Archivum Mathematicum

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Archivum Mathematicum, Vol. 6 (1970), No. 2, 89--113

Persistent URL: http://dml.cz/dmlcz/104714

ON THE EXISTENCE OF GRAPHS WITH A CERTAIN ORDERING OF VERTICES

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(Received July 15, 1968)

1. PRINCIPAL NOTIONS OF THE THEORY OF GRAPHS

A non-empty set G with a symmetric antireflexive relation σ on G (i.e. $\sigma \subset (G \times G)$) is called a graph. We write $\mathfrak{G} = (G, \sigma)$. Elements of the set G are called vertices of the graph $\mathfrak{G}$. If there holds $x\sigma y$ for vertices $x, y \in G$, we may say the vertices x, y determine the edge xy of the graph $\mathfrak{G}$. The edges xy, yx are considered as identical and we say the vertex x, or y resp., to be incidating with the edge xy. The set G being finite, the graph $\mathfrak{G}$ is called finite as well.

If $G' \subset G$, $\sigma' = (G' \times G') \cap \sigma$, it is said (G', σ') to be a subgraph of the graph (G, σ), (see [3] p. 23.) It is denoted $(G', \sigma') \subset (G, \sigma)$.

Sequence $a_0, a_1, \ldots, a_n$ of mutually different vertices of the graph $\mathfrak{G}$ is called a path of length n between the vertices a_0, a_n when $a_i \sigma a_{i+1}$ for $i = 0, 1, \ldots, n - 1$. The vertex a_0 is the initial vertex of this path, the vertex a_n an end one. The graph is called connected if there exists the path between each of its two different vertices. Provided the path exists between the vertices x, y, then there exists between them a path of the smallest length. The length of this path is called the distance of vertices x, y and is denoted $\varrho(x, y)$. If the graph $\mathfrak{G}$ is connected, $\varrho(x, y)$ is a metrics on G.

An order of a vertex is called the number of different edges which the vertex is incidating with. In an finite graph there holds $\sum_{i=1}^{n} s_i = 2h$, where $n = |G|$, $|G| = \text{card } G$, h being the number of edges of the graph $\mathfrak{G}$ and $s_1, \ldots, s_n$ being orders of individual vertices of $\mathfrak{G}$.

The sequence $a_0, a_1, \ldots, a_n$ is called a circle of the length n, if $a_0 = a_n$, the vertices $a_0, \ldots, a_{n-1}$ are mutually different, $a_i \sigma a_{i-1}$ for $i = 1, \ldots, \ldots n - 1, n$ and $n \geqq 2$. A tree is a finite connected graph without circles containing at least two vertices. An arbitrary tree contains at least two vertices of the first order. Let $x_1, \ldots, x_r$ be all vertices of the first order of tree $\mathfrak{G}$. Put $G'_1 = G - \{x_1, \ldots, x_r\}$. The subgraph $\mathfrak{G}'_1 = (G'_1, \sigma'_1) \subset \mathfrak{G}$ is a tree. If it is being continued in the given course, we come after s steps to the subgraph $\mathfrak{G}'_s \subset \mathfrak{G}$, where $\mathfrak{G}'_s = (\{c\}, \emptyset)$, or $\mathfrak{G}'_s = (\{a, b\}, \{(a, b), (b, a)\})$. Then $\mathfrak{G}'_s$ is called the centre of the tree $\mathfrak{G}$.

The vertices of graph $\mathfrak{G}$ are going to be demonstrated as points in a plane and by an abscissa be connected those vertices that are incidating with the same edge.

Next, the graph is going to mean always a finite and connected graph.

2. FORMULATION OF THE PROBLEM

A classic task of the theory of graphs is the problem if a Hamilton line exists in the given graph (G, σ), i.e. if the set G is possible to be ordered into the sequence $a_1, \ldots, a_n$ such that $\varrho(a_i, a_{i+1}) = 1$ for $i = 1, \ldots, n - 1$.

This problem can be generelized. The articles [1], [2], [4], [5] deal with the ordering into the sequence $a_1, \ldots, a_n$ such that $\varrho(a_i, a_{i+1}) \leqq k$ for $i = 1, \ldots, n - 1$. We investigate the ordering of the set G into the sequence $a_1, \ldots, a_n$ such that $\varrho(a_i, a_{i+1}) = k$ for $i = 1, \ldots, n - 1$, where k is the given positive integer. In this study we are not going to investigate the problem if in the given graph (G, σ) such an ordering exists, we are going to study, however, the following problem. The existence of the described ordering of the set G evidently enforces a certain condition for its order. This condition is going to be searched, i.e. the necessary and sufficient condition for n will be studied in order that the graph of n vertices with the described ordering may exist. This problem is going to be solved partly in a general case, partly for trees. The author would like to express his thanks to Doc. M. Sekanina for his valuable assistance in this article.

3. THE SOLUTION OF THE PROBLEM IN A GENERAL CASE

Definition: *Let $\mathfrak{G} = (G, \sigma)$ be a graph, $|G| > 1$, k a positive integer. Say the graph $\mathfrak{G}$ to be k-ordered if there exists the sequence $a_1, \ldots, a_{|G|}$ of all its vertices such that $\varrho(a_i, a_{i+1}) = k$ for $i = 1, \ldots, |G| - 1$. If $\mathfrak{G}$ is, in addition to it, a tree, then we call it a k-graph.*

First we prove some auxiliary assertions.

Lemma 1: *Let $\mathfrak{G} = (G, \sigma)$ be a graph which does not contain a circle of an odd length. Let $x, y \in G$ and let a path C of an even length exist between the vertices x, y. Then the arbitrary path between the vertices x, y is of the even length.*

Proof. Let $C = \{x = a_0, a_1, \ldots, a_n = y\}$. Let $C' = \{x = b_0, b_1, \ldots, b_s = y\}$ be an arbitrary path between the vertices x, y. Let $D =$

$= C \cap C'$ and let $x_1, x_2 \in D$. Then $x_1 = a_i = b_j$, $x_2 = a_r = b_p$, where $0 \leqq i,r \leqq n$, $0 \leqq j,p \leqq s$. Suppose for an arbitrary a_k, $i < k < r$ to hold $a_k \bar{\in} C'$ and for arbitrary b_k, $j < k < p$ to be $b_k \bar{\in} C$. For such vertices x_1, x_2 define $\Delta(x_1, x_2) = \{a_k \in C \mid i < k < r\}$, $\Delta'(x_1, x_2) = \{b_x \in C' \mid j < k < p\}$. The subgraph in $\mathfrak{G}$ formed by the set of vertices $\{x_1, x_2\} \cup \Delta(x_1, x_2) \cup \Delta'(x_1, x_2)$ evidently contains a circle and each point of this set lies on the circle. Then according to the supposition the number $|\Delta(x_1, x_2)| + |\Delta'(x_1, x_2)|$ is even. Denote S, or S' resp., the union of all $\Delta(x_1, x_2)$, or $\Delta'(x_1, x_2)$ resp. Evidently $C = D \cup S$, $C' = D \cup S'$ so that $|C| + |C'| = 2|D| + |S| + |S'|$. According to the previous statement, however, $|S| + |S'|$ is an even number so that the path C' is of the even length.

Lemma 2: *Let $\mathfrak{G} = (G, \sigma)$ be a graph that does not contain a circle of an odd length, let k be an even number. Let $a_0, a_1, \ldots, a_n \in G$, $\varrho(a_i, a_{i+1}) = k$ for $i = 0, 1, \ldots, n - 1$. Then $\varrho(a_0, a_j)$ for $j = 1, \ldots, n$ is the even number.*

Proof. The proof is done by induction. For $j = 1$, according to the supposition, $\varrho(a_0, a_1) = k$ is an even number. Let $\varrho(a_0, a_s)$ be an even number for $0 < s < n - 1$. First we go through the case of lying the vertex a_{s+1} on a path C between the vertices a_0, a_s. Because of $\varrho(a_0, a_s)$ being an even number, there exists a path of even length between the vertices a_0, a_s and according to the lemma 1, the length of the path C is even. Denote C_1, or C_2 a path between the vertices a_{s+1}, a_s, or a_0, a_{s+1} resp. such that $C_1, C_2 \subseteq C$. The length of the path C_1 is even, as $\varrho(a_s, a_{s+1}) = k$ being an even number. In view of the sum of the lengths of paths C_1 and C_2 being the length of the path C, the length of the path C_2 is even as well, so that $\varrho(a_0, a_{s+1})$ is the even number.

Let now a vertex a_{s+1} do not lie on any path between the vertices a_0, a_s. There are two possibilities then, according to the fact, if one of the vertices a_0, a_s lies on a path between the remaining vertex and the vertex a_{s+1}, or not. In a positive case let the vertex a_s lie on a path C between the vertices a_0, a_{s+1} (for the vertex a_0 the proof is done quite analogically). Let C_1, or C_2 be the path between the vertices a_0, a_s, or a_s, a_{s+1} resp., such that $C_1, C_2 \subseteq C$. Numbers $\varrho(a_0, a_s)$, $\varrho(a_s, a_{s+1})$ are even so that the lengths of the paths C_1, C_2 are even as well. The length of the path C is then even like their sum; thus $\varrho(a_0, a_{s+1})$ being an even number.

Let finally any of vertices a_0, a_s do not lie on a path between the remaining vertex and the vertex a_{s+1} and nor the vertex a_{s+1} lie on a path between the vertices a_0, a_s. Then the situation may evidently occur as is illustrated in the figure 1.

Let C_1, C_2, C_3 be paths between vertices a_0, b; b, a_s; b, a_{s+1}, according to figure 1. The sum of the lengths of the paths C_1 and C_2 (C_2 and C_3)

is an even number. From that it follows that the sum of the lenghts of paths C_1 and C_3 is even so that $\varrho(a_0, a_{s+1})$ is an even number as well. And so the proof is finished.

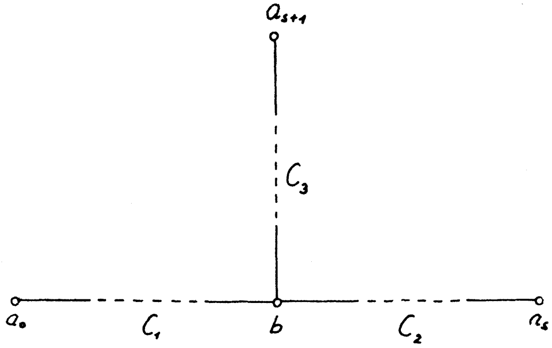

Fig. 1.

Lemma 3: *Let k be an even number and the graph $\mathfrak{G} = (G, \sigma)$ be a k-ordered one. Then $\mathfrak{G}$ contains the circle of an odd length.*

Proof. Let $x_1, x_2 \in G$, $\varrho(x_1, x_2) = 1$. Considering the graph $\mathfrak{G}$ to be k-ordered, the sequence $a_0 = x_1, a_1, \ldots, a_n = x_2$ exists such that $\varrho(a_i, a_{i+1}) = k$ for $i = 0, \ldots, n - 1$. Because of the distance of vertices x_1, x_2 being odd, $\mathfrak{G}$ contains, according to Lemma 2, the circle of an odd length.

Lemma 4: *Let k, m, n be positive integers, $m > n$. If there exists a k-ordered graph of n vertices, then a k-ordered graph of m vertices exists as well.*

Proof. Let $k > 1$, $\mathfrak{G} = (G, \sigma)$ be a k-ordered graph, $|G| = n$. There exists then the sequence $a_1, a_2, \ldots, a_n$ of mutually different vertices of the graph $\mathfrak{G}$ such that $\varrho(a_i, a_{i+1}) = k$ for $i = 1, 2, \ldots, n - 1$. Since $\varrho(a_{n-1}, a_n) = k$ there exists the path C between the vertices a_{n-1}, a_n of the length k. Let a, b be vertices of this path such that $\varrho(a_{n-1}, a) = \varrho(a_n, b) = 1$. Such vertices exist, since $k \geqq 2$ (for $k = 2$ there is $a = b$).

The graph $\mathfrak{G}' = (G', \sigma')$ is constructed in such a way: $G' = G \cup \{a_{n+1}, a_{n+2}, \ldots, a_m\}$; $\sigma' = \sigma \cup \{(a, a_{n+1}), (a_{n+1}, a), (b, a_{n+2}), (a_{n+2}, b), \ldots, (x, a_m), (a_m, x)\}$ where $x = a$, or b, according to the parity of number $m - n$. There is $|G'| = m$. The vertices of graph $\mathfrak{G}'$ be arranged into the sequence $a_1, a_2, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots, a_m$.

There is $\varrho(a_i, a_{i+1}) = k$ for $i = 1, \ldots, n - 1$. According to the construction there is $\varrho(a_i, a_{i+1}) = k$ for $i = n, \ldots, m - 1$. Then the graph is k-ordered.

For $k = 1$ there is $G' = G \cup \{a_{n+1}, \ldots, a_m\}$, $\sigma' = \sigma \cup \{(a_n, a_{n+1}), (a_{n+1}, a_n), \ldots, (a_{m-1}, a_m), (a_m, a_{m-1})\}$. Evidently $| G' | = m$ and the graph (G', σ') is k-ordered.

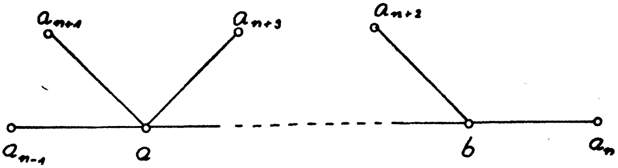

Fig. 2.

Definition: *Let $\mathfrak{G} = (G, \sigma)$ be a graph, k a positive integer. Define the relation τ on G so that $x, y \in G$, $x\tau y$ just when $\varrho(x, y) = k$. Evidently the relation τ is symmetric and antireflexive so that (G, τ) is a graph (it need not be connected). Denote the set of edges of this graph as $\mathscr{S}(\mathfrak{G})$. Elements of the set $\mathscr{S}(\mathfrak{G})$ will be denoted $[x, y]$, what means the edge in (G, τ) determined by vertices $x, y \in G$. An arbitrary subset V of the set $\mathscr{S}(\mathfrak{G})$ with properties:*

v_1) If $[x, y]$, $[x, z] \in V$ and $y \neq z$ so from $[x, t] \in V$ there follows either $t = y$ or $t = z$.

v_2) If V contains the subset $\{[x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n], [x_n, x_1]\}$ so it holds $[x_1, x_2] = \ldots = [x_n, x_1]$, i.e. this subset consists of the only element.

is called a selection in $\mathfrak{G}$. In order to stress V being a selection in $\mathfrak{G}$, we are going to write $V(\mathfrak{G})$.

An arbitrary subset $\{[x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n], [x_n, x_1]\}$ of the set $\mathscr{S}(\mathfrak{G})$, which has at least three elements, be called a cycle in $\mathfrak{G}$. We also say that this cycle is determined by the vertices $x_1, x_2, \ldots, x_n$. The selection $V(\mathfrak{G})$, as the set of edges, determines the graph (G, τ_V) without circles and whose arbitrary vertex has an order extremely of two.

Let $x \in G$, $X \subset \mathscr{S}(\mathfrak{G})$ and (G, τ_X) be a graph with a set of vertices G and a set of edges X. The order of a vertex x in the graph (G, τ_X) be called the degree of the vertex x in the set X and denoted $s_X(x)$. If there is $X = \mathscr{S}(\mathfrak{G})$ we write only $s(x)$. Say the vertex x incidates in X with an element $\alpha \in \mathscr{S}(\mathfrak{G})$, if $\alpha \in X$ and $\alpha = [x, t]$ where $t \in G$ is an arbitrary vertex.

The selection $V(\mathfrak{G})$ is called complete if for an arbitrary selection $W(\mathfrak{G})$, $W(\mathfrak{G}) \supseteq V(\mathfrak{G})$ there is $W(\mathfrak{G}) = V(\mathfrak{G})$.

Let $\mathfrak{H} = (H, \sigma')$ be a subgraph of the graph $\mathfrak{G}$. The restriction $V_H(\mathfrak{G})$ of the selection $V(\mathfrak{G})$ to a subgraph $\mathfrak{H}$ be defined as the set $V_H(\mathfrak{G}) =$

$= \{[x, y] \mid [x, y] \in V(\mathfrak{G}),\ x, y \in H\}$. *Evidently* $V_H(\mathfrak{G})$ *is a selection in* $\mathfrak{G}$ *and* $\mathfrak{H}$ *as well.*

The set of all selections in $\mathfrak{G}$ *be denoted* $\mathscr{V}_{\mathfrak{G}}$. *Evidently* $\mathscr{V}_{\mathfrak{G}}$ *is finite.* There are mentioned now some necessary properties of the selection.

Lemma 5: *Let* $\mathfrak{G} = (G, \sigma)$ *be a graph, V a selection in* $\mathfrak{G}$. *Then it holds*:
1° $\mid V \mid = \frac{1}{2} \sum_{x \in G} s_V(x) \leqq \mid G \mid - 1.$

2° *The set* $W = V \cup \{[x, y]\} - A$, *where* $([x, y] \in \mathscr{S}(\mathfrak{G})) - V$, $A \subset V$ *is a selection in* $\mathfrak{G}$ *just when* $s_W(x)$, $s_W(y) \leqq 2$ *and W does not contain a cycle into which* $[x, y]$ *belongs.*

Proof. 1° The equality follows from the fact, that in the finite graph (G, σ), according to chapter 1, there holds $\sum_{i=1}^{|G|} s_i = 2h$, where h is the number of edges of the graph (G, σ) and s_i are orders of its separate vertices. According to the selection definition, in graph (G, τ_V) there is $h = \mid V \mid$, $\sum_{i=1}^{|G|} s_i = \sum_{x \in G} s_V(x)$ so that $\sum_{x \in G} s_V(x) = 2 \mid V \mid$. Since according to $v_1)$ there is $s_V(x) \leqq 2$ and according to $v_2)$ the graph (G, τ_V) does not contain a circle and so it contains at least two vertices of the first order and then it holds $\sum_{x \in G} s_V(x) \leqq 2 \mid G \mid - 2$.

2° The necessity of the condition is evident. The sufficiency follows from $s_W(z) \leqq s_V(z)$ for $z \in W$, $x \neq z \neq y$ and from the fact that V, as a selection, does not contain a cycle.

Lemma 6: *The graph* $\mathfrak{G} = (G, \sigma)$ *is k-ordered just when a selection* $V(\mathfrak{G})$ *exists such that* $\mid V(\mathfrak{G}) \mid = \mid G \mid - 1$.

Proof. Let $\mathfrak{G} = (G, \sigma)$ be a k-ordered graph. There exists then the sequence $a_1, \ldots, a_{|G|}$ of all its vertices such that $\varrho(a_i, a_{i+1}) = k$ for $i = 1, \ldots, \mid G \mid - 1$. The set $V \subset \mathscr{S}(\mathfrak{G})$ be defined in this way: $V = \{[a_1, a_2], [a_2, a_3], \ldots, [a_{|G|-1}, a_{|G|}]\}$. Evidently V is a selection in $\mathfrak{G}$ and $V \mid = \mid G \mid - 1$.

Let a selection V in $\mathfrak{G}$ exists such that $\mid V \mid = \mid G \mid - 1$. First we prove the vertex of the degree of zero not to exist in V. Suppose the contrary; $t \in G$, $s_V(t) = 0$. According to 1° of lemma 5 there is $\mid G \mid - 1 = \mid V \mid = \frac{1}{2} \sum_{x \in G} s_V(x) = \frac{1}{2} \sum_{x \in G, x \neq t} s_V(x) \leqq \frac{1}{2}[2(\mid G \mid - 1)] = \mid G \mid - 1$ since $s_V(x) \leqq 2$ according to $v_1)$. Then, however, $\sum_{x \in G, x \neq t} s_V(x) = 2(\mid G \mid - 1)$ so that $s_V(x) = 2$ for $x \in G$, $x \neq t$. This means, however, the elements of the set $G - \{t\}$ determine the cycle in V, what is a contradiction with $v_2)$.

Since $\sum_{x \in G} s_V(x) = 2 \mid G \mid - 2$, $1 \leqq s_V(x) \leqq 2$ for each $x \in G$, then there exist just two vertices of the order one in V. Let x_1 be one of them. Construct the sequence $a_1, a_2, \ldots, a_{|G|}$ in this way: $a_1 = x_1$, a_2 is the vertex of G for which $[a_1, a_2] \in V$, $a_3 \neq a_1$ is the vertex of G for which $[a_2, a_3] \in V$ etc., as far as a_m is the vertex of G for which $[a_{m-1}, a_m] \in V$, $a_m \neq a_i$ for $i = 1, \ldots, m-1$ and the vertex $a_{m+1} \in G$ does not exist such that $[a_m, a_{m+1}] \in V$, $a_{m+1} \neq a_i$ for $i = 1, \ldots, m$. If $m < \mid G \mid$, so for $x \in G - \{a_1, \ldots, a_m\} \neq \varnothing$ there is $s_V(x) = 2$ because $s_V(a_1) = s_V(a_m) = 1$. According to the construction of the sequence $a_1, \ldots, a_m$ for each $x \in G - \{a_1, \ldots, a_m\}$, $y \in G$, $[x, y] \in V$ there is $y \in G - \{a_1, \ldots, a_m\}$. Then the vertices of the set $G - \{a_1, \ldots, a_m\}$ determine a cycle in V. This is not possible so that $m = \mid G \mid$ and the graph $\mathfrak{G}$ is k-ordered.

Lemma 7: *Let $\mathfrak{G} = (G, \sigma)$ be a k-ordered graph, $\mathfrak{H} = (H, \sigma')$ its subgraph. Then the inequality $\mid G \mid \geqq 2 \mid H \mid - 1 - \max_{V \in \mathscr{V}_{\mathfrak{H}}} \{\mid V \mid\}$ holds.*

Proof. If there is $\mathfrak{H} = \mathfrak{G}$ then, according to lemma 6, there exists the selection $V(\mathfrak{H})$ such that $\mid V(\mathfrak{H}) \mid = \mid H \mid - 1$. Then $\mid G \mid = 2 \mid H \mid - 1 - \mid H \mid + 1 = \mid H \mid$ and the lemma holds.

Let $\mathfrak{H} \neq \mathfrak{G}$ i.e. $H \subset G$, $H \neq G$. There exist then vertices $a_1, \ldots, a_n$ such that $G = H \cup \{a_1, \ldots, a_n\}$. From lemma 6 there follows the existence of a selection $W(\mathfrak{G})$, for which $\mid W(\mathfrak{G}) \mid = \mid G \mid - 1$. Let $N = W(\mathfrak{G}) - W_H(\mathfrak{G})$. There is $N = N_1 \cup N_2$ where $N_1 = \{[x, y] \in W \diagdown x, y \in H\}$, $N_2 = \{[x, y] \in W \diagdown$ just one of vertices x, y does not lie in $H\}$. There is $N_1 \cap N_2 = \varnothing$.

It holds $\mid G \mid - 1 = \mid W \mid = \frac{1}{2} \sum_{x \in G} s_W(x) = \frac{1}{2} \sum_{x \in H} s_W(x) + \frac{1}{2} \sum_{x \in G - H} s_W(x)$. From the definition of sets N_1, N_2 there follows $\sum_{x \in G - H} s_W(x) = 2 \mid N_1 \mid + \mid N_2 \mid$. Thus $\sum_{x \in H} s_W(x) = 2(\mid G \mid - 1) - \sum_{x \in G - H} s_W(x) = 2 \mid G \mid - 2 - 2 \mid N_1 \mid - \mid N_2 \mid$. Since $W_H(\mathfrak{G})$ is a selection in $\mathfrak{H}$ so there is $\max_{V \in \mathscr{V}_{\mathfrak{H}}} \{\mid V \mid\} \geqq \mid W_H(\mathfrak{G}) \mid = \frac{1}{2} \sum_{x \in H} s_{W_H}(x) = \frac{1}{2} (\sum_{x \in H} s_W(x) - \mid N_2 \mid) = \frac{1}{2} (2 \mid G \mid - 2 - 2 \mid N_1 \mid - \mid N_2 \mid - \mid N_2 \mid) = \mid G \mid - 1 - \mid N_1 \mid - \mid N_2 \mid = \mid G \mid - 1 - \mid N \mid$. Each element of the set N incidates in W with the element of the set $\{a_1, \ldots, a_n\}$. Since $s_W(a_i) \leqq 2$ for $i = 1, \ldots, n$ there is $\mid N \mid \leqq 2n$. According to up to this time proved inequality it holds $\max_{V \in \mathscr{V}_{\mathfrak{H}}} \{\mid V \mid\} \geqq \mid G \mid - 1 - \mid N \mid \geqq$ $\geqq \mid G \mid - 1 - 2n = \mid G \mid - 1 - 2 (\mid G \mid - \mid H \mid) = 2 \mid H \mid - 1 - \mid G \mid$. Hence it directly follows the assertion of the lemma.

In the previous two lemmas we have shown the continuity between the properties of selection and that fact the graph being k-ordered. The criterion of lemma 7 will be especially useful.

Lemma 8. *Let $\mathfrak{G} = (G, \sigma)$ be a k-graph. Then $\mathfrak{G}$ contains as a subgraph the tree $\mathfrak{F} = (F, \sigma')$ of $2k$ vertices such that $\mathfrak{F}$ contains just two vertices of the first order (see fig. 3).*

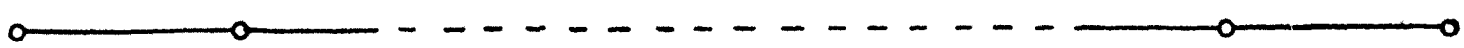

Fig. 3.

Proof. According to the definition of k-graph, $\mathfrak{G}$ is a tree. In chapter I the notion of the centre of tree was introduced. First suppose the centre of the tree $\mathfrak{G}$ consists of one vertex c only. Because of $\mathfrak{G}$ being a k-graph there exists the vertex $x \in G$ such that $\varrho(c, x) = k$. From the definition of the centre of tree it is easily to be deduced the vertex $y \in G$ exists such that $\varrho(c, y) = k$ and the vertex c lies on the path between the vertices x, y. Hence it immediately follows that $\mathfrak{G}$ contains a subgraph $\mathfrak{F}$ of required properties.

Let the centre of the tree $\mathfrak{G}$ be now a couple of vertices a, b connected with an edge. From the supposition $\mathfrak{G}$ being a k-graph there follows the existence of vertex $x \in G$, $\varrho(x, a) = k$. Then there are two possibilities; either $\varrho(x, b) = k - 1$, or $\varrho(x, b) = k + 1$. In the same way as in the first case we are going to show $\mathfrak{G}$ always contains a required subgraph $\mathfrak{F}$.

Now we can come to solving the problem in a general case.

Theorem. *A necessary and sufficient condition for the existence of a k-ordered graph of n vertices is $n \geqq 2k + 1$ for $k > 1$ and $n \geqq 2$ for $k = 1$. For an arbitrary $k > 2$ there exists just one k-ordered graph of $2k + 1$ vertices, viz. the circle of $2k + 1$ vertices. For $k = 2$ such graphs exist three, the circle of five vertices and the graphs from fig. 4 (with indicated ordering of vertices).*

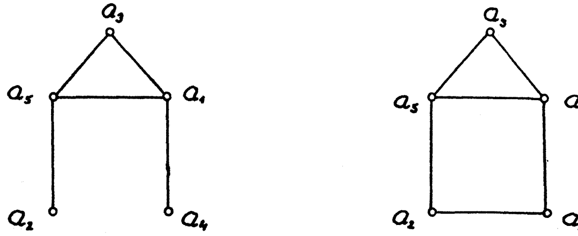

Fig. 4.

Proof. The case $k = 1$ is a trivial one, because the tree of two vertices is 1-ordered and from lemma 4 there follows the existence of 1-ordered graph of n vertices for all $n \geqq 2$.

Let $k > 1$. First prove the sufficiency of the condition in the theorem. Let $\mathfrak{K}$ be a circle of $2k + 1$ vertices, K the set of its vertices. Describe the set $\mathscr{S}(\mathfrak{K})$. To an arbitrary vertex $x \in K$ there exist just two different vertices $y, z \in K$ so that $\varrho(x, y) = \varrho(x, z) = k$. Then no vertex has the degree one in $\mathscr{S}(\mathfrak{K})$ so that to an arbitrary vertex from K there exists a cycle in $\mathscr{S}(\mathfrak{K})$ including this vertex. Let this cycle be determined by vertices $x_1, x_2, \ldots, x_m$. We easily verify that it successively holds $\varrho(x_1, x_2) = k$, $\varrho(x_1, x_3) = 1$, $\varrho(x_1, x_4) = k - 1$, $\varrho(x_1, x_5) = 2$, $\ldots$, $\varrho(x_1, x_s) = \dfrac{s-1}{2}$ provided s being odd, or $\varrho(x_1, x_s) = k - \dfrac{s-2}{2}$, s being even, $\ldots$, $\varrho(x_1, x_m)$. Since the vertices $x_1, \ldots, x_m$ form a cycle there is $\varrho(x_1, x_m) = k$, i.e. either $k = \dfrac{m-1}{2}$, or $k = k - \dfrac{m-2}{2}$ viz. according to the parity of number m. Because $m \geqq 3$, the second case then cannot occur so that $k = \dfrac{m-1}{2}$ i.e. $m = 2k + 1$. We have shown $\mathscr{S}(\mathfrak{K})$ consists just of one cycle. Then the set $V = \mathscr{S}(\mathfrak{K}) - \{[x, y]\}$, where $[x, y] \in \mathscr{S}(\mathfrak{K})$ is an arbitrary element, is a selection in $\mathfrak{K}$ and $|V| = |\mathscr{S}(\mathfrak{K})| - 1 = \frac{1}{2} \sum_{x \in K} s(x) - 1 = |K| - 1$. According to lemma 6, $\mathfrak{K}$ is a k-ordered graph. From lemma 4 it follows there exists a k-ordered graph of n vertices for each $n \geqq 2k + 1$.

We pass to the proof of the necessity of condition. Let $\mathfrak{G} = (G, \sigma)$ be an arbitrary k-ordered graph. Let $\mathfrak{G}$ be first a tree. According to lemma 8 there exists the subgraph $\mathfrak{F}$ of $2k$ vertices in $\mathfrak{G}$, in the there described form. To an arbitrary vertex $x \in F$ there exists just one vertex $y \in F$ such that $\varrho(x, y) = k$. Then $|\mathscr{S}(\mathfrak{F})| = \frac{1}{2} \sum_{x \in F} s_F(x) = \frac{1}{2} |F| = k$. According to lemma 7, $|G| \geqq 2|F| - 1 - \max_{V \in \mathscr{V}_{\mathfrak{F}}} \{|V|\} \geqq 4k - 1 - |\mathscr{S}(\mathfrak{F})| = 3k - 1$. Because $\mathfrak{G}$ does not contain a circle, according to lemma 3, the number k is odd, i.e. $k \geqq 3$. Then $|G| \geqq 3k - 1 > 2k + 1$. If there exists then a k-ordered graph of less than or equal to $2k + 1$ vertices, it must contain a circle.

Let $\mathfrak{K}_m$ be a circle of m vertices, K_m a set of its vertices, $\mathfrak{G} = (G, \sigma)$ a k-ordered graph, $\mathfrak{K}_m \subset \mathfrak{G}$. Provided $|G| \leqq 2k + 1$ and $\mathfrak{G}$ not being identical with the circle of $2k + 1$ vertices, so $m < 2k + 1$. If $m = 2k$ there exists just one vertex of K_m to an arbitrary vertex of K_m having from it the distance equal to k. Therefore $\mathscr{S}(\mathfrak{K}_m) = \dfrac{m}{2} = k$

and according to lemma 7 there is $|G| \geqq 2|K_m| - 1 - |\mathscr{S}(\mathfrak{K}_m)| = 4k - 1 - k = 3k - 1$. If $k = 2$, $\mathfrak{K}_m$ is a circle of four vertices and the only 2-ordered graph of five vertices containing such a circle is evidently the graph from fig. 4b. For $k > 2$ there is $|G| \geqq 3k - 1 > 2k + 1$.

Thus we can suppose $m < 2k$; further suppose $k > 2$. Denote $d_m = \max_{x,y \in K_m} \{\varrho(x, y)\}$. Evidently d_m is the greatest positive integer such that $d_m = m - d_m$, i.e. $d_m = \frac{m}{2}$ for even m and $d_m = \frac{m-1}{2}$ for odd m. Since $m < 2k$ so it is $d_m < k$ and therefore $\mathscr{S}(\mathfrak{K}_m) = \varnothing$. From lemma 7 there follows $|G| \geqq 2|K_m| - 1 = 2m - 1$. In order to hold $|G| \leqq 2k + 1$ it must be $m \leqq k + 1$.

Define the decomposition $\bar{R}$ on the set $G - K_m$ in such a way: $x, y \in G - K_m$ lies in the same class of decomposition $\bar{R}$ just when there exists a path all lying in $G - K_m$ between them (i.e. $\bar{R}$ is the decomposition of the graph $G - K_m$ into connected components). Let A be an arbitrary class of decomposition $\bar{R}$. Denote $A_0 = \{z \in K_m \setminus$ there exists a vertex $x \in A$ so that $\varrho(z, x) = 1\}$. Further denote $A(x) = \{z \in A \setminus \varrho(z, x) \leqq k - d_m - 1\}$, where $x \in A_0$ is an arbitrary fixed chosen vertex. Evidently $\varnothing \neq A_0 \subset K_m$, $A(x) \subset A$. Let $x \in A_0$, $y \in A(x)$, $z \in K_m$. Then $\varrho(y, z) \leqq \varrho(y, x) + \varrho(x, z) \leqq (k - d_m - 1) + d_m < k$ so that $\mathscr{S}(K_m \cup A(x)) = \mathscr{S}(A(x))$ as we have thus early proved that $\mathscr{S}(\mathfrak{K}_m) = \varnothing$.

Next we are going to consider only such classes of the decomposition $\bar{R}$, which contain at least one vertex having the distance k from a vertex of the circle $\mathfrak{K}_m$. The set of such classes be denoted R^*. There is $R^* \neq \varnothing$ because in the opposite case the graph $\mathfrak{G}$ would not be k-ordered.

First suppose R^* is composed of the only class A. Select a vertex $x \in A_0$. A vertex $y \in G$ exists so that $\varrho(x, y) = k$. Evidently $y \in A$. Let $z \in A$, $\varrho(z, x) = 1$. According to the definition of the decomposition $\bar{R}$ there exists a path C' between the vertices z, y, $C' \subset G - K_m$. Let C be a path between the vertices x, y arisen by adding the vertex x to the path C'. Since $\varrho(x, y) = k$, the length of the path C is greater than or equal to k. Let $\mathfrak{H}$ be a subgraph of the graph $\mathfrak{G}$ determined by the set of vertices $H = K_m \cup C$. There is $|H| = |K_m| + |C| \geqq m + k$. If $t \in C$, $\varrho(t, x) \leqq k - d_m - 1$, then according to the previous $s(t) = 0$ in $\mathscr{S}(\mathfrak{H})$; further $s(y) = 1$.

Show that $k - d_m - 1 \geqq 0$. Really $k - d_m - 1 \geqq k - \frac{m}{2} - 1 \geqq k - \frac{k+1}{2} - 1 = \frac{k-3}{2} \geqq 0$, as $d_m \leqq \frac{m}{2}$, $m \leqq k + 1$, $k \geqq 3$.

Therefore we can state that the number of vertices $t \in C$ such that $s(t) = 2$ in $\mathscr{S}(\mathfrak{H})$ is at most $k - 1 - (k - d_m - 1) = d_m$. Since $\mathscr{S}(\mathfrak{K}_m) = \emptyset$ an arbitrary element $\mathscr{S}(\mathfrak{H})$ incidates with the vertex of C. Then $|\mathscr{S}(\mathfrak{H})| = \sum_{x \in C} s(x) \leqq 2d_m + 1$ and according to lemma 7 there is $|G| \geqq 2|H| - 1 - |\mathscr{S}(\mathfrak{H})| \geqq 2(k + m) - 1 - 2d_m - 1 = 2k + 2m - 2d_m - 2$. For an even m there is $d_m = \frac{m}{2}$ and $|G| \geqq 2k + m - 2 > 2k + 1$ because $m \geqq 4$($m \geqq 3$ and m is even). For an odd m there is $|G| \geqq 2k + m - 1 > 2k + 1$.

Thus as far as $|G| \leqq 2k + 1$, R^* contains at least two different classes A, B. Then there exists the vertex $x \in A_0$ ($\bar{x} \in B_0$) such that $|A(x)| \geqq k - d_m - 1$ ($|B(\bar{x})| \geqq k - d_m - 1$) as far as the sets $A(x)$, $B(\bar{x})$ are defined, i.e. in respect of $k - d_m - 1 > 0$. This occur, however, always except the case of m being even and $k = 3$. By the application of lemma 7 to this special case it is easily to be ascertained that it holds in it $|G| > 2k + 1$. Let $H = K_m \cup A(x) \cup B(\bar{x})$, $\mathfrak{H} \subset \mathfrak{G}$ a subgraph with the set of vertices H. There is $|H| = m + |A(x) \cup B(\bar{x})|$. Further $\mathscr{S}(\mathfrak{H}) = \mathscr{S}(A(x) \cup B(\bar{x}))$, because $\mathscr{S}(K_m \cup A(x)) = \mathscr{S}(A(x))$, $\mathscr{S}(K_m \cup B(\bar{x})) = \mathscr{S}(B(\bar{x}))$, $\mathscr{S}(K_m) = \emptyset$. Let V be an arbitrary selection in $A(x) \cup B(\bar{x})$. According to 1° of lemma 5 there is $|V| \leqq |A(x) \cup B(\bar{x})| - 1$. According to lemma 7 there is $|G| \geqq 2|H| - 1 - \max_{V \in \mathscr{V}_{\mathfrak{H}}} \{|V|\} = 2m + 2|A(x) \cup B(\bar{x})| - 1 - (|A(x) \cup B(\bar{x})| - 1) \geqq 2m + 2(k - \mathrm{d}_m - 1) > 2k + 1$ as has been proved in the previous case.

For $k > 2$ the necessity of condition is so testified and at the same time it is proved the only k-ordered graph of $2k + 1$ vertices is a circle of $2k + 1$ vertices.

For $k = 2$ there is $m \leqq k + 1 = 3$, i.e. $m = 3$ and we get the graph from fig. 4a. The proof of the theorem is now finished.

It is seen the existence of k-ordered graphs being linked with the existence of the circle in a graph. A question may arise if there exist k-ordered graphs without circles, i.e. k-graphs. We are coming to the problem to which this paper is dedicated.

4. k-GRAPHS

According to lemma 3 there exist k-graphs only for odd k. Because of the case $k = 1$ having been dealt with in the previous theorem, in the next it will be always supposed $k > 1$ an odd number.

Definition. *Let $\mathfrak{G} = (G, \sigma)$ be a tree, $x, y \in G$. By an interval $\langle x, y \rangle$*

we call the set of all vertices on the path joining vertices x and y. Intervals $\langle x, y)$, $(x, y\rangle$ and (x, y) are deduced from $\langle x, y\rangle$ in a usual way.

Lemma 9. *There exists a k-graph of $\frac{7k-5}{2}$ vertices.*

Proof. Let be $\mathfrak{G}_k = (G_k, \sigma_k)$ a tree constructed in this way:

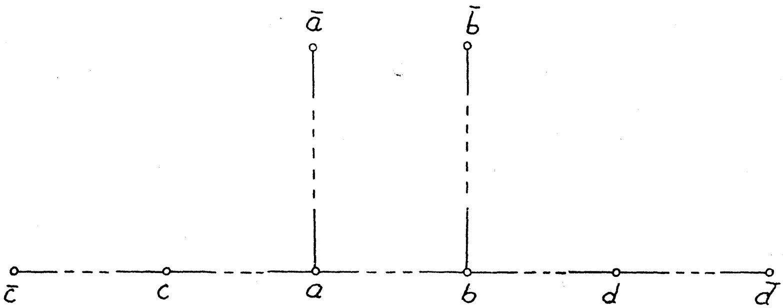

Fig. 5.

where $\varrho(\bar{c}, \bar{d}) = \frac{5k-5}{2}$, $\varrho(\bar{c}, a) = \varrho(b, \bar{d}) = k - 1$, $\varrho(a, \bar{a}) = \varrho(b, \bar{b}) = \frac{k-1}{2}$, $\varrho(c, a) = \varrho(b, d) = \frac{k-1}{2}$. Then $|G_k| = \varrho(\bar{c}, \bar{d}) + 1 + \varrho(a, \bar{a}) + \varrho(b, \bar{b}) = \frac{7k-5}{2}$. We are going to prove $\mathfrak{G}_k$ to be a k-graph.

Denote $G^0 = \langle a, b\rangle$, $G^1 = (d, \bar{d}\rangle$, $G^2 = \langle \bar{b}, b)$, $G^3 = (a, c\rangle$, $G^4 = \langle d, b)$, $G^5 = (a, \bar{a}\rangle$, $G^6 = \langle \bar{c}, c)$. The sets G^i for $i = 0, 1, \ldots, 6$ are two by two disjunctive and there holds $G_k = \bigcup_{i=0}^{6} G^i$. There is $\varrho(\bar{c}, c) = \varrho(d, \bar{d}) = \varrho(b, \bar{d}) - \varrho(b, d) = \frac{k-1}{2}$. From these two distances and the distances mentioned in the definition $\mathfrak{G}_k$ it follows that $|G^0| = \frac{k+1}{2}$, $|G^i| = \frac{k-1}{2}$ for $i = 1, 2, \ldots, 6$.

Let $y_i \in G^i$ be the vertex defined in this way: $G^i = \langle z, z'\rangle$, or $G^i = \langle z, z')$, or $G^i = (z, z'\rangle$, then $y_i = z$, or $y_i = z$, or $y_i = z''$ resp., where z'' is the vertex for which $z'' \in (z, z'\rangle$ and $\varrho(z'', z) = 1$. To an arbitrary vertex $x \in G_k$ we put a couple of indices i, j, like this: x has the first index i, if $x \in G^i$; x has the second index j, if $\varrho(x, y_i) = j$. Arrange the vertices of the tree $\mathfrak{G}_k$ into the sequence $x_{0,0}, x_{1,0}, x_{2,0}, \ldots, x_{6,0}, x_{0,1}, x_{1,1}, \ldots, x_{6,1}, x_{0,2}, \ldots, x_{6,2}, \ldots, x_{0,\frac{k-3}{2}}, x_{1,\frac{k-3}{2}}, \ldots, x_{6,\frac{k-3}{2}}, x_{0,\frac{k-1}{2}}$.

From the earlier mentioned case it follows to be the matter of arranging all vertices of the graph $\mathfrak{G}_k$, where each vertex occurs only once in this sequence. Show the distance of neighbouring vertices in this sequence to be equalled to k and so prove that $\mathfrak{G}_k$ is a k-graph.

Let j_0 be an arbitrary second index. Show that $\varrho_i = \varrho(x_{i,j_0}, x_{i+1,j_0}) = k$ for $i = 0, 1, \ldots, 5$. The proof is done for $i = 0$, in other cases is analogical. There is $\varrho_0 = \varrho(a, d) - \varrho(x_{0,j_0}, a) - \varrho(x_{1,j_0}, d) = \varrho(a, d) - j_0 + j_0 + 1 = k$. Further on there is $\varrho = (x_{6,j}, x_{0,j+1}) = \varrho(\bar{c}, a) - j_0 + j_0 + 1 = k$ for $j = 0, \ldots, \frac{k-3}{2}$. Since the distance of arbitrary neighbouring vertices in the sequence being of the one of types $\varrho_0, \varrho_1, \ldots, \varrho_5, \varrho$, lemma 9 is being proved.

Definition. *Let $\mathfrak{G} = (G, \sigma)$ be a k-graph. According to lemma 8 $\mathfrak{G}$ includes a subgraph $\mathfrak{F} = (F, \sigma')$ from lemma 8. The vertices of the set $G - F$ are called compensation vertices of the graph $\mathfrak{G}$ with regard to the subgraph $\mathfrak{F}$. Select the subgraph $\mathfrak{F}$ firmly once for ever. Let V be an arbitrary selection in $\mathfrak{G}$. Denote $M(V) = \{[x, y] \in V \diagdown x \in F, y \in G - F\}$. Define on the set $G - F$ a decomposition into classes T, U, P in this manner: $T(V) = \{x \in G - F \diagdown s_M(x) = 2\}$, $U(V) = \{x \in G - F \diagdown s_M(x) = 1\}$, $P(V) = \{x \in G - F \diagdown s_M(x) = 0\}$. Denote further $d(V) = k - |V_F(\mathfrak{G})|$. There is $d(V) \geqq 0$, for $|\mathscr{S}(\mathfrak{F})| \leqq k$. We call the selection V to be good if $d(V) = 0$. Denote $\mu(V) = 4k - 2 - |T(V)| + |P(V)| + 2d(V)$.*

Lemma 10. *Let $\mathfrak{G} = (G, \sigma)$ be a k-graph. Then $|G| \geqq \min\limits_{V \in \mathscr{V}_{\mathfrak{G}}} \{\mu(V)\}$.*

Proof. From lemma 6 there follows the existence of the selection $V \in \mathscr{V}_{\mathfrak{G}}$ for which $\sum\limits_{x \in G} s_V(x) = 2|V| = 2(|G| - 1)$. Then there exist at most two vertices of the degree less than two in V so that $\sum\limits_{x \in F} s_V(x) \geqq 2(|F| - 1) = 4k - 2$. Next according to the preceding definition there is $|G| = |F| + |T(V)| + |U(V)| + |P(V)|$. It holds $\sum\limits_{x \in F} s_V(x) = 2|V_F| + |M(V)|$; $|V_F| = k - d(V)$, $|M(V)| = 2|T(V)| + |U(V)|$. From these inequalities there follows $4k - 2 \leqq \sum\limits_{x \in F} s_V(x) = 2(k - d(V)) + 2|T(V)| + |U(V)|$ so that $|G| = |F| + |T(V)| + |U(V)| + |P(V)| \geqq 2k + |T(V)| + |P(V)| + (2k - 2 - 2|T(V)| + 2d(V)) = 4k - 2 - |T(V)| + |P(V)| + 2d(V) = \mu(V) \geqq \min\limits_{V \in \mathscr{V}_{\mathfrak{G}}} \{\mu(V)\}$.

Lemma 11. *Let $\mathfrak{G} = (G, \sigma)$ be a k-graph, V a selection in $\mathfrak{G}$. Then a good selection W in $\mathfrak{G}$ exists such that $\mu(W) \leqq \mu(V)$.*

Proof. The proof be done by induction to number $d(V) = |\mathscr{S}(\mathfrak{F}) - V_F|$. For $d(V) = 0$ the selection V is good and the lemma holds. Be supposed $d(V) > 0$. Then there exists an element $[x, y] \in \mathscr{S}(\mathfrak{F}) - V_F$. Denote $D_U(x, y)$ the set of elements of a selection U, incidating just with one of vertices x, y. According to v_1) there is $| D_U(x, y) | \leqq 4$. Show that a selection W_1 in $\mathfrak{G}$ exists such that $[x, y] \in W_1$ and $| D_{W_1}(x, y) | \geqq | D_V(x, y) | - 2$. If $D_V(x, y) = \emptyset$ then $W_1 = V \cup \{[x, y]\}$ is the selection according to 2° of lemma 5. Provided $0 < | D_V(x, y) | \leqq 3$ we choose $\alpha \in D_V(x, y)$ and the set W_1 be defined as $W_1 = (V \cup \{[x, y]\}) - A$, where $A = D_V(x, y) - \{\alpha\}$. There are $s_{W_1}(x) \leqq 2$, $s_{W_1}(y) \leqq 2$ and in W_1 a cycle containing $[x, y]$ does not exist since the degree of at least one of vertices x, y being less than two. According to 2° of lemma 5 W_1 is a selection in $\mathfrak{G}$ and $| D_{W_1}(x, y) | \geqq | D_V(x, y) | - 2$.

The case $| D_V(x, y) | = 4$. Denote α_1, α_2, or β_1, β_2 the elements of the set $D_V(x, y)$ incidating with a vertex x, or y resp. in V. A choice of indices i, j $(i, j = 1, 2)$ exists such that $W^{i,j} = V \cup \{[x, y]\} - \{\alpha_i, \beta_j\}$ being the selection in $\mathfrak{G}$. It holds $s_{W^{i,j}}(y) \leqq 2$, $s_{W^{i,j}}(x) \leqq 2$ for arbitrary indices i, j. Further suppose that for an arbitrary choice i, j the set $W^{i,j}$ contains the cycle $C^{i,j}$ including $[x, y]$. Then $\alpha_2, \beta_2 \in C^{1,1}$, $\alpha_1, \beta_1 \in C^{2,2}$. Then the elements of the set $C = C^{1,1} \cup C^{2,2} - \{[x, y]\}$ form a cycle in $\mathscr{S}(\mathfrak{G})$ and $C \subset V$, which is a contradiction (see fig. 6).

Fig. 6.

Let i_0, j_0 be that choice of indices i, j for which the set W^{i_0,j_0} does not contain a cycle to which the element $[x, y]$ belongs. According to 2° of lemma 5, W^{i_0,j_0} is a selection in $\mathfrak{G}$ and denote it W_1. There is $[x, y] \in W_1$, $| D_{W_1}(x, y) | = | D_V(x, y) | - 2$.

There is $| M(W_1) | - | D_{W_1}(x, y) | = | M(V) | - | D_V(x, y) | \geqq | M(V) | - | D_{W_1}(x, y) | - 2$. Hence it follows that $| M(W_1) | \geqq | M(V) | - 2$, i.e. $2 | T(W_1) | + | U(W_1) | \geqq 2 | T(V) | + | U(V) | - 2$. Since $| U(V) | = | G | - | F | - | T(V) | - | P(V) |$ and an analogical relation holds for the selection W_1 as well, there is $| T(W_1) | - | P(W_1) | \geqq | T(V) | - | P(V) | - 2$. Since $d(W_1) = d(V) - 1$ there is $\mu(W_1) = 4k - 2 - | T(W_1) | + | P(W_1) | + 2d(W_1) \leqq 4k - 2 - | T(V) | + | P(V) | + 2 + 2d(V) - 2 = \mu(V)$. We have constructed the selection W_1 such that $\mu(W_1) \leqq \mu(V)$ and $d(W_1) = d(V) - 1$. And so the proof of the lemma is finished for after s steps we get a good selection W_s such that $\mu(W_s) \leqq \mu(V)$.

Definition. *Let $\bar{R}$ be a decomposition of the set $G - F$ such that two compensation vertices lie in the same class of the decomposition $\bar{R}$ just then if at most one vertex of F lies on the path between them. A class of this decomposition $\bar{R}$ is called a compensation, if there exists a vertex lying in it and having the distance k from two different vertices of F.*

Let e, f be vertices of the first order in $\mathfrak{F}$. We are going to define certain characteristic vertices for an arbitrary compensation K. Denote by a this vertex $a \in F$ for which there exists a vertex $x \in K$ such that $\varrho(a, x) = 1$. Because of $\mathfrak{G}$ being a tree there exists, according to the definition of decomposition $\bar{R}$, just one vertex of this property.

Further denote by b this vertex of the set F for which $\varrho(a, b) = k - 1$ and $b \in \langle a, e \rangle$, or $b \in \langle a, f \rangle$ according to the fact if $\varrho(a, e) > \varrho(f, a)$, or $\varrho(e, a) < \varrho(f, a)$.

The set of vertices of the path between the vertex a and an arbitrary vertex $x \in K$, taken without the vertex a, is called a chain in the compensation K. The length $l(K)$ of the compensation K is meant the number $l(K) = \max_{x \in K} \{\varrho(x, a)\}$. Denote by g, g' these vertices of the set F for which $\varrho(g, a) = \varrho(g', a) = k - l(K)$, provided $l(K) < k$ and $g = g' = a$ as far as $l(K) \geqq k$.

Verify the existence of vertices b, g, g'. The vertex b always exists because $\varrho(e, f) = 2k - 1$. Further on, as long as to the vertex $x \in K$ there exist two different vertices of F being in the distance k from the vertex x, then this property is shared with an arbitrary vertex $y \in K$ for which $k > \varrho(y, a) > \varrho(x, a)$. According to this fact and the definition of compensation the vertices g, g' exist.

To the compensation K put an index i such that $i = k - \min \{\varrho(a, e), \varrho(a, f)\}$. The compensation K be called the i-th compensation and is going to be denoted by K_i. The characteristic vertices of the compensation K defined before will be denoted by an index i below, i.e. a_i, b_i etc.

If $\varrho(e, a_i) > \varrho(a_i, f)$ then it holds $\varrho(e, b_i) = i$. Really according to the definition of vertex b_i and index i there is $b_i \in \langle a_i, e \rangle$ and $i = k - \varrho(a_i, f)$. Thus $i = k - (\varrho(e, f) - \varrho(e, a_i)) = 1 - k + \varrho(e, a_i)$ because $\varrho(e, f) = 2k - 1$. Then $\varrho(e, a_i) = k - 1 + i$ and $\varrho(e, b_i) = \varrho(e, a_i) - (k - 1) = i$. Analogically we verify that, under the supposition of $\varrho(e, a_i) < \varrho(a_i, f)$, there is $\varrho(f, b_i) = i$.

Denote $O_i(\varrho) = \{x \in K_i \setminus \varrho(x, a_i) = \varrho\}$. The sets $O_i(\varrho_1)$ and $O_i(\varrho_2)$ be called associated if there is $\varrho_1 + \varrho_2 = k$. Arbitrary vertices $x, y \in K_i$ be called associated if $x \in O_i(\varrho_1)$, $y \in O_i(\varrho_2)$ and the sets $O_i(\varrho_1)$, $O_i(\varrho_2)$ are associated.

Let K_i' be an arbitrary chain in K_i. Denote by $s(K_i'; x, y)$ the set formed *by associated vertices x, y and by these vertices of set F which are in the distance equalled to k from some of the vertices. Evidently there is* $|s(K_i'; x, y)| \leqq 6$.

Lemma 12. *Let $\mathfrak{G} = (G, \sigma)$ be a k-graph, K_i a compensation in $\mathfrak{G}$ such that $l(K_i) \leqq k$, K_i' a chain in K_i. Let $\mathfrak{H}$ be a subgraph of the graph $\mathfrak{G}$ determined by the set of vertices $H = F \cup K_i'$. Then it holds*
$1°$ *Let $x \in K_i'$. Then for the degree of the vertex x in $\mathscr{S}(\mathfrak{H})$ it holds: $s(x) = 1$ if $\varrho(x, a_i) < i$ and $s(x) = 2$ if $\varrho(x, a_i) \geqq i$.*
$2°$ *An arbitrary set $s(K_i'; x, y)$ for which $|s(K_i'; x, y)| = 6$ determines the cycle in $\mathscr{S}(\mathfrak{H})$ and any cycle in $\mathscr{S}(\mathfrak{H})$ is determined by this set.*
$3°$ *Let V be an arbitrary selection in $\mathfrak{H}$. Then* $|V| \leqq k + 1 - i + 2|K_i'| - \min\left\{\frac{k+1}{2} - i, |K_i'| - \frac{k-1}{2}\right\}$.

Proof. $1°$ Since we suppose $l(K_i) \leqq k$ there is $|K_i'| \leqq k$ and therefore an arbitrary vertex of H being in the distance k from some vertex of the chain K_i' belongs to F. Let $x \in K_i'$, $\varrho(x, a_i) < i$ and suppose $\varrho(a_i,e) < \varrho(a_i, f)$. According to introducing the index i there is $\varrho(a_i,e) = k - i < k - \varrho(x, a_i)$, i.e. $\varrho(x, e) = \varrho(x, a_i) + \varrho(a_i, e) < k$. To the vertex x there exists then at most one vertex of F being in the distance k from it. Because at least one vertex like this always exists there is $s(x) = 1$. Provided $\varrho(a_i, e) > \varrho(a_i, f)$, the consideration is analogical. Let now $x \in K_i'$, $\varrho(x, a_i) \geqq i$. There is $\varrho(a_i, e) \geqq k - i \geqq k - \varrho(x, a_i)$, i.e. $\varrho(x, e) \geqq k$ and analogically $\varrho(x, f) \geqq k$. Therefore $s(x) = 2$ holds. $2°$ Let $x, y \in K_i'$ be associated vertices such that $|s(K_i'; x, y)| = 6$. Then there exist mutually different vertices x', x'', y', y'' such that $\varrho(x, x') = \varrho(x, x'') = \varrho(y, y') = \varrho(y, y'') = k$. There is $s(K_i'; x, y) = \{x, y, x', x'', y', y''\}$. Without a loss in generality we can suppose $x', y' \in \langle e, a_i\rangle$, $x'', y'' \in \langle a_i, f\rangle$. There is $\varrho(x', y'') = \varrho(x', a_i) + \varrho(a_i, y'') = k - \varrho(x, a_i) + k - \varrho(y, a_i) = k$ since the vertices x, y are associated. In the same way we prove that $\varrho(x'', y') = k$. Then the elements of the set $s(K_i'; x, y)$ determine the cycle $\{[x, x'], [x', y''], [y'', y], [y, y'], [y', x''], [x'', x]\}$ in $\mathscr{S}(\mathfrak{H})$.

Let C be an arbitrary cycle in $\mathscr{S}(\mathfrak{H})$, denote by C' the set of vertices determining C. Then C' must contain a vertex of K_i'; be it designated by x. Because, according to $1°$, there is $s(x) \leqq 2$ it must be $s(x) = 2$. Let $x', x'' \in F$, $\varrho(x, x') = \varrho(x, x'') = k$. Then $x', x'' \in C'$. Let $y', y'' \in F$, $\varrho(y', x'') = \varrho(y'', x') = k$. There is $y', y'' \in C'$. Suppose $x' \in \langle e, a_i\rangle$, $x'' \in \langle a_i, f\rangle$. Then $y' \in \langle e, a_i\rangle$, $y'' \in \langle a_i, f\rangle$. Thus if it were not like this, e.g. for the vertex y', then $s(y') = 1$, which is not possible. Thus $\varrho(y', a_i) = \varrho(y'', a_i)$ and C' contains besides vertices x, x', x'', y', y'' only the vertex $y \in K_i'$ such that $\varrho(y, y') = \varrho(y, y'') = k$. Then the vertices x, y are associated and $C' = s(K_i'; x, y)$.

3° Let V be an arbitrary selection in $\mathfrak{H}$. First form an estimate of the order of the set $\mathscr{S}(\mathfrak{H})$. There is $|\mathscr{S}(\mathfrak{H})| = |\mathscr{S}(\mathfrak{F})| + \sum_{x \in K_i'} s(x) = k + 2\,|K_i'| - (i - 1)$ according to 1°. Since for $x \in H$ there is $s(x) \leqq 2$, an arbitrary subset in $\mathscr{S}(\mathfrak{H})$, which does not contain a cycle, being a selection in $\mathfrak{H}$. An cardinal number of this subset differs from $|\mathscr{S}(\mathfrak{H})|$ at least of the number of cycles in $\mathscr{S}(\mathfrak{H})$, because in an arbitrary cycle there must be at least one element, which does not belong into this subset. According to 2° the number of cycles in $\mathscr{S}(\mathfrak{H})$ is equal to the number of couples of associated vertices x, y for which $\varrho(x, a_i) \geqq i$, $\varrho(y, a_i) \geqq i$. Because at least one of arbitrary two associated vertices has the distance greater than $\frac{k-1}{2}$ from the vertex a_i, the number of couples of associated vertices is equal to the number $|K_i'| - \frac{k-1}{2}$.

If x, y are associated vertices and $\varrho(x, a_i) > k - i$, then it is $\varrho(y, a_i) < i$. From the preceding fact it follows that the number of cycles is equal to $\min\left\{\frac{k+1}{2} - i, |K_i'| - \frac{k-1}{2}\right\}$. Thus $|V| \leqq \mathscr{S}(\mathfrak{H}) - \min\left\{\frac{k+1}{2} - i, |K_i'| - \frac{k-1}{2}\right\} = k + 1 - i + 2\,|K_i'| - \min\left\{\frac{k+1}{2} - i, |K_i'| - \frac{k-1}{2}\right\}$.

Lemma 13. *Let* $\mathfrak{G} = (G, \sigma)$ *be a k-graph,* $|G| \leqq \frac{7k-5}{2}$, K_i *a compensation in* $\mathfrak{G}$. *Then it holds:*

1° $i - 1 < l(K_i) < k - i$

2° $i \leqq \frac{k-1}{2}$

3° *For an arbitrary good selection* V *in* $\mathfrak{G}$ *there is* $|T(V) \cap K_i| \leqq \frac{k+1}{2} - i$.

Proof. 1° According to the definition, K_i must contain a vertex being in the distance k from two different vertices in F. According to 1° of lemma 12 there is $l(K_i) > i - 1$.

Let $l(K_i) \geqq k - i$. Then K_i contains the chain K_i' of $k - i$ vertices. Let $\mathfrak{H}$ be a subgraph of the graph $\mathfrak{G}$ determined by the set of vertices $H = F \cup K_i'$. There is $|H| = 3k - i$. According to 3° of lemma 12 there is $\max\limits_{V \in \mathscr{V}_{\mathfrak{H}}} \{| V |\} \leqq k + 1 - i + 2 \mid K_i' \mid - \min \left\{\frac{k+1}{2} - i, | K_i' | - \right.$ $\left. - \frac{k-1}{2}\right\} = \frac{5k+1}{2} - 2i$. According to lemma 7 there is $| G | \geqq$ $\geqq 2(3k - i) - 1 - \left(\frac{5k+1}{2} - 2i\right) = \frac{7k-3}{2} > \frac{7k-5}{2}$; a contradiction with the supposition.

2° According to 1° there is $i - 1 < k - i$, i.e. $i < \frac{k+1}{2}$, i.e. $i \leqq \frac{k-1}{2}$.

3° Let V be a good selection in $\mathfrak{G}$, i.e. $| V_F | = k$. From the definition $O_i(\varrho)$ and $T(V)$ it follows that $O_i(\varrho)$ contains at most one vertex of the set $T(V)$. From 1° of lemma 12 it follows that $O_i(\varrho) \cap T(V) = \varnothing$, provided $\varrho < i$. We are going to show that if the sets $O_i(\varrho_1)$, $O_i(\varrho_2)$ are associated there is $| [O_i(\varrho_1) \cup O_i(\varrho_2)] \cap T(V) | \leqq 1$. Suppose the contrary, i.e. let there exist vertices $x, y \in T(V)$, $x \in O_i(\varrho_1)$, $y \in O_i(\varrho_2)$. If the vertices x, y lie on the same chain K_i' of the compensation K_i there is $| s(K_i'; x, y) | = 6$. Let C be a cycle in $\mathscr{S}(\mathfrak{H})$ determined by the set $s(K_i'; x, y)$ according to 2° of lemma 12. Since $x, y \in T(V)$ and $V_F(\mathfrak{G}) = \mathscr{S}(\mathfrak{F})$, C is a cycle in V, which is a contradiction. The same cycle exist in V although the vertices x, y do not lie on the same chain of compensation K_i.

From the preceding it follows that on K_i there are so many vertices of the set $T(V)$ how many mutually different sets $O_i(\varrho_j)$ are such that $\varrho_j \geqq i$ and at same time between $O_i(\varrho_j)$ any two mutually associated sets do not exist. According to 1° there is $l(K_i) < k - i$ so that a set $O_i(\varrho)$ associated with $O_i(\varrho')$ does not exist, if $\varrho' < i$. Therefore $| T(V) \cap K_i | \leqq \frac{k+1}{2} - i$.

Definition. *Let $\mathfrak{G} = (G, \sigma)$ be a k-graph, K_i, K_j its two mutually different compensations. Let $y \in K_j$. As y_1 denote the vertex $y_1 \in F$ such that $\varrho(y, y_1) = k$ and $y_1 \in \langle e, a_j \rangle$ or $y_1 \in \langle a_j, f \rangle$ according to that if $a_i \in \langle a_j, f \rangle$ or $a_i \in \langle e, a_j \rangle$.*

Let now $x \in K_i$, $y \in K_j$. Say that $x \to y$ just when there exist the vertex y_1 and a vertex $\bar{x}$ associated with x and it holds $\varrho(\bar{x}, y_1) = k$. From now on, an index one below and a stripe above will designate exclusively the above designated vertices.

Lemma 14. *Let $\mathfrak{G} = (G, \sigma)$ be a k-graph, K_i, K_j its two different compensations such that there exist vertices $x \in K_i$, $y \in K_j$ with the property*

$x \to y$. *Then* $|G| > \frac{7k-5}{2}$.

Proof. Let $|G| \leqq \frac{7k-5}{2}$ hold.

(1) Show some relations between vertices x, y, $\bar{x}$, y_1. According to the previous definition $a_j \in \langle a_i, y_1 \rangle$. Therefore $\varrho(y_1, a_i) = \varrho(a_i, a_j) + \varrho(a_j, y_1)$. Designate $\delta = \varrho(a_i, a_j)$. Since $\varrho(y_1, a_j) = k - \varrho(y, a_j)$ there is $\varrho(\bar{x}, a_i) = k - \varrho(a_i, y_1) = \varrho(y, a_j) - \delta$.

(2) Denote by Y the set $Y = \{t \in K_j \setminus$ there exists $z_t \in K_i$ so that $z_t \to t\}$. Further let $\delta' = \min_{t \in Y} \{\varrho(t, a_j)\}$. Let $t' \in Y$ such that $\varrho(t', a_j) = \delta'$. There is $Y \neq \emptyset$ because $y \in Y$. We shall prove $0 < \delta' - \delta \leqq \frac{k-1}{2}$.

For an arbitrary $t \in Y$ there is $0 < \varrho(\bar{z}_t, a_i) = \varrho(t, a_j) - \delta$ according to (1). Thus $\delta' - \delta > 0$. Suppose $\delta' - \delta > \frac{k-1}{2}$. Show that then not even one of the relations $\varrho(\bar{z}_{t'}, e) = k$, $\varrho(\bar{z}_{t'}, f) = k$, $\varrho(\bar{z}_{t'}, b_i) = k$ holds. Provided it would be $\varrho(\bar{z}_{t'}, e) = k$ then $\delta' - \delta = \varrho(\bar{z}_{t'}, a_i) = k - \varrho(e, a_i) \leqq k - (k - i) = i \leqq \frac{k-1}{2}$, according to 2° of lemma 13. This is, however, a contradiction with the supposition $\delta' - \delta > \frac{k-1}{2}$. Here we used the relation $i = k - \min\{\varrho(e, a_i), \varrho(f, a_i)\}$. Analogically for the vertex f. If $\varrho(\bar{z}_{t'}, b_i) = k$ then according to the definition of the vertex b_i there is $\varrho(\bar{z}_{t'}, a_i) = 1$, what again resists to the supposition $\delta' - \delta > \frac{k-1}{2}$.

Since $z_{t'} \to t'$ then $\varrho(\bar{z}_{t'}, t_1') = k$ and according to the preceding $t_1' \neq e, f, b_i$. Hence it follows that for the vertex $u \in K_j$ such that $\varrho(u, a_j) = \delta' - 1$ there exists the vertex $\bar{z}_u \in K_i$ for which $\varrho(\bar{z}_u, u_1) = k$. It holds $\varrho(\bar{z}_u, a_i) = \varrho(\bar{z}_{t'}, a_i) - 1 = \delta' - \delta - 1 \geqq \frac{k-1}{2}$. If $\varrho(\bar{z}_u, a_i) > \frac{k-1}{2}$, then there exists a vertex $z_u \in K_i$ associated to the vertex $\bar{z}_u$. Provided $\varrho(\bar{z}_u, a_i) = \frac{k-1}{2}$, the vertices $\bar{z}_u$, $\bar{z}_{t'}$ are associated. In both cases the vertex u has the property of existing the vertex $z_u \in K_i$ such that $z_u \to u$. This is a contradiction with the definition of vertex t'. Thus $\delta' - \delta \leqq \frac{k-1}{2}$.

(3) The vertex $z_{t'}$ is associated with the vertex $\bar{z}_{t'}$, for which there holds $\varrho(\bar{z}_{t'}, a_i) = \delta' - \delta$. Therefore $\varrho(z_{t'}, a_i) = k - \delta' + \delta$ and the compensation K_i contains the chain K_i' of $k - \delta' + \delta$ vertices. Since $\varrho(t', a_j) = \delta'$, the compensation K_j contains a chain of δ' vertices.

Let $\mathfrak{H}$ be a subgraph of the graph $\mathfrak{G}$ determined by the set of vertices $H = F \cup K_i' \cup K_j'$, where K_j' being a chain in K_j such that $|K_j'| = \min\left\{\delta', \frac{k-1}{2}\right\}$. There is $|H| = 3k - \delta' + \delta + \min\left\{\delta', \frac{k-1}{2}\right\}$.

(4) Let $u, v \in H$ be vertices of these properties: $u, v \in F \cup K_i'$, $a_j \in \langle u, v\rangle$, $\varrho(u, v) = k$. Then a pair of vertices $u', v' \in K_j'$ does not exist such that $\varrho(u, u') = \varrho(v, v') = k$.

Assume that the assertion does not hold.

Since the number k being odd there is $u' \neq v'$. E.g. let $\varrho(u', a_j) < \varrho(v', a_j)$. Then it holds $\varrho(u, a_j) > \varrho(v, a_j)$. Since $\varrho(u, v) = k$ and $a_j \in \langle u, v\rangle$ then it is $\varrho(v, a_j) \leqq \frac{k-1}{2}$, i.e. $\varrho(v', a_j) > \frac{k-1}{2}$. This is a contradiction with the supposition of $|K_j'| = \min\left\{\delta', \frac{k-1}{2}\right\}$.

(5) Let V be an arbitrary selection in $\mathfrak{H}$. Denote by $B(V)$ a set of these vertices $x \in K_j'$ with properties: $s_V(x) = 2$, $\varrho(x, a_j) < \delta'$ and if there exists the vertex x_1 then exists the vertex $\bar{z}_x \in K_i'$, $\varrho(\bar{z}_x, x_1) = k$.

Let V_0 be an arbitrary and firmly determined selection in $\mathfrak{H}$. Show a selection W in $\mathfrak{H}$ exists such that $|V_0| = |W|$ and $B(W) = \varnothing$.

The proof be done by induction to the order of the set $B(V_0)$. Let $x \in B(V_0)$. First suppose the vertex x_1 exists and $[x_1, x] \in V_0$. According to the definition of the set $B(V_0)$ there exists the vertex $\bar{z}_x \in K_i'$ for which $\varrho(\bar{z}_x, x_1) = k$. Denote by x_3 this vertex of F for which $\varrho(x_1, x_3) = k$. Such a vertex exists only one. Further denote by α such of elements $[x_1, x_3]$, $[x_1, \bar{z}_x]$ which does not belong to V_0. Since, according to v_1), there is $s_{V_0}(x_1) \leqq 2$ and $[x_1, x] \in V_0$, then such an element exists. Define the set W_1 in this way: $W_1 = (W \cup \{\alpha\}) - \{[x, x_1]\}$. Show the W_1 to be a selection. According to 2° of lemma 5 we must verify $s_{W_1}(x_1) \leqq 2$, $s_{W_1}(\bar{z}_x) \leqq 2$, or $s_{W_1}(x_3) \leqq 2$ according to this if $\alpha = [x_1, \bar{z}_x]$, or $\alpha = [x_1, x_3]$ resp., and show that a cycle containing α does not exist in W_1. The relation $s_{W_1}(x_1) \leqq 2$ holds according to the definition W_1. Further on, according to (4), a vertex $z \in K_j'$ does not exist such that $\varrho(\bar{z}_x, z) = k$ because $\varrho(\bar{z}_x, x_1) = k$ and $a_j \in \langle \bar{z}_x, x_1\rangle$, since $a_j \in \langle a_i, x_1\rangle$ and $\varrho(x_1, x) = k$ and $x \in K_j'$. Thus $s_{W_1}(\bar{z}_x) \leqq 2$. As far as a vertex $z \in K_i'$ exists such that $\varrho(z, x_3) = k$, then the vertices z, $\bar{z}_x$ are associated and consequently it holds $z \to x$. With regard to the supposition $\varrho(x, a_j) < \delta'$ there is a contradiction. Therefore $s(x_3) \leqq 2$.

The first part of condition 2° is being verified. Suppose now that there exists a cycle C in W_1 such that $\alpha \in C$; denote by $\bar{C}$ the set of vertices determining the cycle C. Since $x_1 \in \bar{C}$, $[x, x_1] \bar{\in} W_1$ it holds that $[x_1, \bar{z}_x]$, $[x_1, x_3] \in C$. As we have proved that there does not exist a vertex $z \in K_i'$, for which $\varrho(z, x_3) = k$, a vertex $u \in K_j'$ exists such that $\varrho(x_3, u) = k$. A necessary supposition of the existence of such a vertex is $a_j \in \langle x_3, x_1 \rangle$ so that we get a contradiction with (4). Thus W_1 is a selection and $|B(W_1)| = |B(V_0)| - 1$, $|W_1| = |V_0|$.

A case remains of existing the vertices x_2, x_4 such that $x_2 \in F$, $x_4 \in K_i'$ and $[x_2, x]$, $[x_4, x] \in V_0$, where $x_2 \neq x_1$, i.e. $a_j \bar{\in} \langle a_i, x_2 \rangle$. Denote by x_5 the vertex $x_5 \in F$, $\varrho(x_4, x_5) = k$, where $a_j \in \langle x_4, x_5 \rangle$. Show this vertex really exists. Suppose e.g. $\varrho(a_j, e) < \varrho(a_j, f)$. There is $\varrho(a_j, e) = k - j \geqq \geqq \frac{k+1}{2}$ according to 2° of lemma 13. Further on, $\varrho(x, a_j) \leqq \frac{k-1}{2}$ according to the definition of the chain K_j'. There exist then two different vertices of F such that their distance from the vertex a_j equals to $\varrho(a_j, x)$. One of these vertices is the vertex x_5. Since $\varrho(x, x_4) = \varrho(x, x_2) = k$, then it holds $a_i \in \langle x_2, x \rangle$. Therefore $\varrho(x_2, x_5) = \varrho(x_5, a_j) + \varrho(a_i, a_j) + \varrho(a_i, x_2) = \varrho(x, a_j) + \varrho(a_i, a_j) + \varrho(a_i, x_4) = \varrho(x, x_4) = k$.

Let $[x_4, x_5] \bar{\in} V_0$. Define the set $W_1 = (V_0 \cup \{[x_4, x_5]\}) - \{[x, x_4]\}$. Considering the fact there is $s_{W_1}(x_4) \leqq 2$. As the vertices x_4, x_5 fulfil the suppositions of assertion (4), a vertex being in the distance k from the vertex x_5 does not exist on K_j'. Therefore $s(x_5) \leqq 2$. Suppose W_1 containing the cycle C such that $[x_4, x_5] \in C$. Because $s(x_5) \leqq 2$, there is $[x_5, x_2] \in C$. Let $C_1 = [C - \{[x_4, x_5], [x_5, x_2], [x_2, x]\}] \cup \{[x_4, x]\}$. There is $C_1 \subset V_0$ and C_1 is the cycle in V_0; a contradiction. Then according to 2° of lemma 5, W_1 is a selection in $\mathcal{S}(\mathfrak{H})$ and at the same time there holds $|B(W_1)| = |B(W_0)| - 1$, $|W_1| = |V_0|$.

Let $[x_4, x_5] \in V_0$. Since $[x_4, x] \in V_0$, $[x, x_2] \in V_0$ then if follows from v_2) that $[x_2, x_5] \bar{\in} V_0$. Define $W_1 = (V_0 \cup \{[x_2, x_5]\}) - \{[x, x_2]\}$. Quite analogical as in the previous case we are going to show W_1 to be a selection and $|B(W_1)| = |B(V_0)| - 1$, $|W_1| = |V_0|$.

We went through all possible cases so that we may conclude a selection W in $\mathcal{S}(\mathfrak{H})$ exists such that $|W| = |V_0|$ and $B(W) = \varnothing$.

(6) According to (5) it is sufficient in estimating $\max\limits_{V \in \mathcal{V}_{\mathfrak{H}}} \{|V|\}$ to consider only the selections V for which $B(V) = \varnothing$. Take such an arbitrary selection. There is $|V| = |V_{F \cup K_i'}| + |V - V_{F \cup K_i'}| = |V_{F \cup K_i'}| + \sum\limits_{x \in K_j'} s_V(x)$. The restriction $V_{F \cup K_i'}$ is a selection on the graph that is determined by the set of vertices $F \cup K_i'$. According to 3° of lemma 12 there is $|V_{F \cup K_i'}| \leqq k + 1 - i + 2 |K_i'| \quad \min \left\{ \frac{k+1}{2} - \right.$

$-i, |K_i'| - \frac{k-1}{2}\Big\}$. If $|K_i'| - \frac{k-1}{2} > \frac{k+1}{2} - i$ then $|K_i'| >$ $> k - i$ and we get a contradiction with 1° of lemma 13.

It holds $\min\left\{\frac{k+1}{2} - i, \; |K_i'| - \frac{k-1}{2}\right\} = |K_i'| - \frac{k-1}{2}$.

Consequently $|V_F \cup K_i'| \leqq \frac{5k+1}{2} - \delta' + \delta - i$. Since $B(V) = \varnothing$ then it holds $s_V(x) = 2$ for $x \in K_j'$ only for one of these supposition: either $\varrho(x, a_j) = \delta'$, or there exists the vertex $x_1 \in F$ and does not exist the vertex $\bar{z}_x \in K_i'$ such that $\varrho(x_1, \bar{z}_x) = k$. There is $|K_i'| = k - \delta' + \delta \geqq \frac{k+1}{2} > |K_j'|$ because, according to (2), $\delta' - \delta \leqq \frac{k-1}{2}$.

Let g_i, g_i' be previous defined vertices, i.e. such vertices that $\varrho(g_i, a_i) = \varrho(g_i', a_i) = k - |K_i'|$. If the vertex $x \in K_j'$ exists such that $x_1 \in (g_i, g_i')$ then it holds $a_j \in (g_i, g_i')$, because $a_j \in \langle a_i, x_1\rangle$. Accordingly, however, $|K_j'| \geqq \varrho(x, a_j) \geqq k - (k - |K_i'|) = |K_i'|$, which is a contradiction. Thus for a vertex $x \in K_j'$, for which there exists the vertex x_1 and does not exist $\bar{z}_x \in K_i'$, $\varrho(\bar{z}_x, x_1) = k$, it holds $x_1 \in \langle e, b_i)$, or $x_1 \in (b_i, f\rangle$ according to the fact if $\varrho(e, a_i) > \varrho(f, a_i)$, or $\varrho(f, a_i) > \varrho(e, a_i)$ resp. Without loss of generality it may be supposed $\varrho(e, a_i) > \varrho(f, a_i)$. Then $\varrho(e, b_i) = i$ so that the number of vertices $x \in K_j'$ such that $x_1 \in \langle e, b_i)$ is at most i.

Thus the number of vertices $x \in K_j'$, for which $s_V(x) = 2$, is at most i for $\delta' > \frac{k-1}{2}$ and at most $i+1$ for $\delta' \leqq \frac{k-1}{2}$. Therefore $\sum_{x \in K_j'} s_V(x) \leqq |K_j'| + i + 1$ as to $\delta' \leqq \frac{k-1}{2}$ and $\sum_{x \in K_j'} s_V(x) \leqq |K_j'| + i$, if $\delta' > \frac{k-1}{2}$.

Let $\delta' \leqq \frac{k-1}{2}$. Then $|K_j'| = \delta'$ and $|H| = 3k + \delta$, $\max\limits_{V \in \mathscr{V}_{\mathfrak{H}}} \{|V|\} \leqq \frac{5k+1}{2} - \delta' + \delta - i + (\delta' + i + 1) = \frac{5k+3}{2} + \delta$. According to lemma 7 there is $|G| \geqq \frac{7k-5}{2} + \delta > \frac{7k-5}{2}$, which is a contradiction.

Let $\delta' > \frac{k-1}{2}$, i.e. $|K_j'| = \frac{k-1}{2}$. Then $|H| = \frac{7k-1}{2} -$

$-\delta' + \delta$ and $\max\limits_{V\in\mathscr{V}_{\mathfrak{H}}} \{|V|\} \leqq \frac{5k+1}{2} - \delta' + \delta - i + \left(\frac{k-1}{2} + i\right) =$ $= 3k - \delta' + \delta$. According to lemma 7 there is $|G| \geqq 4k - 2 - \delta' +$ $+ \delta \geqq \frac{7k-3}{2}$ because with regards to (2) there is $\delta' - \delta \leqq \frac{k-1}{2}$. Thus again $|G| > \frac{7k-5}{2}$. This is a contradiction and the proof of lemma is finished.

Now we can come to the proof of the final theorem.

Theorem. *Let $k > 1$ be an odd number. Then a k-graph of n vertices exists just when $n \geqq \frac{7k-5}{2}$.*

Proof. The sufficiency of the condition follows directly from lemmas 9 and 4. We prove the necessity. Let $\mathfrak{G} = (G, \sigma)$ be a k-graph and suppose $|G| < \frac{7k-5}{2}$. We must come to a contradiction.

Let V be an arbitrary good selection in $\mathfrak{G}$, K_i a compensation in $\mathfrak{G}$ such that for an arbitrary compensation K_j there holds $l(K_j) \leqq l(K_i)$. Without loss of generality it may be supposed that $\varrho(e, a_i) > \varrho(a_i, f)$.

Define the decomposition of the set $T(V)$ onto the classes T_1, T_2, T_3 in this way: $T_1 = \{x \in T(V) \setminus x', x'' \in (b_i, e\rangle\}$, $T_2 = \{x \in T(V) \setminus x' \in \langle f, b_i\rangle, x'' \in (b_i, e\rangle\}$, $T_3 = \{x \in T(V) \setminus x', x'' \in \langle f, b_i\rangle\}$, where $x', x'' \in F$ being such vertices that $[x, x'], [x, x''] \in V$. Estimate from above the numbers $|T_i|$, $i = 1, 2, 3$.

(1) Let $y \in T_1$. Let K_j be this compensation for which $y \in K_j$. Evidently $K_j \neq K_i$. Since $a_j \in \langle y', y''\rangle$ and $y', y'' \in (b_i, e\rangle$ then $a_j \in (b_i, e\rangle$, $\varrho(b_i, a_j) >$ > 1. There is $\varrho(b_i, e) = i$ so that $\varrho(a_j, e) < i - 1$. According to 2° of lemma 13 there is $i \leqq \frac{k-1}{2}$ and therefore $\varrho(a_j, e) < \frac{k-3}{2}$.

Because $j = k - \varrho(a_j, e) > \frac{k+3}{2}$, then according to 2° of lemma 13 there is $|G| > \frac{7k-5}{2}$, which is a contradiction. Thus $T_1 = \emptyset$.

(2) From the equality $\varrho(b_i, e) = i$ it follows that the interval $(b_i, e\rangle$ contains of i vertices. Because of the selection V being good, then to an arbitrary vertex $x \in F$ there exists at most one vertex $y \in G - F$ such that $[x, y] \in V$. Therefore $|T_2| \leqq i$.

(3) With regards to the definition of the set T_3 there is $T(V) \cap \cap K_i \subseteq T_3$. From 3° of lemma 13 it follows that $| T(V) \cap K_i | \leqq \leqq \frac{k+1}{2} - i$. Denote by $A(V)$ a set of these vertices $x \in K_i$ with properties: $\varrho(x, a_i) \geqq i$, $x \,\bar{\in}\, T(V)$, and the vertex $\bar{x}$ associated with x does not exist. It holds $\frac{k+1}{2} - i - | T(V) \cap K_i | \geqq | A(V) |$.

We are going to show there holds $| T_3 | \leqq | P(V) | + \frac{k+1}{2} - i$.

Let $y \in T_3 \cap (G - K_i)$. Let K_j be this compensation for which $y \in K_j$. Let y_1 be one of vertices y', y'' in the sense of definition before lemma 14. Show to hold $y_1 \,\bar{\in}\, (g_i, g_i')$. In the opposite case there is $a_j \in \in (g_i, g_i')$ and thus $l(K_j) \geqq \varrho(y, a_j) = k - \varrho(y_1, a_j) > k - \varrho(a_i, g_i) = = l(K_i)$, which is a contradiction. There exists then the vertex $\bar{t} \in K_i$ such that $\varrho(\bar{t}, y_1) = k$ since according to the previous fact and definition of the set T_3 it holds that $y_1 \in \langle f, b_i \rangle - \langle g_i, g_i' \rangle$. There are these possibilities:

α) $\varrho(\bar{t}, a_i) < i$. Because of the selection V being good and $[y, y_1] \in V$ there is $[\bar{t}, y_1] \,\bar{\in}\, V$. Then $\bar{t} \in P(V)$.

β) $\varrho(\bar{t}, a_i) \geqq i$ and there is no vertex $t \in K_i$ associated with the vertex $\bar{t}$. Similarly as in α) there is $[\bar{t}, y_1] \,\bar{\in}\, V$. Then $\bar{t} \in A(V)$.

γ) $\varrho(\bar{t}, a_i) \geqq i$ and the vertex $t \in K_i$ associated with the vertex $\bar{t}$ exists. Then, however, $t \to y$ and according to lemma 14 there is $| G | > > \frac{7k-5}{2}$, which is a contradiction. The case γ) then cannot occur.

In the preceding we practically introduced a mapping φ: $T_3 \cap \cap (G - K_i) \to K_i$ such that $\varphi(y) = \bar{t}$. Let $y, \tilde{y} \in T_3 \cap (G - K_i)$, $y \neq \tilde{y}$, where $\varphi(y) = \varphi(\tilde{y}) = z$. There is $y_1 \neq \tilde{y}_1$ since the selection V being good. Since $[z, y_1] \,\bar{\in}\, V$, $[z, \tilde{y}_1] \,\bar{\in}\, V$ there is $k > \varrho(z, a_i) \geqq i$. Therefore it holds $z \in A(V) \cap T(V)$.

Define the decomposition of the set $\varphi[T_3 \cap (G - K_i)]$ onto the classes R_1, R_2, R_3 in this way: $R_1 = \{x \in \varphi[T_3 \cap (G - K_i)] \diagdown | \varphi^{-1}(x) | = = 1, \varrho(x, a_i) < i\}$, $R_2 = \{x \in \varphi[T_3 \cap (G - K_i)] \diagdown | \varphi^{-1}(x) | = 1, \varrho(x, a_i) \geqq \geqq i\}$, $R_3 = \{x \in \varphi[T_3 \cap (G - K_i)] \diagdown | \varphi^{-1}(x) | = 2\}$. There is $| T_3 \cap \cap (G - K_i) | = | R_1 | + | R_2 | + 2 | R_3 |$ and at the same time $| P(V) | \geqq \geqq | R_1 | + | R_3 |$, $| A(V) | \geqq | R_2 | + | R_3 |$. By substituing we get

$$| T_3 \cap (G - K_i) | \leqq | P(V) | + \frac{k+1}{2} - i - | T(V) \cap K_i |.$$

Since $T(V) \cap K_i \subseteq T_3$ and $T_3 \subset T(V)$ there is $T_3 \cap K_i = T(V) \cap K_i$.

Then it holds $|T_3| = |T_3 \cap (G - K_i)| + |T_3 \cap K_i| = |T_3 \cap (G - K_i)| + |T(V) \cap K_i| \leqq |P(V)| + \frac{k+1}{2} - i$, which we wanted to be proved.

Because of the selection V being good there is $d(V) = 0$. Then $\mu(V) = 4k - 2 - |T(V)| + |P(V)| + 2d(V) = 4k - 2 - |T(V)| + |P(V)|$. According to (1), (2), (3) there is $|T(V)| = |T_1| + |T_2| + |T_3| \leqq i + |P(V)| + \frac{k+1}{2} - i = |P(V)| + \frac{k+1}{2}$. After substituing we have $\mu(V) \geqq \frac{7k-5}{2}$. From lemma 11 and the fact V being an arbitrary good selection it follows that $\min\limits_{W \in \mathscr{V}_{\mathfrak{G}}} \{\mu(W)\} \geqq \frac{7k-5}{2}$. According to lemma 10 there is $|G| \geqq \frac{7k-5}{2}$, which is a contradiction. By this assertion the theorem is being proved.

It remains a problem how to find out the description of k-graphs of $\frac{7k-5}{2}$ vertices. No other such graphs than the graphs from lemma 9 have been found by the author. There is a probability these graphs are determined univocally.

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