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INDUCED ALGEBRAS

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The paper concerns algebras induced in a certain natural way by generalized algebras, i.e. by sets with a system of multivalued operations. Connections between congruence relations, homomorphisms and closed subsets of generalized algebras and those of algebras induced by them are studied. Further, a special case of generalized algebras in investigated, namely generalized algebras with one unary operation and algebras induced by them. At the end of the paper the sum and the product of generalized algebras are introduced and it is shown that the algebra induced by the sum of generalized algebras is isomorphic to the product of the respective induced algebras.

1. GENERALIZED ALGEBRAS

1.1. Definition. Let A be a non-void set, K a set. A mapping $(a_\varkappa)_{\varkappa\in K}$ of the set K into the set A is called a *sequence of type* K *in* A or shortly a K*-sequence in* A. The family of all K-sequences in A is denoted by A^K. A mapping f of the family A^K of all K-sequences in A into the family 2^A of all subsets of the set A is called a *generalized operation of type* K *on* A (a K*-operation on* A).

1.2. Remark. We shall use the denotation $f(a_\varkappa|\varkappa \in K)$ instead of $f((a_\varkappa)_{\varkappa\in K})$ for the value of a generalized operation f of type K on A at a K-sequence $(a_\varkappa)_{\varkappa\in K}$.

1.3. Definition. Let A be a non-void set, I a set, $(K_\iota)_{\iota\in I}$ a system of sets, $(f_\iota)_{\iota\in K}$ a system of generalized operations on A such that the generalized operation f_ι is of type K_ι for each $\iota \in I$. Then the ordered pair $(A, (f_\iota)_{\iota\in I})$ is called a *generalized algebra of type* $(K_\iota)_{\iota\in I}$.

1.4. Definition. Let $(A, (f_\iota)_{\iota\in I})$ be a generalized algebra, ϑ an equivalence relation on A. ϑ is called a *congruence relation on* $(A, (f_\iota)_{\iota\in I})$ if and only if for each $\iota \in I$ and for arbitrary K_ι-sequences $(a_\varkappa)_{\varkappa\in K_\iota}$, $(b_\varkappa)_{\varkappa\in K_\iota}$ in A with the property $a_\varkappa \vartheta b_\varkappa$ for all $\varkappa \in K_\iota$ there exists to each $x \in f_\iota(a_\varkappa|\varkappa \in K_\iota)$ such $y \in f_\iota(b_\varkappa|\varkappa \in K_\iota)$ that $x\vartheta y$ holds.

1.5. Definition. Let $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$ be generalized algebras of the same type. Let φ be a mapping of A into B fulfilling the condition

$$\varphi[f_\iota(a_\varkappa|\varkappa \in K_\iota)] = g_\iota(\varphi(a_\varkappa)|\varkappa \in K_\iota)$$

for all $\iota \in I$ and for all K_ι-sequences $(a_\varkappa)_{\varkappa\in K_\iota}$ in A. Then φ is called a *homomorphism of* $(A, (f_\iota)_{\iota\in I})$ *into* $(B, (g_\iota)_{\iota\in I})$.

1.6. Definition. Let $(A, (f_\iota)_{\iota\in I})$ be a generalized algebra, $B \subseteq A$ a set. We say B is a *closed set* in $(A, (f_\iota)_{\iota\in I})$ if and only if $f_\iota(a_\varkappa|\varkappa \in K_\iota) \subseteq B$ holds for each $\iota \in I$ and for each K_ι-sequence $(a_\varkappa)_{\varkappa\in K_\iota}$ in B.

2. INDUCED ALGEBRAS

2.1. Definition. Let $(A, (f_\iota)_{\iota\in I})$ be a generalized algebra. Define for each $\iota \in I$ an (ordinary) operation F_ι of type K_ι on the set 2^A of all subsets of A in the following way:

$F_\iota(A_\varkappa|\varkappa \in K_\iota) = \{a|a \in f_\iota(a_\varkappa|\varkappa \in K_\iota),\ a_\varkappa \in A_\varkappa$ for all $\varkappa \in K_\iota\}$, where $(A_\varkappa)_{\varkappa\in K_\iota}$ is a K_ι-sequence in 2^A. The algebra $(2^A, (F_\iota)_{\iota\in I})$ is called *induced* by the generalized algebra $(A, (f_\iota)_{\iota\in I})$.

2.2. Theorem. *Let A be a non-void set. An algebra $(2^A, (F_\iota)_{\iota\in I})$ is induced by a generalized algebra if and only if for arbitrary $\iota \in I$ the following implication holds:*

If $A_\varkappa = \bigcup_{B_\varkappa\in\mathfrak{A}_\varkappa} B_\varkappa$ for all $\varkappa \in K_\iota$, then $F_\iota(A_\varkappa \mid \varkappa\in K_\iota) = \{a \mid a\in F_\iota(B_\varkappa \mid \varkappa\in K_\iota)$, $B_\varkappa \in \mathfrak{A}_\varkappa$ for all $\varkappa \in K_\iota\}$.

Proof. 1. Let the algebra $(2^A, (F_\iota)_{\iota\in I})$ be induced by a generalized algebra $(A, (f_\iota)_{\iota\in I})$. Let $\iota \in I$, $A_\varkappa = \bigcup_{B_\varkappa\in\mathfrak{A}_\varkappa} B_\varkappa$ for all $\varkappa \in K_\iota$. Then $F_\iota(A_\varkappa \mid \varkappa \in K_\iota) = \{a \mid a \in f_\iota(a_\varkappa \mid \varkappa \in K_\iota),\ a_\varkappa \in A_\varkappa$ for all $\varkappa \in K_\iota\} = \{a \mid a \in f_\iota(a_\varkappa \mid \varkappa \in K_\iota),\ a_\varkappa \in B_\varkappa \in \mathfrak{A}_\varkappa$ for all $\varkappa \in K_\iota\} = \{a \mid a \in F_\iota(B_\varkappa|\varkappa \in K_\iota),\ B_\varkappa \in \mathfrak{A}_\varkappa$ for all $\varkappa \in K_\iota\}$.

2. Let the algebra $(2^A, (F_\iota)_{\iota\in I})$ have the property mentioned in the theorem. Put $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = F_\iota(\{a_\varkappa\} \mid \varkappa \in K_\iota)$ for all $\iota \in I$, $(a_\varkappa)_{\varkappa\in K_\iota} \in A^{K_\iota}$. Then the algebra $(2^A, (F_\iota)_{\iota\in I})$ is induced by the generalized algebra $(A, (f_\iota)_{\iota\in I})$. We have namely $A_\varkappa = \bigcup_{\{a_\varkappa\}\in\mathfrak{A}_\varkappa} \{a_\varkappa\}$, where $\mathfrak{A}_\varkappa$ is the family of all one-element subsets of $A_\varkappa$ for each $\varkappa \in K_\iota$. Hence $F_\iota(A_\varkappa \mid \varkappa \in K_\iota) = \{a \mid a \in F_\iota(\{a_\varkappa\} \mid \varkappa \in K_\iota), \{a_\varkappa\} \in \mathfrak{A}_\varkappa$ for all $\varkappa \in K_\iota\} = \{a \mid a \in f_\iota(a_\varkappa \mid \varkappa \in K_\iota),\ a_\varkappa \in A_\varkappa$ for all $\varkappa \in K_\iota\}$ for arbitrary $\iota \in I$, $(A_\varkappa)_{\varkappa\in K_\iota} \in (2^A)^{K_\iota}$.

2.3. Remark. In [2] it is stated incorrectly that an algebra $(2^A, (F_\iota)_{\iota\in I})$ is induced by a generalized algebra if the following condition is fulfilled:

$$A_\varkappa \subseteq B_\varkappa \text{ for all } \varkappa \in K_\iota \text{ implies } F_\iota(A_\varkappa \mid \varkappa \in K_\iota) \subseteq F_\iota(B_\varkappa \mid \varkappa \in K_\iota) \text{ for each } \iota \in I,\ (A_\varkappa)_{\varkappa\in K_\iota},\ (B_\varkappa)_{\varkappa\in K_\iota} \in (2^A)^{K_\iota}.$$

This condition is merely necessary for $(2^A, (F_\iota)_{\iota\in I})$ to be induced by a generalized algebra, which follows immediately from 2.2, but it is not sufficient. It is possible to show it by the following example:

$A = \{a, b, c\}$, i. e. $2^A = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$.

Define a unique unary operation F on 2^A in the following way:

$$F(\emptyset) = \emptyset,\ F(\{a\}) = \{a\},\ F(\{b\}) = \{b\},\ F(\{c\}) = \{c\},\ F(B) = A$$

for

$$B = \{a, b\}, \{a, c\}, \{b, c\}, A.$$

The algebra $(2^A, F)$ clearly fulfils the condition mentioned in the remark, but it is induced by no generalized algebra (A, f).

2.4. Definition. Let A be a non-void set, ϑ an equivalence relation on A. Define an equivalence relation Θ on 2^A in the following way: $B\Theta C$ if and only if there exists to each $b \in B$ such $c \in C$ and to each $c' \in C$ such $b' \in B$ that $b\vartheta c$, $b'\vartheta c'$.

The equivalence relation Θ is called *induced* by the equivalence relation ϑ.

2.5. Theorem. *Let $(A, (f_\iota)_{\iota\in I})$ be a generalized algebra, $(2^A, (F_\iota)_{\iota\in I})$ the algebra induced by it. Then the following statements are equivalent:*

(A) *An equivalence relation Θ on 2^A is induced by a congruence relation on $(A, (f_\iota)_{\iota\in I})$.*

(B) *An equivalence relation Θ on 2^A is a congruence relation on $(2^A, (F_\iota)_{\iota\in I})$ fulfilling the condition: $B\Theta C$ if and only if there exists to each $b \in B$ such $c \in C$ that $\{b\}\,\Theta\{c\}$ holds.*

Proof. 1. Let (A) hold. Let ϑ be the respective congruence relation on $(A, (f_\iota)_{\iota\in I})$. Let $\iota \in I$, $(A_\varkappa)_{\varkappa\in K_\iota}$, $(B_\varkappa)_{\varkappa\in K_\iota} \in (2^A)^{K_\iota}$, let $A_\varkappa \Theta B_\varkappa$ hold for all $\varkappa \in K_\iota$. Then there exists to each $a_\varkappa \in A_\varkappa$ such $b_\varkappa \in B_\varkappa$ that $a_\varkappa \vartheta b_\varkappa$ holds for each $\varkappa \in K_\iota$. Let $a_0 \in F_\iota(A_\varkappa \mid \varkappa \in K_\iota) = \{a \mid a \in f_\iota(a_\varkappa \mid \varkappa \in K_\iota),$ $a_\varkappa \in A_\varkappa$ for all $\varkappa \in K_\iota\}$. Consequently there exist $a_\varkappa \in A_\varkappa$ such that $a_0 \in f_\iota(a_\varkappa \mid \varkappa \in K_\iota)$. By the preceding, there exist $b_\varkappa \in B_\varkappa$ such that $a_\varkappa \vartheta b_\varkappa$ holds for each $\varkappa \in K_\iota$. From this it follows that there exists $b_0 \in f_\iota(b_\varkappa \mid \varkappa \in K_\iota) \subseteq \{b \mid b \in f_\iota(b_\varkappa \mid \varkappa \in K_\iota),\ b_\varkappa \in B_\varkappa$ for all $\varkappa \in K_\iota\} = F_\iota(B_\varkappa \mid \varkappa \in K_\iota)$ such that $a_0 \vartheta b_0$ holds. Therefore Θ is a congruence relation on $(2^A, (F_\iota)_{\iota\in I})$. By 2.4, we have $B\Theta C$ if and only if there exists to each $b \in B$ such $c \in C$ and to each $c' \in C$ such $b' \in B$ that $b\vartheta c$, $b'\vartheta c'$. But since Θ is an equivalence relation, consequently a symmetric relation, it is possible to omit the second condition. Therefore $B\Theta C$ if and only if there exists to each $b \in B$ such $c \in C$ that $b\vartheta c$, i.e. if and only if there exists to each $b \in B$ such $c \in C$ that $\{b\}\Theta\{c\}$.

2. Let (B) hold. Put $a\vartheta b$ if and only if $\{a\}\Theta\{b\}$. Let $B\Theta C$. Then there exists to each $b \in B$ such $c \in C$ that $\{b\}\Theta\{c\}$, i.e. $b\vartheta c$. Since Θ is an

equivalence relation, consequently a symmetric relation, there exists also to each $c' \in C$ such $b' \in B$ that $\{b'\}\Theta\{c'\}$, i.e. $b'\vartheta c'$. Hence Θ is induced by ϑ. It remains to show that ϑ is a congruence relation on $(A, (f_\iota)_{\iota\in I})$. Let $\iota \in I$, $(a_\varkappa)_{\varkappa\in K_\iota}$, $(b_\varkappa)_{\varkappa\in K_\iota} \in A^{K_\iota}$, let $a_\varkappa\vartheta b_\varkappa$ hold for all $\varkappa \in K_\iota$. By the definition of ϑ, we have $\{a_\varkappa\}\Theta\{b_\varkappa\}$ for all $\varkappa \in K_\iota$. Hence $F_\iota(\{a_\varkappa\} \mid \varkappa \in K_\iota)\ \Theta F_\iota(\{b_\varkappa\} \mid \varkappa \in K_\iota)$, consequently there exists to each $a \in f_\iota\ (a_\varkappa \mid \varkappa \in K_\iota)$ such $b \in f_\iota(b_\varkappa \mid \varkappa \in K_\iota)$ that $a\ \vartheta\ b$.

2.6. Corollary. *Let $(A, (f_\iota)_{\iota\in I}$ be a generalized algebra, $(2^A, (F_\iota)_{\iota\in I}$ the algebra induced by it. Let ϑ be an equivalence relation on A, Θ the equivalence relation on 2^A induced by ϑ. Then the following statements are equivalent:*

(A) *ϑ is a congruence relation on $(A, (f_\iota)_{\iota\in I})$.*

(B) *Θ is a congruence relation on $(2^A, (F_\iota)_{\iota\in I})$.*

Proof. 1. Let (A) hold. Then also (B) holds by 2.5.

2. Let (B) hold. By 2.4, we have $B\ \Theta\ C$ if and only if there exists to each $b \in B$ such $c \in C$ and to each $c' \in C$ such $b' \in B$ that $b\ \vartheta\ c$, $b'\ \vartheta\ c'$, i.e. $\{b\}\ \Theta\ \{c\}$, $\{b'\}\ \Theta\ \{c'\}$. Considering that Θ is a congruence relation on $(2^A, (F_\iota)_{\iota\in I})$, consequently it is a symmetric relation and the second condition can be omitted. Hence $B\ \Theta\ C$ if and only if there exists to each $b \in B$ such $c \in C$ that $\{b\}\ \Theta\ \{c\}$. Consequently Θ is a congruence relation on $(2^A, (F_\iota)_{\iota\in I})$ fulfilling the condition in 2.5. By 2.5, ϑ is a congruence relation on $(A, (f_\iota)_{\iota\in I})$.

2.7. Definition. Let A, B be non-void sets, φ a mapping of A into B. Define a mapping Φ of 2^A into 2^B in the following way: $\Phi(C) = \varphi[C]$ for each $C \subseteq A$. The mapping Φ is called *induced* by the mapping φ.

2.8. Theorem. *Let $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$ be generalized algebras of the same type, $(2^A, (F_\iota)_{\iota\in I})$, $(2^B, (G_\iota)_{\iota\in I})$ the algebras induced by them. Then the following statements are equivalent:*

(A) *A mapping Φ of 2^A into 2^B is induced by a homomorphism of $(A, (f_\iota)_{\iota\in I})$ into $(B, (g_\iota)_{\iota\in I})$.*

(B) *A mapping Φ of 2^A into 2^B is a homomorphism of $(2^A, (F_\iota)_{\iota\in I})$ into $(2^B, (G_\iota)_{\iota\in I})$ fulfilling the conditions:*

a) card $\Phi(\{a\}) = 1$ *for all* $a \in A$;

b) $x \in \Phi(C)$ *if and only if there exists* $c \in C$ *such that* $\Phi(\{c\}) = \{x\}$ *holds.*

Proof. 1. Let (A) hold. Let φ be the respective homomorphism of $(A, (f_\iota)_{\iota\in I})$ into $(B, (g_\iota)_{\iota\in I})$. Let $\iota \in I$, $(A_\varkappa)_{\varkappa\in K_\iota} \in (2^A)^{K_\iota}$. Then $\Phi(F_\iota(A_\varkappa \mid \varkappa \in K_\iota)) = \varphi[F_\iota(A_\varkappa \mid \varkappa \in K_\iota)] = \varphi[\{a \mid a \in f_\iota(a_\varkappa \mid \varkappa \in K_\iota)$. $a_\varkappa \in A_\varkappa$ for all $\varkappa \in K_\iota\}] = \{\varphi(a) \mid a \in f_\iota(a_\varkappa \mid \varkappa \in K_\iota), a_\varkappa \in A_\varkappa$ for all $\varkappa \in K_\iota\} =$ $= \{b \mid b \in g_\iota(\varphi(a_\varkappa) \mid \varkappa \in K_\iota), a_\varkappa \in A_\varkappa$ for all $\varkappa \in K_\iota\} = G_\iota(\varphi[A_\varkappa] \mid \varkappa \in K_\iota) =$ $= G_\iota(\Phi(A_\varkappa) \mid \varkappa \in K_\iota)$, so that Φ is a homomorphism of $(2^A, (F_\iota)_{\iota\in I}$, into $(2^B, (G_\iota)_{\iota\in I})$. Further, card $\Phi(\{a\}) =$ card $\varphi[\{a\}] = 1$ for each $a \in A)$

Lastly, we have $x \in \Phi(C) = \varphi[C]$ if and only if there exists $c \in C$ such that $\varphi(c) = x$, i.e. $\Phi(\{c\}) = \{x\}$.

2. Let (B) hold. Put $\varphi(a) = b$ if and only if $\Phi(\{a\}) = \{b\}$. By a), it is assigned to each $a \in A$ a unique $b \in B$ in this way, i.e. φ is a mapping of A into B. By b), we have $\Phi(C) = \{x \mid x \in A$, there exists $c \in C$, $\Phi(\{c\}) = \{x\}\} = \{x \mid x \in A$, there exists $c \in C$, $\varphi(c) = x\} = \varphi[C]$, so that Φ is induced by φ. It remains to show that φ is a homomorphism of $(A, (f_\iota)_{\iota\in I})$ into $(B, (g_\iota)_{\iota\in I})$. Let $\iota \in I$, $(a_\varkappa)_{\varkappa\in K_\iota} \in A^{K_\iota}$. Then $\varphi[f_\iota(a_\varkappa \mid \varkappa \in K_\iota)] = \varphi[F_\iota(\{a_\varkappa\} \mid \varkappa \in K_\iota)] = \Phi(F_\iota(\{a_\varkappa\} \mid \varkappa \in K_\iota)) = G_\iota(\Phi(\{a_\varkappa\}) \mid \varkappa \in K_\iota) = g_\iota(\varphi(a_\varkappa) \mid \varkappa \in K_\iota)$.

2.9. Corollary. *Let $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$ be generalized algebras of the same type, $(2^A, (F_\iota)_{\iota\in I})$, $(2^B, (G_\iota)_{\iota\in I})$ the algebras induced by them. Let φ be a mapping of A into B, Φ the mapping of 2^A into 2^B induced by φ. Then the following statements are equivalent:*

(A) *φ is a homomorphism of $(A, (f_\iota)_{\iota\in I})$ into $(B, (g_\iota)_{\iota\in I})$.*

(B) *Φ is a homomorphism of $(2^A, (F_\iota)_{\iota\in I})$ into $(2^B, (G_\iota)_{\in I})$.*

Proof 1. Let (A) hold. Then (B) holds by 2.8.

2. Let (B) hold. By 2.7, we have $\Phi(C) = \varphi[C]$. From this it follows that the conditions a) and b) in the statement (B) of 2.8 are fulfilled, so that φ is a homomorphism by 2.8, for Φ is a homomorphism.

2.10. Theorem. *Let $(A, (f_\iota)_{\iota\in I})$ be a generalized algebra, $(2^A, (F_\iota)_{\iota\in I})$ the algebra induced by it. Let $B \subseteq A$. Then the following statements are equivalent:*

(A) *B is a closed subset in $(A, (f_\iota)_{\iota\in I})$.*

(B) *2^B is a closed subset in $(2^A, (F_\iota)_{\iota\in I})$.*

Proof. 1. Let (A) hold. Let $\iota \in I$, $(A_\varkappa)_{\varkappa\in K_\iota} \in (2^B)^{K_\iota}$. Then $F_\iota(A_\varkappa \mid \varkappa \in K_\iota) = \{a \mid a \in f_\iota(a_\varkappa \mid \varkappa \in K_\iota),\ a_\varkappa \in A_\varkappa$ for all $\varkappa \in K_\iota\} \subseteq \{a \mid a \in f_\iota(a_\varkappa \mid \varkappa \in K_\iota)$, $a_\varkappa \in B$ for all $\varkappa \in K_\iota\} \subseteq B$, for B is a closed subset in $(A, (f_\iota)_{\iota\in I})$. Thus we have $F_\iota(A_\varkappa \mid \varkappa \in K_\iota) \in 2^B$, so that 2^B is a closed subset in $(2^A, (F_\iota)_{\iota\in I})$.
2. Let (B) hold. Let $\iota \in I$, $(a_\varkappa)_{\varkappa\in K_\iota} \in B^{K_\iota}$. Then $(\{a_\varkappa\})_{\varkappa\in K_\iota} \in (2^B)^{K_\iota}$, so that $F_\iota(\{a_\varkappa\} \mid \varkappa \in K_\iota) \in 2^B$. Hence $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = F_\iota(\{a_\varkappa\} \mid \varkappa \in K_\iota) \subseteq B$ and B is a closed subset in $(A, (f_\iota)_{\iota\in I})$.

2.11. Remark. The theorems analogous to 2.9 and 2.10 are stated in [2]. But the author assumes that $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \neq \varnothing$ holds for arbitrary $\iota \in I$ and for arbitrary $(a_\varkappa)_{\varkappa\in K_\iota} \in A^{K_\iota}$.

2.12. Remark. The construction of the algebra induced by a generalized algebra has an analogy in automata theory, namely when constructing the deterministic automaton generating the same language as a given nondeterministic automaton. If we do not regard to initial and final states, the given nondeterministic automaton can be considered as a generalized algebra $(S, (f_a)_{a \in A})$, where S is the set of states, A the alphabet. Each operation f_a is unary and defined by the relation $f_a(s) = f(s, a)$, where $f(s, a)$ is the function of transitions of the automaton. Then the respective deterministic automaton can be constructed by means of the induced algebra $(2^S, (F_a)_{a \in A})$ and its function of transitions is given by the relation $F(T, a) = F_a(T)$ for each $T \subseteq S$. (See [3].)

ALGEBRAS INDUCED BY SETS WITH A BINARY RELATION

3.1. Definition. Let A be a non-void set with a binary relation R. Define a unary generalized operation f on A in the following way:

$$f(a) = \{b \mid b \in A,\ (a, b) \in R\} \text{ for each } a \in A.$$

The unary algebra $(2^A, F)$ induced by the unary generalized algebra (A, f) is called *induced by the set A with the relation R*, the unary operation F is called *induced by the relation R*.

3.2. Theorem. *Let A be a non-void set with a binary relation R. Then the relation R is reflexive if and only if the induced operation F fulfils the condition $B \subseteq F(B)$ for each $B \subseteq A$.*

Proof. 1. Let R be reflexive, $x \in B \subseteq A$. Then $x \in f(x) \subseteq \{a \mid a \in f(b), b \in B\} = F(B)$, so that $B \subseteq F(B)$ holds for each $B \subseteq A$.

2. Let $B \subseteq F(B)$ holds for each $B \subseteq A$. Let $x \in A$. Then $\{x\} \subseteq F(\{x\}) = f(x)$, so that $x \in f(x)$ and R is reflexive.

3.3. Theorem. *Let A be a non-void set with a binary relation R. Then the relation R is symmetric if and only if the induced operation F fulfils the condition: $B \cap F(C) \neq \emptyset$ implies $C \cap F(B) \neq \emptyset$ for arbitrary $B, C \subseteq A$.*

Proof 1. Let R be symmetric, B, $C \subseteq A$, $x \in B \cap F(C)$. Then $x \in B$, $x \in F(C)$, thus there exists $y \in C$ such that $x \in f(y)$. Hence $y \in f(x) \subseteq \{a \mid a \in f(b), b \in B\} = F(B)$, so that $y \in C \cap F(B)$.

2. Let $B \cap F(C) \neq \emptyset$ implies $C \cap F(B) \neq \emptyset$ for arbitrary B, $C \subseteq A$. Let $x \in f(y)$. Then $\{x\} \cap F(\{y\}) = \{x\} \cap f(y) \neq \emptyset$. Hence $\{y\} \cap F(\{x\}) = \{y\} \cap f(x) \neq \emptyset$, so that $y \in f(x)$ and R is symmetric.

3.4. Theorem. *Let A be a non-void set with a binary relation R. Then the relation R is transitive if and only if the induced operation F fulfils the condition $F^2(B) = F(F(B)) \subseteq F(B)$ for each $B \subseteq A$.*

Proof 1. Let R be transitive, $B \subseteq A$, $x \in F^2(B) = \{a \mid a \in f(b), b \in F(B)\}$. Then there exists $y \in F(B)$ such that $x \in f(y)$. Further there exists $z \in B$ such that $y \in f(z)$. Hence $x \in f(z) \subseteq \{a \mid a \in f(b),\ b \in B\} = F(B)$, so that $F^2(B) \subseteq F(B)$.

2. Let $F^2(B) \subseteq F(B)$ for each $B \subseteq A$, $x \in f(y)$, $y \in f(z)$. Then $x \in F(\{y\}) \subseteq \{a \mid a \in f(b),\ b \in F(\{z\})\} = F^2(\{z\}) \subseteq F(\{z\}) = f(z)$ and R is transitive.

3.5. Theorem. *Let A be a non-void set with a binary relation R. Then the induced operation F has the following properties:*

1. $F(\emptyset) = \emptyset$.
2. *If B, $C \subseteq A$, then $F(B \cup C) = F(B) \cup F(C)$.*

Proof 1. Obvious. 2. By 2.2, we have $F(B \cup C) = \{a \mid a \in F(X),\ X = B, C\} = F(B) \cup F(C)$.

3.6. Theorem. *Let A be a non-void set with a binary relation R. Then the induced operation F fulfils the condition $F^2(B) = F(B)$ for each $B \subseteq A$ if and only if the relation R is transitive and x, $y \in A$, xRy imply the existence of $z \in A$ such that zRy, xRz.*

Proof 1. Let $F^2(B) = F(B)$ hold for each $B \subseteq A$. By 3.4, R is transitive. Let $y \in f(x) = F(\{x\})$. Then $y \in F^2(\{x\}) = \{a \mid a \in f(b), b \in F(\{x\})\}$, thus there exists $z \in f(x)$ such that $y \in f(z)$.

2. Let R be transitive and let $y \in f(x)$ imply the existence of $z \in A$ such that $z \in f(x)$, $y \in f(z)$. By 3.4, we have $F^2(B) \subseteq F(B)$ for all $B \subseteq A$. Let $B \subseteq A$, $u \in F(B) = \{a \mid a \in f(b),\ b \in B\}$. We have $u \in f(w)$ for some $w \in B$. Hence, it follows that there exists $v \in A$ such that $u \in f(v)$, $v \in f(w)$. Consequently, $u \in \{a \mid a \in f(b),\ b \in f(w)\} \subseteq \{a \mid a \in f(b),\ b \in F(B)\} = F^2(B)$ and $F^2(B) = F(B)$ holds.

3.7. Remark. In the remaining part of this paragraph topological concepts will be used in the sense of [1].

3.8. Theorem. *Let A be a non-void set with a binary relation R. Then the induced operation F is a closure if and only if the relation R is reflexive.*

Proof. It follows from 3.2 and 3.5.

3.9. Theorem. *Let A be a non-void set with a binary relation R. Then the induced operation F is a topological closure if and only if the relation R is a quasiordering.*

Proof. It follows from 3.4 and 3.8.

3.10. Theorem. *Let A be a non-void set with a binary relation R. Then the induced operation F is a feebly semi-separating closure if and only if the relation R is reflexive and antisymmetric.*

Proof. Regarding 3.8 it suffices to show that $x \in F(\{y\})$, $y \in F(\{x\})$ imply $x = y$ if and only if R is antisymmetric. But the foregoing implication holds if and only if $x \in f(y)$, $y \in f(x)$ imply $x = y$, i.e. if and only if R is antisymmetric.

3.11. Theorem. *Let A be a non-void set with a binary relation R. Then the induced operation F is a semi-separating closure if and only if A is an antichain with respect to the relation R.*

Proof. 1. Let F be a semi-separating closure, $x \in A$. Then $F(\{x\}) = f(x) = \{x\}$, so that xRy implies $x = y$ and A is an antichain with respect to R.

2. Let A be an antichain with respect to R. Then F is a feebly semi-separating closure by 3.10. Besides, we have $F(\{x\}) = f(x) = \{x\}$ for each $x \in A$, so that F is even a semi-separating closure.

3.12. Remark. The induced operation F is obviously a semi-separating closure if and only if it is a discrete closure. So if it is a semi-separating closure then it is also a separating, regular and normal closure.

4. SUM AND PRODUCT OF GENERALIZED ALGEBRAS

4.1. Definition. Let $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$ be generalized algebras of the same type and such that $A \cap B = 0$. For arbitrary $\iota \in I$ define a generalized operation $(f \cup g)_\iota$ on $A \cup B$ in the following way:

$$(f \cup g)_\iota\,(a_\varkappa \mid \varkappa \in K_\iota) = \begin{cases} \emptyset & \text{if there exist } \varkappa,\ \varkappa' \in K_\iota \text{ such} \\ & \text{that } a_\varkappa \in A,\ a_\varkappa' \in B; \\ f_\iota(a_\varkappa \mid \varkappa \in K_\iota) & \text{if } a_\varkappa \in A \text{ for all } \varkappa \in K_\iota; \\ g_\iota(a_\varkappa \mid \varkappa \in K_\iota) & \text{if } a_\varkappa \in B \text{ for all } \varkappa \in K_\iota. \end{cases}$$

The generalized algebra $(A \cup B, ((f \cup g)_\iota)_{\iota\in I})$ is called the *sum* of the generalized algebras $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$.

4.2. Definition. Let $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$ be generalized algebras of the same type. For arbitrary $\iota \in I$ define a generalized operation $(f \times g)_\iota$ on $A \times B$ in the following way:

$$(f \times g)_\iota\,((a_\varkappa, b_\varkappa) \mid \varkappa \in K_\iota) = (f_\iota(a_\varkappa \mid \varkappa \in K_\iota),\ g_\iota(b_\varkappa \mid \varkappa \in K_\iota)).$$

The generalized algebra $(A \times B, ((f \times g)_\iota)_{\iota\in I})$ is called the *product* of the generalized algebras $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$.

4.3. Theorem. *Let $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$ be generalized algebras of the same type and such that $A \cap B = \emptyset$. Let $(2^A, (F_\iota)_{\iota\in I})$, $(2^B, (G_\iota)_{\iota\in I})$, $(2^{A\cup B}, ((F \cup G)_\iota)_{\iota\in I})$ be the algebras induced by $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$, $(A \cup B, ((f \cup g)_\iota)_{\iota\in I})$. Then the algebras $(2^{A\cup B}, ((F \cup G)_\iota)_{\iota\in I})$, $(2^A \times 2^B, ((F \times G)_\iota)_{\iota\in I})$ are isomorphic.*

Proof. Both algebras are obviously of the same type. Define a bijective mapping φ of $2^A \times 2^B$ onto $2^{A\cup B}$ in the following way:

$$\varphi(C, D) = C \cup D \text{ for arbitrary } C \subseteq A,\ D \subseteq B.$$

The mapping φ is a homomorphism of $(2^A \times 2^B, ((F \times G)_\iota)_{\iota\in I})$ onto $(2^{A\cup B}, ((F \cup G)_\iota)_{\iota\in I})$. In fact, let $\iota \in I$, $(A_\varkappa)_{\varkappa\in K_\iota} \in (2^A)^{K_\iota}$, $(B_\varkappa)_{\varkappa\in K_\iota} \in (2^B)^{K_\iota}$. Then $\varphi((F\times G)_\iota\ ((A_\varkappa, B_\varkappa) \mid \varkappa \in K_\iota)) = \varphi(F_\iota(A_\varkappa \mid \varkappa \in K_\iota), G_\iota(B_\varkappa \mid \varkappa \in K_\iota)) = F_\iota(A_\varkappa \mid \varkappa \in K_\iota) \cup G_\iota(B_\varkappa \mid \varkappa \in K_\iota) = \{a \mid a \in f_\iota(a_\varkappa \mid \varkappa \in K_\iota), a_\varkappa \in A_\varkappa$ for all $\varkappa \in K_\iota\} \cup \{a \mid a \in g_\iota(a_\varkappa \mid \varkappa \in K_\iota), a_\varkappa \in B_\varkappa$ for all $\varkappa \in K_\iota\} = \{a \mid a \in (f \cup g)_\iota\,(a_\varkappa \mid \varkappa \in K_\iota), a_\varkappa \in A_\varkappa \cup B_\varkappa$ for all $\varkappa \in K_\iota\} = (F \cup G)_\iota\,(A_\varkappa \cup B_\varkappa \mid \varkappa \in K_\iota) = (F \cup G)_\iota\,(\varphi(A_\varkappa, B_\varkappa) \mid \varkappa \in K_\iota)$. Since φ is a bijective homomorphism, it is an isomorphism.

REFERENCES

[1] Čech E., *Topological spaces*, Academia, Prague 1966.
[2] Pickett H. E., *Homomorphisms and subalgebras of multialgebras*, Pacific J. Math. 21 (1967), 327—342.
[3] Rabin M. O., Scott D., *Finite automata and their decision problems*, IBM Journal 3 (1959), 114—125.
[4] Schmidt J., *Algebraic operations and algebraic independence in algebras with infinitary operations*, Math. Jap. VI, 77—112.

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