}_j\boldsymbol{A}_j = \boldsymbol{u}_j\boldsymbol{A}_j,$$

$$\tilde{i}\ \boldsymbol{u}_i\boldsymbol{A}_i \cup \tilde{j}\ \boldsymbol{u}_j\boldsymbol{A}_j = P;$$

consequently, if there is some $A_0 \subset A$ such that $A_0 \in \mathfrak{A}_i$, then $A_0 \subset \subset \tilde{i}\ u_i A_i$, and again $A_0 = u_i A_i = A$, i.e. there holds

$$\begin{cases} A \\ B \end{cases} \text{is the} \begin{cases} \text{smallest} \\ \text{greatest} \end{cases} \text{element of} \begin{cases} \mathfrak{A}_i \\ \mathfrak{A}_j \end{cases} \quad (\text{under} \subset).$$

In this situation the aim $\mathbf{A}_i$ ($\mathbf{A}_j$) is called *active* (*passive*). (This terminology is chosen for certain connections with the pay-off functions which Berge ([1]) has introduced at so called active (passive) players, compare § 9.)

If instead of (P), (A) the following stronger conditions

(P*) $B \in \mathfrak{A}_j \Rightarrow$ there exists $\boldsymbol{\sigma}_j \in \tilde{j}\ S(u_j)$ such that $\mathrm{s}(x, \boldsymbol{\sigma}_j) \subset \mathbf{A}_j$ for all $x \in B$,

(A*) there exist $A \in \mathfrak{A}_i$, $\boldsymbol{\sigma}_i \in \tilde{i}\ S(u_i)$ such that $\mathrm{s}(x, \boldsymbol{\sigma}_i) \subset \mathbf{A}_i$ for all $x \in A$,

are satisfied, then clearly moreover there holds:

a ($\tilde{\iota}$, $\{\mathbf{A}_\iota\}$)-absolute u_ι-strategy exists for $\iota = 1, 2$.

In this form the idea of active and passive aims will be applied in § 6.

12. In this work *two possibilities of applications* of the idea of active and passive aims will be used — after choosing $\mathfrak{A}_1$, $\mathfrak{A}_2$ (usually as the collections of all fixpoints of suitable correspondences) and proving the satisfaction of (C), (P*) to verify the satisfaction of (A*) in such a way:

either to put $A := \tilde{i}\ u_i \mathbf{A}_i$ and to verify $A \in \mathfrak{A}_i$, usually by means of constructions of suitable $\tilde{i}$-acceptable u_i-strategies; for these constructions there are used results of § 3b, in particular, those giving the existence of a ($\tilde{\iota}$, $\{\mathbf{A}_i\}$)-absolute u_i-strategy,

or to construct suitable $A \in \mathfrak{A}_i$, $\boldsymbol{\sigma}_i \in \tilde{i}\ S(u_i)$, and to verify $\mathrm{s}(x, \boldsymbol{\sigma}) \subset \subset \mathbf{A}_i$ for all $x \in A$, usually in such a way: $\mathfrak{A}_i$ has been given as the set of all fixpoints of some correspondence, and A is constructed as its smallest fixpoint by means of "successive approximations", (compare § 5d), moreover, at this construction also a suitable $\boldsymbol{\sigma}_i$ is constructed.

Let us note that (at fixed (u_1, u_2), $(\mathbf{A}_1, \mathbf{A}_2)$, $(\mathfrak{A}_1, \mathfrak{A}_2)$) often both the possibilities can be used; proofs at the second of them are more complicated, but the first possibility gives the existence of the "active" ($\tilde{i}$, $\{\mathbf{A}_i\}$)-absolute u_i-strategy only non-constructively.

d) Fixpoints of correspondences. Successive approximations

13. In this part **d** let P be a set, X, $Y \subset P$, $w \in \mathrm{Corr}\ (P, \exp P)$, let $(w_m)_{m \in M}$ be a system of elements of Corr $(P, \exp P)$. $A \parallel B$ (for sets A, B) means $A \not\subset B \wedge B \not\subset A$, $A \nparallel B$ means $A \subset B \vee B \subset A$.

14. We say that A is a *fixpoint* of w (a *common fixpoint* of $(w_m)_{m\in M}$) iff $A = wA$ ($A = w_m A$ for all $m \in M$). The meanings of phrases of the type "A is the smallest (greatest) (common) fixpoint of ..." are clear ($\subset$ acts as the partial ordering). Evidently

$$A \text{ is } \begin{cases} \text{a} \\ \text{the smallest} \\ \text{the greatest} \end{cases} \text{common fixpoint of } (w_m)_{m\in M} \Leftrightarrow$$

$$\Leftrightarrow P - A \text{ is } \begin{cases} \text{a} \\ \text{the greatest} \\ \text{the smallest} \end{cases} \text{common fixpoint of } (\overline{w_m})_{m\in M}$$

Thus, if in § 5.11 e.g. $\mathfrak{A}_i$ is given as the set of all common fixpoints of $(w_m)_{m\in M}$, then $\mathfrak{A}_j (= \{P - X \mid X \in \mathfrak{A}_i\})$ is the set of all common fixpoints of $(\overline{w_m})_{\in mM}$.

15.0. First of all let us consider the case of a single M-correspondence (denoted by w). This can be considered also as the case of an isotone mapping of the complete lattice (exp P, $\subset$) into itself (compare § 2.8). According to well-known results (see e.g. [13], chap. IV, Theorem 8), the *extreme fixpoints* of w exist, namely, $\bigcup\limits_{\substack{X\subset P \\ X\subset wX}} X$ ($\bigcap\limits_{\substack{X\subset P \\ X\supset wX}} X$) is the smallest (greatest) fixpoint of w. But for our purposes (see § 5.12) we must introduce the "*successive approximations*" (by means of a certain transfinite iterating of w) of these fixpoints (this idea is well-known, too; cf. e.g. [13], ch. IV, § 1, exercise 8).

15.1 Let w be an *M-correspondence*. Letters ξ, η will denote ordinal numbers, ∞ is an auxiliary symbol which is *not* an ordinal number, ζ denotes either an ordinal number or ∞. If $X \subsetneqq wX$, then $w^\xi X$ is defined for all ξ (but w^ξ need not be defined, compare § 5e) by the transfinite induction:

$$w^0 X := X,$$

$$w^\xi X := \mathop{\mathcal{O}}_{0\leq\eta<\xi} ww^\eta X \qquad (\xi > 0),$$

where $\mathcal{O} = \begin{cases}\bigcup \\ \bigcap\end{cases}$ (if $X = wX$, then $X = w^\xi X$ for all ξ in both choices of $\mathcal{O}$). Let us mention several important properties:

$$\xi_1 \leqslant \xi_2 \Rightarrow w^{\xi_1} X \subsetneqq w^{\xi_2} X, \tag{1}$$

therefore, we may write

$$w^\xi X = \lim_{0\leq\eta<\xi} ww^\eta X \qquad (\xi > 0) \tag{2}$$

(see [13], chap. IV, § 8);

(3) $$ww^{\xi}X = w^{\xi+1}X,$$

(4) $$w^{n}X = \underbrace{w\ \ldots\ldots\ wX}_{n\ \text{times}} \qquad (0 < n < \omega_0),$$

(5) $$w^{\xi}X = \lim_{0 \leqq \eta < \xi} w^{\eta}X \qquad \text{if} \qquad \xi \text{ is limit},$$

(6) $$w^{\xi_1+\xi_2}X = w^{\xi_2}w^{\xi_1}X,$$

(7) $$w^{\xi_1}X = w^{\xi_2}X,\quad \xi_1 < \xi_2 \Rightarrow w^{\xi}X = w^{\xi_1}X \quad \text{for all } \xi \geqslant \xi_1,$$

(8) $$w^{\xi_1}X = w^{\xi_2}X,\ w^{\xi_3}X = w^{\xi_4}X,\ \xi_1 < \xi_2,\ \xi_3 < \xi_4 \Rightarrow w^{\xi_1}X = w^{\xi_3}X,$$

(9) $$w^{\xi_1}X \neq w^{\xi_2}X,\quad \xi_1 < \xi_2 \Rightarrow w^{\xi}X \neq w^{\xi_2}X \quad \text{for all} \quad \xi < \xi_2,$$

(10) $$\operatorname{card} \xi > \operatorname{card} \exp P \Rightarrow w^{\xi}X = w^{\xi+1}X.$$

Thus we can denote

$$w^{\infty}X := w^{\xi_0}X\ (= w^{\xi_0+1}X),$$

where ξ_0 is arbitrary such that $w^{\xi_0}X = w^{\xi_1+1}X$; therefore

(11) $$w^{\zeta}w^{\infty}X = w^{\infty}X,$$

in particular, $ww^{\infty}X = w^{\infty}X = w^{\infty}w^{\infty}X$.

(We have not presented proofs, as they are simple.)

15.2. Thus we have defined $w^{\zeta}X$ if $X \mathrel{\not\parallel} wX$. It can be easily proved (cf. (2))

(12) $$X \mathrel{\not\parallel} wX,\ Y \mathrel{\not\parallel} wY,\ X \subset Y \Rightarrow w^{\zeta}X \subset w^{\zeta}Y \text{ for all } \zeta$$

Therefore

(13) $$X \subsetneqq wX \Rightarrow w^{\infty}X \text{ is the } \begin{cases}\text{smallest}\\ \text{greatest}\end{cases} \text{of } Y \text{ such that } Y = wY,\ Y \supsetneqq X$$

(namely, if e.g. $X \subset wX$, then $w^{\infty}X = ww^{\infty}X$, $w^{\infty}X \supset w^{\circ}X = X$, and if $Y = wY$, $Y \supset X$, then $Y = w^{\infty}Y \supset w^{\infty}X$, see (11), (1), (12)). Especially

(13′) $$\begin{cases}w^{\infty}\emptyset\\ w^{\infty}P\end{cases} \text{is the} \quad \begin{cases}\text{smallest}\\ \text{greatest}\end{cases} \text{fixpoint of } w.$$

By a transfinite induction one can easily obtain

(14) $$X \subsetneqq wX \Rightarrow w^{\xi}X = \begin{cases}(1 \cup w)^{\xi}X,\\ (1 \cap w)^{\xi}X.\end{cases}$$

15.3. $\bar{w}$ is an M-correspondence, too (§ 4 (26)); there holds: $P - X \subseteq w(P - X)$ iff $X \supseteq \bar{w}X$, hence $\bar{w}^{\xi}X$ is defined iff $w^{\xi}(P - X)$ is defined, and clearly (see (1), (2), § 4.10)

$$\bar{w}^{\xi}X = P - w^{\xi}(P - X) \qquad (\text{where } X \nparallel \bar{w}X). \tag{15}$$

16.0. Let us note that if w is an M-correspondence, $X \subseteq wX$, then $w^{\infty}X$ is the $\begin{cases}\text{smallest}\\ \text{greatest}\end{cases}$ common fixpoint of the two M-correspondences w, $\begin{cases}\mathbf{1} \cup X\\ \mathbf{1} \cap \underline{\overline{X}}\end{cases}$ (§ 2.11). Of course, there are non-empty systems of M-correspondences of Corr $(P, \exp P)$ having no common fixpoint. Nevertheless, sometimes the common fixpoints of a system of correspondences are exactly the fixpoints of a suitable correspondence. We shall need only the following simple result (which can be easily generalized):

16.1. Lemma. *If* $w_1, w_2 \in$ Corr $(P, \exp P)$,

$$w_2 \,.\, w_1 \subset w_1 \,.\, w_2 \,.\, w_1 \subset w_2 \,.\, w_1 \,.\, w_2 \,.\, w_1,$$

then X *is a common fixpoint of* w_1, w_2 *iff* X *is a fixpoint of* $w_2 \,.\, w_1$.

Proof. Let $w := w_2 \,.\, w_1$. If $X = w_1X = w_2X$, then $X = wX$. On the other hand, if $X = wX$, then $wX \subset w_1wX \subset wwX = wX = X$, hence $X = w_1wX = w_1X$, $w_2X = w_2w_1X = wX = X$. Q.E.D.

e) Transfinite iterations of certain correspondences

17.1. In § 5.17 let $w \in$ Corr $(P, \exp P)$ be an *M-correspondence.* If $w \nparallel \mathbf{1}$ (i.e. if either $w \supset \mathbf{1}$ or $w \subset \mathbf{1}$), then $w^{\zeta}X$ is defined for all $X \subset P$ and all ζ, and we can define $w\zeta$ as the element of Corr $(P, \exp P)$ for which

$$(w^{\zeta})\,X = w^{\zeta}X,$$

(where $(w^{\zeta})\,X$ is defined by § 2.3, but $w^{\zeta}X$ is defined by § 5.15.1) for all $X \subset P$. It is easy to see that if $w \subseteq \mathbf{1}$, then we get valid formulae from (1)—(11) (of § 4d) by omitting X and replacement of ww^{η}, ww^{ξ}, $w^{\xi_2}w^{\xi_1}$, $w^{\zeta}w^{\infty}$ by $w \,.\, w^{\eta}$, $w \,.\, w^{\xi}$, $w^{\xi_2} \,.\, w^{\xi_1}$, $w^{\zeta} \,.\, w^{\infty}$, respectively, in them. (12) implies that

$$w^{\xi} \text{ is an } M\text{-correspondence} \qquad (\text{where } \mathbf{1} \nparallel w). \tag{16}$$

By means of (13) we conclude

$$\mathbf{1} \subseteq w \Rightarrow w^{\infty} \text{ is the } \begin{cases}\text{smallest}\\ \text{greatest}\end{cases} \text{ of all } v \text{ such that } v = w \,.\, v,\ v \supseteq \mathbf{1}, \tag{17}$$

where $v \in$ Corr $(P, \exp P)$.

Here let us yet mention some interesting remark: Let $Q := P \times \exp P$, let φ be the trivial mapping (§ 2.4) of $\exp\,(P \times \exp P)$ onto $\mathrm{Corr}\,(P, \exp P)$. For any $u \in \mathrm{Corr}\,(P, \exp P)$ let $W_u \in \mathrm{Corr}\,(Q, \exp Q)$ be such that for each $\nu \in \exp Q$ $W_u \nu = \varphi^{-1}(u \,.\, \varphi(\nu))$. Then (among others) the construction of w^∞ can be also given as (in substance) that introduced in 15.1 but taken with "exchanges" $P := Q$, $w := W_w$, $X := \varphi^{-1}(\mathbf{1})$: if w is an M-correspondence, then W_w is also an M-correspondence (this is clear), and if $\mathbf{1} \subseteq w$, then $\varphi^{-1}(\mathbf{1}) \subseteq \varphi^{-1}(w)$, $(W_w)^\infty\, \varphi^{-1}(\mathbf{1})$ is defined, but (13) and (17) easily show that $\varphi^{-1}(w^\infty) = (W_w)^\infty\, \varphi^{-1}(\mathbf{1})$; hence $w^\infty = \varphi((W_w)^\infty\, \varphi^{-1}(\mathbf{1}))$. (If one identifies each $\nu \subset Q$ with $\varphi(\nu)$, then it would be possible to write directly $w^\infty = (W_w)^\infty \mathbf{1}$.)

17.2. Let us note that

$$(18)\quad \mathbf{1} \subseteq w \Rightarrow w^\infty \text{ is the } \begin{cases} \text{smallest} \\ \text{greatest} \end{cases} \text{ of all } v \text{ such that } v \,.\, v = v,\ v \supseteq w,$$

where $v \in \mathrm{Corr}\,(P, \exp P)$. (In fact, $w^\infty \,.\, w^\infty = w^\infty$ (see above), and if $v \,.\, v = v \supseteq w$, then $w^0 = \mathbf{1} \subseteq w \subseteq v$; if $w^\eta \subseteq v$ for $0 \leqslant \eta < \xi$, then $w^\xi = \lim\limits_{0 \leqq \eta < \xi} w \,.\, w^\eta \subseteq w \,.\, v \subseteq v \,.\, v = v$, hence $w^\infty \subseteq v$.)

17.3. $\bar{w}$ is an M-correspondence, too (§ 4 (26)), and if $w \subseteq \mathbf{1}$, then $\bar{w} \subseteq \bar{\mathbf{1}} = \mathbf{1}$ (§ 4 (23), (47)), therefore $(\bar{w})^\zeta$ is defined (for all ζ), and (15) implies

$$(19)\qquad (\bar{w})^\zeta = \overline{w^\zeta} \qquad (\text{where } \mathbf{1} \nparallel w).$$

18.1. Let $u \in \mathrm{Corr}\,(P, \exp P)$ be an *M-correspondence* in § 5.18. Then we can put

$$u^\triangle := (\mathbf{1} \cup u)^\infty,$$

$$u_\nabla := (\mathbf{1} \cap u)^\infty,$$

$$u_\triangle := (\tilde{u})_\nabla = (\mathbf{1} \cap \tilde{u})^\infty,$$

$$u^\nabla := (\tilde{u})^\triangle = (\mathbf{1} \cup \tilde{u})^\infty,$$

as $\mathbf{1} \cup u$, $\mathbf{1} \cap u$ are M-correspondences comparable with $\mathbf{1}$, $\tilde{u}$ is an M-correspondence (§ 2; § 4 (26); § 5.17.1). There holds

$$(20)\qquad \overline{u^\triangle} = (\bar{u})_\nabla, \qquad \overline{u_\nabla} = (\bar{u})^\triangle,$$

(in fact, $\overline{u^\triangle} = \overline{(\mathbf{1} \cup u)^\infty} = (\overline{\mathbf{1} \cup u})^\infty = (\mathbf{1} \cap \bar{u})^\infty = (\bar{u})_\nabla$, see (19), § 4 (48), hence $(\bar{u})^\triangle = \overline{\overline{(\bar{u})^\triangle}} = \overline{(\bar{\bar{u}})_\nabla} = \overline{u_\nabla}$, see § 4 (22)),

$$(21)\qquad u_\triangle = \overline{(u')^\triangle}, \qquad u^\nabla = \overline{(u')_\nabla},$$

(e.g. $u_\triangle = \overline{\overline{\overline{(\mathbf{1} \cap \bar{u}')^\infty}}} = \overline{(\mathbf{1} \cup \overline{\overline{u'}})^\infty} = \overline{(\mathbf{1} \cup u')^\infty} = \overline{(u')^\triangle}$, see § 4 (22), § 2.5.3, § 5 (19), § 4 (48), similarly $u^\nabla = \overline{(u')_\nabla}$). Further $(\mathbf{1} \cap \tilde{u})\, P = (\mathbf{1} \cup u)\, P = (\mathbf{1} \cup \tilde{u})\, P = P$ (as $\tilde{u}P = P$, see § 4 (33)), $(\mathbf{1} \cap \tilde{u})\, \emptyset = (\mathbf{1} \cap u)\, \emptyset = \emptyset$, therefore

$$u_\triangle P = u^\triangle P = u^\nabla P = P, \qquad u_\triangle \emptyset = u_\nabla \emptyset = \emptyset. \tag{22}$$

18.2. If moreover $u\emptyset = \emptyset$ (i.e. if u is an *RM-correspondence*), then $u'' = u$ (§ 4 (27)), hence $u^\triangle = \overline{\overline{(u'')^\triangle}} = \overline{(u')_\triangle}$, $u_\nabla = \overline{\overline{(u'')_\nabla}} = \overline{(u')^\nabla}$ (§ 4 (22), § 5 (21)), further $u^\triangle \emptyset = \overline{(u')_\triangle} \emptyset = P - (u')_\triangle P = P - P = \emptyset$ (§ 4 (30), § 5 (22)):

$$u\emptyset = \emptyset \Rightarrow u^\triangle = \overline{(u')_\triangle}, \qquad u_\nabla = \overline{(u')^\nabla}, \qquad u^\triangle \emptyset = \emptyset. \tag{23}$$

19.0. Let us mention several notes on some concepts considered at *general topologies* (compare § 2.8, § 2.18), e.g. in papers [6], [17]; [3], [15].

19.1. If some $v \in \operatorname{Corr}(P, \exp P)$ is considered as a general topology (on P) — we may say as a general "*closure-operation*" —, then $\bar{v}$ is the corresponding *interior-operation* ($\bar{v}A = P - v(P - A)$ is the interior of A, see [17], Def. 2); for $x \in P$ $x\bar{v}$ is the set of all *neighbourhoods* of x (a set A is a neighbourhood of B ($A, B \subset P$) iff $B \cap u(P - A) = \emptyset$, i.e. iff $B \subset \bar{u}A$; a set A is a neighbourhood of x iff A is a neighbourhood of $\{x\}$, i.e. iff $x \in \bar{u}A$; see [6], 4.1).

19.2. We say that $v \in \operatorname{Corr}(P, \exp P)$ is an *U-correspondence* iff $v \,.\, v = v$ (i.e. iff v is idempotent under .); compare e.g. with papers [3], [15] (in which, nevertheless, only Čech topological spaces are considered), where under a *U-topology* an idempotent Čech topology is meant. If $u \in \operatorname{Corr}(P, \exp P)$ is an RM-correspondence (i.e. — if $P \neq \emptyset$ — a game correspondence), then $\mathbf{1} \cup u$ is a Čech topology (§ 2.18.2; especially, if moreover u is a Čech derivative — which has a special "game-meaning", see § 2.18.3 —, then $\mathbf{1} \cup u$ is the Čech topology belonging to u, see § 2.18.2), $u^\triangle = (\mathbf{1} \cup u)^\infty$ is the smallest of all U-correspondences being greater than or equal to $\mathbf{1} \cup u$ (§ 5 (18)), and $u^\triangle$ is a Čech topology (§ 5 (16), (18), (23)), therefore $u^\triangle$ is the *upper U-modification* (in the sense given in [15]) of $\mathbf{1} \cup u$.

Further, the interior-operation to the upper U-modification $u^\triangle$ is $\overline{u^\triangle} = (u')_\triangle$ (19.1; (23); § 4 (22)).

19.3. Let us note that in § 6 certain "*game interpretations*" of $u^\triangle$, $u_\triangle$ etc. will be given, and this can be used for proving some properties of these "modifications" by means of game methods (as we have mentioned in § 0).

(*To be continued*)

REFERENCES

[1]—[12]: see the first part of this discourse.

[13] Birkhoff, G., Lattice theory. New York 1948.

[14] Gale, D., and Stewart, F. M., Infinite games with perfect information. Annals of Math. Studies *28* (1953, Princeton), 245—266.

[15] Koutský, K., Polák, V., and Sekanina, M., On the commutativity of modifying. Publ. Fac. Sci. Univ. Brno, No *454* (1964), 275—292.

[16] Мыцельский, Ян и Земба А., О бесконечных играх. Бюлл. Польской А. Н., отд. III, т. *III*, No 3 (1955), 135—138.

[17] Sekanina, M., Úplné systémy okolí množin v obecných topologických prostorech. Spisy přír. fak. univ. v Brně, č. *374* (1956), 1—8.

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