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MONOTONIC PROPERTIES OF ZEROS AND EXTREMANTS OF THE DIFFERENTIAL EQUATION $y'' + q(t)\,y = 0$

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In honour of the 70th birthday anniversary of Prof. O. Borůvka.

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1. The question of the distribution of zeros of solutions of the linear differential equation of the 2nd order plays an important role and has been studied by many authors. At the last time, by means of methods elaborated by O. Borůvka ([1], [2]), several problems of this kind have been solved. There are given certain properties of the distribution of zeros of oscillating solutions of the linear differential equation

$$y'' + q(t)\,y = 0 \tag{0.1}$$

and all differential equations of the form (0.1) (thus all continuous functions $q(t)$) the solutions of which having the required properties are to be determined (see e.g. [7]). The already mentioned methods, however, are not suitable enough for the investigation of sequences of zeros of solutions of the concrete equation (0.1).

Lee Lorch and Peter Szego have deduced a simple, sufficient condition for the monotonicity of order n in the sequence of differences of zeros of any solution of the equation (0.1) ([5]), which they later formulated on the basis of Hartman s paper [3] directly for the function $q(t)$ ([6]). In the paper [10] it is shown that the same condition is a sufficient one also for the monotonicity of order $n - 2$ of the sequence of differences of extremants of any solution of the equation (0.1).

This work studies more completely the sequences $\{t_k\}$ of zeros of solutions of the equation (0.1), compares it with the equidistant sequence in the case of the equation (0.1) being oscillatory and the function $q'(t)$ being monotonic of order n. Further there are studied sequences $\{T_k - t_k\}$, where $\{T_k\}$ or $\{t_k\}$, respectively, denotes the sequence of consecutive zeros or extremants of any solution of

$$Y'' + Q(t)\,Y = 0 \tag{0.2}$$

in the case of function $[q(t) - Q(t)]$ being monotonic of order $n + 1$.

The mentioned results enable to divide all equations of the form (0.1) onto the classes consisting of the equations which have "asymptotically equal" distribution of zeros.

In § 6 the achieved results are then applied to the Bessel equation for $|\nu| \geq 1/2$. In spite of these results being possible to be applied to a number of equations of the similar type, in this case the Bessel equation was chosen from this reason that its theory was considerably worked out and application is thus sufficiently telling. Besides the number of new results for the Bessel functions, some known properties of the Bessel functions, the direct derivation of which reaches far to the theory of the Bessel functions, follow from the particular theorems, whereas they are deduced, in such a way, relatively simply as the properties of solutions of the equation (0.1).

2. The extremant of the function $y(t) \in C_2$ is meant each number t_k where the function $y(t)$ assumes the locally extreme value (only proper local extremes are considered).

The function $F(t)$ is said to be of the class $M_n(a, b)$ or monotonic of the order n in (a, b) (see [3], p. 164), if it has n $(n \geqq 0)$ of continuous derivatives $F^{(0)}, F', \ldots, F^{(n)}$ satisfying

$$(0.3) \qquad (-1)^i F^{(i)}(t) \geqq 0 \text{ for } t \in (a, b) \qquad i = 0, 1, \ldots, n.$$

If the preceding inequalities are fulfilled for $i = 0, 1, \ldots$ then the function $F(t)$ is called a complete monotonic in (a, b) and is denoted by $F(t) \in \in M_\infty(a, b)$. The symbol M_n denotes $M_n(0, \infty)$.

The function $F(t)$ is called the function of class $M_{n,m}(T_0, \infty)$ if there is $F(t) \in M_n(T_0, \infty)$ and $F(t)$ has for $t > T_0$ m derivatives for which there hold

$$(0.4) \qquad F^{(i)}(t) \to 0 \text{ for } T \to \infty, \qquad i = 0, 1, \ldots, m.$$

(Evidently $F \in M_{n0}(T_0, \infty)$ implies $F \in M_{n,n-1}(T_0, \infty)$ for $n \geqq 1$.) Analogously $M_{nm} = M_{nm}(0, \infty)$.

Further let $\{t_k\}$ denote the sequence and $\Delta^n t_k$ an n-th difference of the sequence $\{t_k\}$, so that

$$(0.5) \qquad \Delta^0 t_k = t_k \qquad \Delta t_k = t_{k+1} - t_k, \ldots, \Delta^n t_k = \Delta^{n-1} t_{k+1} - \Delta^{n-1} t_k$$
$$k = 0, 1, 2, \ldots, \quad n = 1, 2, \ldots$$

The sequence $\{t_k\}$ is called monotonic of order n if

$$(0.6) \qquad (-1)^i \Delta^i t_k \geqq 0 \qquad k = 0, 1, \ldots, \quad i = 1, 2, \ldots, n.$$

If there is $n = \infty$ then the sequence $\{t_k\}$ is called a complete monotonic.

The notion of the sequence of zeros, resp. of extremants of the solution $y(t)$ is understood any increasing sequence of consecutive zeros, resp. of extremants of the solution $y(t)$. In all the work, provided it is not explicitly mentioned otherwise, we suppose $t > 0$.

3. If $x(t)$, $y(t)$ denotes a pair of independent solutions of the equation (0.1) and $w = xy' - x'y = \text{const} \neq 0$, it is possible, by the transformation

(0.7) $$y(t) = [v(t)]^{1/2}\, u(s), \qquad t'(s) = v(t), \qquad v(t) = x^2(t) + y^2(t),$$

to transform the equation (0.1) onto the equation with constant coefficients

(0.8) $$u''(s) + w^2 u(s) = 0$$

([2], p. 229; [5], p. 59.)

Further it is known that, under the supposition of $0 < q(\infty) < \infty$, $q' \in M_n$ ($n \geqq 0$), the equation (0.1) has such a pair of solutions $x(t)$, $y(t)$ to any constant $c_1 > 0$ that for $v(t) = x^2 + y^2$ there holds $[v(t) - c_1] \in \in M_{n,n+2}$. The function $v(t)$ is determined uniquely. The triad x^2, xy, y^2 denotes then a fundamental system of solutions of Appel's equation

(0.9) $$v''' + 4q(t)\, v' + 2q'(t)\, v = 0$$

which has a unique solution $v(t)$ of the already mentioned properties ([3], p. 182).

4. In this work there is substantially made full use of theorems proved by P. Hartman, Lee Lorch and P. Szego, which we mention, even though in a little adapted form.

Lemma 0.1. ([3] *Theorem* 12.1_n, 12.2_n, *p.* 171)

Let $n \geqq 0$, $k \geqq 1$ *and let the following conditions are fulfilled*

α) $q(t)$ possess a derivative $q'(t)$ of class M_n, $0 < q(\infty) < \infty$, $f(t) \in \in M_{n+1,0}$ and $q_j(t) \in M_{n+1}$, for $j = 0, 1, \ldots, k-1$.

Further let there exist positive continuous functions $\varepsilon_1(t), \ldots, \varepsilon_k(t)$ such that for $t \to \infty$ and $m = 1, \ldots, k$ there holds

β) $$\int_t^\infty s^{m-1}[q(s)]^{-1}\, F(s)\, ds = O(\varepsilon_m(t))$$

γ) $$\sum_{j=0}^{k-1} \int_t^\infty s^{m-1}[q(s)]^{-1}\, q_j(s)\varepsilon_{k-j}(s)\, ds = o(\varepsilon_m(t))$$

Then the differential equation

(0.10) $$v^{(k+2)} + q(t)\, v^{(k)} - \sum_{j=0}^{k-1} (-1)^{j+k}\, q_j(t)\, v^{(j)} = (-1)^k f(t)$$

has a unique solution $v = v(t)$ of class $M_{n+k,n+k+2}$, satisfying

(0.11) $$v^{(j)}(t) = O(\varepsilon_{k-j}(t)) \text{ as } t \to \infty, \qquad j = 0, 1, \ldots, k-1$$

Remark 0.1. If we choose $\varepsilon_j(t) = t^{j-k}$ for $j = 1, \ldots, k$ then the conditions β), γ) turn into the conditions

β') $\quad |\int^{\infty} t^{k-1} f(t) \, [q(+)]^{-1} \, dt \,| < \infty$

γ') $\quad |\int^{\infty} t^{k-j-1} q_j(t) \, [q(+)]^{-1} \, dt \,| < \infty$ for $j = 0, \ldots, k-1$.

The solution $v(t)$ satisfying $v(t) \in M_{n+k,n+k+2}$ is, in this case, unique.

Lemma. 0.2. ([5], *Lemma* 5.1, *p.* 64)

Let an open interval I_s *of the variable* s *be mapped onto an open interval* I_t, *resp.* I_T *by a mapping* $t'(s) = v(t)$, *resp.* $T'(s) = V(T)$, *where* $v(t)$, $V(T)$ *are the given positive functions such that* $v^{(n)}(t)$, $V^{(n)}(T)$ *exist in an open interval* $I_{tT} = I_t \cap I_T$. *Further suppose that the inequalities*

$$(0.12) \qquad (-1)^i \, V^{(i)}(T) \geqq (-1)^i \, v^{(i)}(t) > 0 \qquad i = 0, 1, \ldots, n$$

are fulfilled for t, $T \in I_{tT}$, *where* t, T *in* (0,12) *correspond to the same value* s. *Then*

$$(0.13) \qquad (-1)^i \, T^{(i+1)}(s) \geqq (-1)^i \, t^{(i+1)}(s) > 0 \qquad i = 0, 1, \ldots, n$$

for $s \in I_s$.

Remark 0.2. ([4], p. 70, Lemma 2.2, p. 58)

Lemma 0.2. remains valid if we replace, in the right part of the inequalities (0.12) and (0.13), the sign "$>$" simultaneously by the sign "$\geqq$". Similarly the sign "$\geqq$" in the left side of the inequalities (0.12) and (0.13) may be replaced at the same time by the sign "$>$", resp. "$=$".

In addition to it we mention some simple properties of the monotonic functions of the order n, more often used in the following.

Lemma 0.3. *Let* $[f(t) - c] \in M_{n,0}$ $(n \geqq 1)$, *where* $0 \leqq c < \infty$ *and* $f'(t) \neq 0$ *for* $t \in (0, \infty)$. *Then*

$$(0.14) \qquad (-1)^i f^{(i)}(t) > 0 \qquad i = 0, 1, \ldots, n-1$$

Proof: Suppose that there exist the integer $N \leqq n-1$ and the number $\xi > 0$ such that $f^{(N)}(\xi) = 0$. Because the function $(-1)^{(N)} \cdot f^{(N)}(t)$ is non-negative and non-increasing, $f^{(N)}(t) \equiv 0$ holds for $t \geqq \xi$. Consequently, on the interval $\langle \xi, \infty)$ there is $f(t)$ polynomial of the degree maximum $N-1$. Because of $[f(t) - c] \in M_{n,0}$ there is $f(\infty) = c < \infty$. Therefore it is necessary $f(t) \equiv c$ for $t \geqq \xi$, which is in contradiction with the supposition $f'(t) \neq 0$. This proved the lemma.

Remark 0.3. If there is $n \geqq 0$, $q' \in M_n$, $q'(\xi) = 0$ $(\xi > 0)$, then equally $q'(t) = 0$ for $t \geqq \xi$ and for the solution $v(t) \in M_n$ of the equation (0.9) there holds $v(t) \equiv$ const. for $t \geqq \xi$. Similarly if there is $v(t) = x^2 + y^2 =$ const. $\neq 0$ for $t \geqq \xi$, then $q(t)$ is also a constant ([5], p. 72).

I. FUNCTION $v(t)$

As it was stated, if $x(t)$, $y(t)$ denote two linearly independent solutions of the equation (0.1), then x^2, xy, y^2 form a fundamental system of solutions of equation (0.9). We prove now the existence of the integral $v = v(t)$ of equation (0.9), which is of the form $v = x^2 + y^2$ and fulfils the second part of the inequalities (0.12). From the properties of the function $v(t)$, namely with regard to the transformation (0.7) it is possible to deduce the distribution of zeros of solutions of (0.1).

Lemma 1.1. *Let $q(t)$ possess a derivative $q'(t)$ of class M_n $(n \geqq 1)$, $q'(t) \neq 0$ for $t \in (0, \infty)$ $0 < q(\infty) < \infty$ and $c_1 > 0$ be an arbitrary constant. Then the equation* (0.1) *has a pair of solutions $x(t)$, $y(t)$ such that*

$$v(t) = x^2(t) + y^2(t) > 0 \tag{1.1}$$

satisfies

$$[v(t) - c_1] \in M_{n,n+2} \tag{1.2}$$

$$(-1)^i \, v^{(i)}(t) > 0 \qquad i = 0, 1, \ldots, n-1. \tag{1.3}$$

The pair of solutions (x, y) is unique up to their replacement by $(ax + by, cx + dy)$, where a, b, c, d are constants such that $a^2 + c^2 = b^2 + d^2 = 1$, $ab + cd = 0$.

Proof: The equation (0.9) fulfils the suppositions of Lemma 0.1 and P. Hartman has shown ([3], p. 182) that this unique solution $v(t) \in M_n$ of the equation (0.9) is of the form (1.1) inclusive of uniqueness and conditions for a, b, c, d. It remains to prove the validity of the inequality (1.3). This, of course, directly follows from (0.9) and lemma 0.3 because $q'(t) \neq 0$.

It is necessary to know the value of the Wronskian $w = xy' - x'y$ for every transformation function $v = x^2 + y^2$ in order that we may study the transformation (0.7) in more details.

Lemma 1.2. *Let $q(t)$ possess a derivative $q'(t) \in M_n$ $(n \geqq 1)$ and $0 < q(\infty) < \infty$. Let (x, y) denote a pair of solutions of the equation* (0.1) *satisfying*

$$v = x^2 + y^2, \qquad [v(t) - c_1] \in M_{n,n+2}. \tag{1.4}$$

If $w(x, y) = xy' - x'y$ denotes their Wronskian, then for all pairs of solutions (x_1, y_1) of the equation (0.1) *which satisfies*

$$x_1^2 + y_1^2 = x^2 + y^2 = v(t) \in M_n \tag{1.5}$$

there hold

$$(1.6) \qquad | w(x, y) | = | w(x_1, y_1) | = w \ (= \text{const})$$

$$(1.7) \qquad w = v(\infty) [q(\infty)]^{1/2}.$$

Proof: Solutions x, y form a fundamental system of solutions of equation (0.1). Thus there hold

$$x_1 = a_{11}x + a_{12}y, \qquad y_1 = a_{21}x + a_{22}y$$

Therefore

$$w(x_1, y_1) = x_1y_1' - x_1'y_1 =$$
$$= (a_{11}x + a_{12}y)(a_{21}x' + a_{22}y') - (a_{11}x' + a_{12}y')(a_{21}x + a_{22}y) =$$
$$= (a_{11}a_{22} - a_{12}a_{21})(xy' - x'y) = (a_{11}a_{22} - a_{12}a_{21})\, w(x, y).$$

The first equality in (1.5), with regard to Lemma 1.1, implies

$$a_{11}^2 + a_{21}^2 = a_{12}^2 + a_{22}^2 = 1, \qquad a_{11}a_{12} + a_{21}a_{22} = 0.$$

Under these conditions the expression $(a_{11}a_{22} - a_{12}a_{21})$ can assume only the values ± 1, and therefore (1.6) holds.

Denote now $\varrho(t) = x^2 + y^2$, where (x, y) denote any pair of independent solutions of the equation (0.1). Then ϱ complies the differential equation

$$(1.8) \qquad \varrho'' = -q(t)\, \varrho + w^2 \varrho^{-3}$$

where $w = xy' - yx'$ denotes the Wronskian of a pair (x, y). ([1], p. 201.) Let now $v(t) = \varrho^2(t)$. From (1.8) it follows that $v = v(t)$ fulfils the identity (Mamman's identity)

$$(1.9) \qquad v''v - \frac{1}{2} v'^2 + 2q(t)\, v^2 = 2w^2.$$

For the integral $v(t)$ satisfying (1.4), however, there hold $v(\infty) = c_1$, $v'(\infty) = 0$, $v''(\infty) = 0$ and therefore from (1.9) there follows (1.7).

Lemma 1.3. *Let $q(t)$ possess a derivative $q'(t) \in M_1$ and $0 < q(\infty) < \infty$. If $v(t)$ denotes the solution of* (0.9), *for which $[v(t) - c_1] \in M_{1,3}$ holds, then the integrals*

$$(1.10) \qquad \int^{\infty} [q(\infty) - q(t)]\, dt, \qquad \int^{\infty} [v(t) - v(\infty)]\, dt$$

converge or diverge at the same time.

Proof. The identity (1.9) is possible to be written in the form

$$(1.11) \qquad q(t)\, v^2(t) - w^2 = \frac{1}{4} [v'(t)]^2 - \frac{1}{2} v''(t)\, v(t).$$

By the integration (1.11) we get, for $v(t) \in M_1$ and t, sufficiently large

$$J(t) = \int_t^\infty [q(s)\, v^2(s) - w^2]\, ds = \frac{1}{4}\, [v(s)\, v'(s)]_t^\infty - \frac{3}{4} \int_t^\infty v(s)\, v''(s)\, ds \tag{1.12}$$

Because $[v(t) - c_1] \in M_{1,3}$ we have

$$0 \leqq \int_t^\infty v(s)\, v''(s)\, ds \leqq v(t) \int_t^\infty v''(s)\, ds = -v(t)\, v'(t). \tag{1.13}$$

From (1.12) and (1.13) it follows that $|\, J(t)\, | \leqq -v(t)\, v'(t) < \infty$ that is integral $J(t)$ is convergent.

On the other hand there identically holds

$$q(t)\, [v^2(t) - v^2(\infty)] = [q(\infty) - q(t)]\, v^2(\infty) + q(t)\, v^2(t) - q(\infty)\, v^2(\infty) \tag{1.14}$$

Hence using (1.7) we get

$$\begin{aligned} v(t) - v(\infty) = {} & v^2(\infty)\, [q(\infty) - q(t)]\, \{q(t)\, [v(t) + v(\infty)]\}^{-1} + \\ & + [q(t)\, v^2(t) - w^2]\, \{q(t)\, [v(t) + v(\infty)]\}^{-1}. \end{aligned} \tag{1.15}$$

Since for $t > t_0$, t_0 sufficiently large

$$0 \leqq 2q(t_0)\, v(\infty) \leqq q(t)\, [v(t) + v(\infty)] \leqq 2q(\infty)\, v(t_0) < \infty, \tag{1.16}$$

holds, by the integration (1.15) in the interval (t, ∞) we get

$$\begin{aligned} 0 \leqq \int_t^\infty [v(s) - v(\infty)]\, ds \leqq {} & [2q(t_0)]^{-1}\, v(\infty) \int_t^\infty [q(\infty) - q(s)]\, ds + \\ & + [2q(t_0)\, v(\infty)]^{-1}\, |\, J(t)\, |. \end{aligned} \tag{1.17}$$

If there is $0 \leqq \int_t^\infty [q(\infty) - q(s)]\, ds < \infty$, then with regard to the fact that the integral $J(t)$ converges, also the integral $\int_t^\infty [v(s) - v(\infty)]\, ds$ converges.

Similarly

$$\begin{aligned} \int_t^\infty [v(s) - v(\infty)]\, ds \geqq {} & v^2(\infty)\, [2q(\infty)\, v(t_0)]^{-1} \int_t^\infty [q(\infty) - q(s)]\, ds - \\ & - [2q(\infty)\, v(t_0)]^{-1}\, |\, J(t)\, |. \end{aligned} \tag{1.18}$$

If $\int_t^\infty [q(\infty) - q(s)]\, ds$ diverges, then with regard to the convergence of the integral $J(t)$, the divergence of the integral $\int_t^\infty [v(s) - v(\infty)]\, ds$ follows from (1.18). This proved the Lemma.

II. THE DISTRIBUTION OF ZEROS OF SOLUTIONS OF THE EQUATION (0.1)

We compare now the sequence $\{t_k\}$ of zeros of any solution of (0.1) with an equidistant sequence. To this purpose we first deduce the lemma proving necessary properties of the mapping (0.7).

Lemma 2.1. *For the functions* $s(t)$, $S(T)$, *let* $S(T_0) = s(t_0) = 0$ *hold, where* $0 < T_0 < t_0 < \infty$ *and*

$$(2.1) \quad S(t) > s(t) > 0 \quad s'(t) > S'(t) > 0 \quad s''(t) \geqq 0 \text{ for } t \in (0, \infty).$$

Let further hold

$$0 \leqq \lim_{s\to\infty} [S(t) - s(t)] = A < \infty, \qquad 0 < s'(\infty) = S'(\infty) = a < \infty$$

and let $t(s) = s^{-1}(t)$, *resp.* $T(S) = S^{-1}(T)$ *denote an inversion function to the function* $s(t)$, *resp.* $S(T)$. *Then*

$$(2.2) \qquad t(s) > T(s) > 0 \quad \text{for} \quad s \in (0, \infty)$$

$$(2.3) \qquad \lim_{s\to\infty} [t(s) - T(s)] = \frac{1}{a} \lim_{T\to\infty} [S(T) - s(T)] = \frac{A}{a}$$

holds.

Proof. Since $0 < s'(\infty) = S'(\infty) < \infty$ and consequently $s(\infty) = S(\infty) = \infty$, the functions $t(s)$, $T(S)$ are defined for s, $S \in (0, \infty)$. Since $S(T_0) = s(t_0) = 0$ there is $T_0 = T(0)$, $t_0 = t(0)$ and with regard to the fact that $t_0 > T_0 > 0$ there holds $0 < T(0) < t(0)$. From the continuity of functions $T(S)$, $t(s)$ it follows that $T(s) < t(s)$ for $s \in (0, \xi)$ where $\xi > 0$ is a suitable number, or $\xi = \infty$. Suppose that such $\xi < \infty$ exists that $T(\xi) = t(\xi) = \eta$ ($\eta < \infty$, because $o < t'(\infty) = T'(\infty) < \infty$). Then it holds $\xi = S(T(\xi)) = s(t(\xi))$, thus there exists the number $\eta < \infty$ such that $S(\eta) = s(\eta)$, which is in contradiction with the supposition $S(t) > s(t)$ for $t \in (0, \infty)$. Because $T(0) = T_0 > 0$ and $T(S)$ is increasing, (2.2) holds.

Let t, T correspond to the same value s, resp. S. Since $s''(t) \geqq 0$, for $s \geqq 0$ and $t \geqq t_0$, $T \geqq T_0$ with regard to (2.1), it holds

$$(2.4) \qquad s'[T] \leqq [S(T) - s(T)]\,[t(S) - T(S)]^{-1} \leqq s'(t).$$

Since t, $T \to \infty$ for $s \to \infty$ and $s'(\infty) = a$, from (2.4) there follows

$$(2.5) \qquad [S(T) - s(T)]\,[t(S) - T(S)]^{-1} \to a \text{ for } s \to \infty$$

and hence (2.3). By that the lemma is proved.

Now we prove the first result concerning the distribution of zeros of the equation (0.1), which is, at the same time, the main result of this paragraph.

Theorem 2.1. *Let $q(t)$ possess a derivative $q'(t) \in M_n$ $(n \geqq 1)$, $q'(t) \neq 0$ for $t \in (0, \infty)$ and $0 < q(\infty) = a^{-2} < \infty$ $(a > 0)$. If $\{t_k\}$ denotes the sequence of zeros of any solution of* (0.1), *then there exists such number $M > 0$ that*

$$(2.6)\quad (-1)^i \Delta^i[M + k\pi a - t_k] > 0; \quad (-1)^{n+1} \Delta^{n+1}[M + k\pi a - t_k] \geqq 0$$

$$k = 0, 1, \ldots, \quad i = 0, 1, \ldots, n$$

$$(2.7)\qquad \{M + k\pi a - t_k\} \to 0 \text{ for } k \to \infty$$

hold if and only if $\int^{\infty} [q(\infty) - q(t)]\, \mathrm{d}t$ *converges.*

Proof. The proof of the theorem is done by comparing the sequences of zeros of solutions $Y(T)$ of the equation

$$(2.8)\qquad Y''(T) + Q(T)\, Y(T) = 0$$

with the sequence of zeros of a suitable solution $y(t)$ of the equation

$$(2.9)\qquad y''(t) + a^{-2} y(t) = 0.$$

If we put $T = t$ and $Q(t)$ replaced by $q(t)$, the equation (2.8) will play the role of the equation (0.1) in Theorem 2.1. Consequently, let further $\{T_k\}$ denote the sequence of zeros of any solution of (2.8) and $\{t_k\} = \{M + k\pi a\}$ denote the sequence of zeros of any solution of (2.9).

1. According to Lemma 1.1 the equation (2.8) has solutions $X(T)$, $Y(T)$ such that for $V(T) = X^2 + Y^2$

$$(2.10)\quad [V(T) - a] \in M_{n,n+2}; \quad (-1)^i\, V^{(i)}(T) > 0, \quad i = 0, 1, \ldots, n - 1$$

holds.

Thus, for $W = XY' - X'Y$ it holds according to Lemma 1.2

$$(2.11)\qquad W^2 = V^2(\infty)\, Q(\infty) = a^2 \,.\, a^{-2} = 1.$$

Similarly the equation (2.9) has solutions

$$(2.12)\qquad x(t) = a^{1/2} \sin a^{-1}t, \qquad y(t) = a^{1/2} \cos a^{-1}t$$

for which, as it may be instantly seen,

$$(2.13)\qquad v(t) = x^2(t) + y^2(t) = a \qquad w = xy' - x'y = 1$$

hold.

2. Transformations

$$(2.14)\qquad Y(T) = [V(T)]^{1/2}\, U(S); \quad T'(s) = V(T)$$

$$y(t) = a^{1/2} u(s); \qquad t'(s) = a$$

transform the equations (2.8) and (2.9) onto the equations

$$U''(S) + U(S) = 0 \qquad u''(s) + u(s) = 0. \tag{2.15}$$

Let us choose the integration constants in (2.14) so that

$$S(T) = \int_{T_0}^{T} \frac{\mathrm{d}r}{V(r)} \qquad s(t) = \int_{M}^{t} \frac{\mathrm{d}r}{a} = \frac{1}{a}(t - M) \tag{2.16}$$

where $T_0 > 0$ is an arbitrary but fixed number and $M > T_0$ is, for the time being, closely non-specified. Then

$$S_k = S(T_k) = s(t_k) = s(M + k\pi a) = k\pi = s_k \qquad k = 0, 1, \ldots \tag{2.17}$$

($\{S_k\} = \{s_k\} = \{k\pi\}$ are the sequences of zeros of suitable solutions of (2.15)).

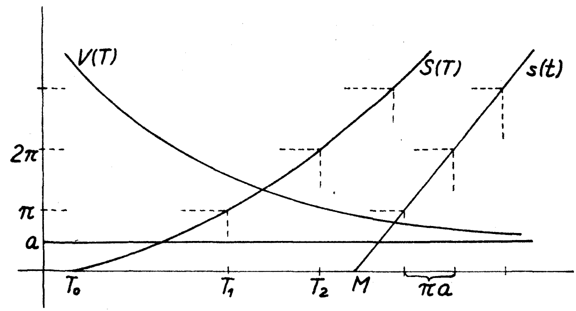

Fig. 1

Because $M > T_0$, with regard to (2.10)

$$S(M) = \int_{T_0}^{M} \frac{\mathrm{d}t}{V(t)} > \int_{M}^{M} \frac{\mathrm{d}t}{v(t)} = S(M) = 0 \tag{2.18}$$

and further

$$0 < S'(t) = V^{-1}(t) \leqq a^{-1} = s'(t) \tag{2.19}$$

where equality occurs only for $t = \infty$. Thus, the function $[S(t) - s(t)]$ is continuous, decreasing and positive in an interval $\langle M, \xi)$ where $\xi > M$ is a suitable number, or $\xi = \infty$ so that $S(t) > s(t) > 0$ for $t \in \langle M, \xi)$.

3. Suppose now that the integral $\int^{\infty} [Q(\infty) - Q(t)]\, \mathrm{d}t$ converges. Then the integral $\int^{\infty} [a^{-1} - V^{-1}(t)]\, \mathrm{d}t$ converges as well, because $V(\infty) = a$ and

$$(2.20) \qquad \int^{\infty} \left\{\frac{1}{a} - \frac{1}{V(t)}\right\} \mathrm{d}t = \int^{\infty} \frac{V(t) - a}{a V(t)}\, \mathrm{d}t \leqq \frac{1}{a^2} \int^{\infty} [V(t) - a]\, \mathrm{d}t$$

holds, where the last integral converges according to Lemma 1.3.

The function $S(M) = \int_{T_0}^{M} [V(t)]^{-1}\, \mathrm{d}t$ (see (2.16)) is continuous and increasing $S(T_0) = 0$, $S(\infty) = \infty$. Define now the function

$$(2.21) \qquad \varphi(M) = \int_{M}^{\infty} \left\{\frac{1}{a} - \frac{1}{V(t)}\right\} \mathrm{d}t.$$

$\varphi(M)$ is continuous, positive, decreasing and $\varphi(\infty) = 0$, because the mentioned integral converges. There exists then unique number $M > T_0$ such that $S(M) = \varphi(M)$, thus there exist a unique solution of the equation

$$(2.22) \qquad \int_{T_0}^{M} \frac{\mathrm{d}t}{V(t)} = \int_{M}^{\infty} \left\{\frac{1}{a} - \frac{1}{V(t)}\right\} \mathrm{d}t.$$

If M is the solution of equation (2.22), hence it follows

$$(2.23) \qquad \lim_{t\to\infty} \left\{ \int_{T_0}^{t} [V(r)]^{-1}\, \mathrm{d}r - \int_{M}^{t} a^{-1}\right\} \mathrm{d}r = \lim_{t\to\infty} [S(t) - s(t)] = 0.$$

If $\int^{\infty} [Q(\infty) - Q(t)]\, \mathrm{d}t$ diverges, then according to Lemma 1.3, the integral $\int^{\infty} [V(t) - a]\, \mathrm{d}t$ also diverges and hence the divergence of integral (2.21) follows. The equation (2.22) cannot be therefore fulfilled for any finite number M.

Thus we have shown, that if and only if the integral $\int^{\infty} [Q(\infty) - Q(t)]\, \mathrm{d}t$ converges, there exists the solution M of the equation (2.22). According to Lemma 2.1 it follows from (2.23)

$$(2.24) \qquad [t(s) - T(s)] > 0 \ \text{ for } s \in (0, \infty), \qquad \lim_{s\to\infty} [t(s) - T(s)] = 0$$

where $t(s)$, resp. $T(S)$ denote the inversion function to $s(t)$, resp. $S(T)$.

Let us choose now such a solution $\bar{Y}(T)$, resp. $\bar{y}(t)$ of the equation

(2.8), resp. (2.9) that there holds $\bar{Y}(T_0) = \bar{y}(M) = 0$. $\{T_k\}$, resp. $\{t_k\} = \{M + k\pi a\}$ denotes now the sequence of zeros of the solution $\bar{Y}(T)$, resp. $\bar{y}(t)$. According to (2.17) there is $S_k = s_k = k\pi$ and since $t_k = t(s_k)$, $T_k = T(S_k)$ it holds true

$$M + k\pi a - T_k = t_k - T_k = t(s_k) - T(s_k). \tag{2.25}$$

Since $s_k \to \infty$ for $k \to \infty$, it follows from (2.25), (regarding (2.24) and in the proof introduced the denotation) the assertion (2.6) for $i = 0$ and the assertion (2.7).

4. Let us denote now

$$\Delta_\pi F(s) = F(s + \pi) - F(s); \qquad \Delta_\pi^i F(s) = \Delta_\pi[\Delta_\pi^{i-1} F(s)]. \tag{2.26}$$

According to [9], p. 73, there exists a number $\Theta (0 < \Theta < 1)$ such that

$$\Delta_\pi^i F(s) = \pi^i F^{(i)}(s + i\pi\Theta) \tag{2.27}$$

provided $F^{(i)}(s)$ exist in the open interval $(s, s + i\pi)$ and the lower derivatives are continuous in the corresponding closed interval.

From (2.25) it follows

$$(-1)^i \Delta^i(M + k\pi a - T_k) = (-1)^i \Delta^i[t(s_k) - T(S_k)] \tag{2.28}$$

and hence, if we put $i = 1$ $(t'(s) = a)$

$$\Delta(M + k\pi a - T_k) = \pi[a - T'(k\pi + \pi\Theta) \qquad k = 0, 1, \ldots \tag{2.29}$$

Because of $T'(S) = V(T)$ and $V(T) > a$ for $T \in (0, \infty)$, there is $\Delta[M + k\pi a - T_k] < 0$ and (2.6) holds also for $i = 1$.

5. Let now $i = 2, 3, \ldots, n$. Because of $t^{(i)}(s) \equiv 0$ for $s > 0$, from (2.28) it follows

$$\begin{aligned}(-1)^i \Delta^i(M + ka\pi - T_k) &= (-\pi)^{i-1} T^{(i)}(k\pi + \pi i\Theta) = \\ &= (-\pi)^{i-1} [T'(k\pi + i\pi\Theta)]^{(i-1)} = (-\pi)^{i-1} V^{(i-1)}(\tau)\end{aligned} \tag{2.30}$$

where $\tau \in (T_k, T_{k+1})$ is a suitable number. According to Lemma 1.1, however, there hold

$(-1)^i V^{(i)}(T) > 0$; $(-1)^n V^{(n)}(T) \geqq 0$, $\qquad i = 0, 1, \ldots, n-1 \quad T > 0$

and thus

$$(-1)^i \Delta^i[M + ka\pi - T_k] = (-\pi)^{i-1} V^{(i-1)}(\tau) > 0 \qquad i = 0, 1, \ldots, n$$

$$(-1)^{n+1} \Delta^{n+1}[M + ka\pi - T_k] \geqq 0.$$

Hence there follows, with regard to the denotation introduced in the proof, the validity of (2.6) for $i = 2, 3, \ldots, n$.

By this the proof is finished.

Corollary 2.1. *If the conditions of Theorem 2.1 are satisfied, then to any numbers T_0, $\alpha \in (0, \infty)$ there exists $M > 0$ sufficiently large such that there holds* (2.6) *and*

$$\{M + k\pi a - t_k\} \to \alpha \text{ for } k \to \infty \tag{2,31}$$

if and only if the integral $\int^{\infty} [q(\infty) - q(t)]\,\mathrm{d}t$ *converges.*

Proof. If $\bar{M}$ is the solution of (2.22), we may see from the proof of the theorem that, if we choose $M = \bar{M} + \alpha$, then it is necessary to replaced (2.7) by (2.31).

Remark 2.1. Some steps in the proof of Theorem 2.1 are mentioned analogously to the proof of Theorem 5.1 in [5]. This theorem, however, cannot be used because of having stronger suppositions and in this direction substantionally weaker assertion.

Remark 2.2. Provided we omit the condition $q'(t) \neq 0$ in Theorem 2.1, the inequalities in (2.6) pass to non-sharp. It may be directly seen from the proof of the theorem.

III. FUNCTION $z(t) = V(t) - v(t)$

This paragraph is devoted to the "monotonic dependence" of the function $v(t)$, from the paragraph 1, upon the "monotonic change" of the coefficient $q(t)$ of equation (0.1).

Lemma 3.1. *Let $q(t)$ possess the derivative $q'(t) \in M_{n+1}$ $(n \geqq 1)$ $0 < q(\infty) < \infty$, $Q(t)$ possess the derivative $Q'(t) \in M_n$, $[q(t) - Q(t)] \in \in M_{n+1}$, and $c \geqq 0$ be any constant. Furthermore, let $v(t)$ be any function defined by the relation* (1.1) *satisfying $v(t) \in M_{n+1}$. (According to Lemma* 1.1 *this function really exists.)*

Then the differential equation

$$z''' + 4Q(t)\,z' + 2Q'(t)\,z = -F(t) \tag{3.1}$$

where

$$F(t) = 4v'(t)\,[Q(t) - q(t)] + 2v(t)\,[Q'(t) - q'(t)] \tag{3.2}$$

has a unique solution $z = z(t)$ such that

$$[z(t) - c] \in M_{n,n+2}. \tag{3.3}$$

Proof. We are going to show that, for the equation (3.1), the suppositions α), β'), γ') of Lemma 0.1 and Remark 0.1 are fulfilled, if $q(t)$ is replaced by $4Q(t)$, $q_0(t) = 2Q'(t)$, $f(t) = F(t)$, $k = 1$ and n is replaced by $n - 1$.

1. From suppositions of the lemma instantly follows that the condition α) is fulfilled, provided we show that $F(t) \in M_{n,0}$.

For any functions $\varphi(t) \in M_n$, $\psi(t) \in M_n$ ther holds with regard to the identity

$$[\varphi(t) \cdot \psi(t)]^{(i)} = \sum_{k=0}^{i} \binom{i}{k} \varphi^{(k)}(t) \cdot \psi^{(i-k)}(t) \tag{3.4}$$

that the product $\varphi \cdot \psi$ is also a function of class M_n.

Since $-(Q-q) \in M_{n+1}$, $-v' \in M_n$, there holds $[v'(Q-q)] \in M_n$. Furthermore there holds $v'(\infty) = 0$, $0 < q(t) - Q(t) < \infty$ for $t > 0$ and thus $[v'(Q-q)] \in M_{n,0}$.

Analogously for the second member of the expression (3.2) there holds $[v(Q'-q')] \in M_{n,0}$, because $v \in M_{n+1}$, $(Q'-q') \in M_n$, $0 < v(\infty) < < \infty$ and $Q'(\infty) = q'(\infty) = 0$.

Thus, for the function $F(t)$ defined in (3.2) there holds $F(t) \in M_{n,0}$ and the condition α) is fulfilled.

2. We are going to prove now the fulfilment of conditions β'). Let us choose $t_0 > 0$ so that $q(t_0) > 0$, $Q(t_0) > 0$. Then for $t \in (t_0, \infty)$

$$0 \leqq [q(t) - Q(t)]\,[Q(t)]^{-1} \leqq [q(t_0) - Q(t_0)]\,[Q(t_0)]^{-1} = k < \infty.$$

holds.

Therefore

$$\begin{aligned} 0 \leqq -\int_{t_0}^{\infty} v'(t)\,[q(t) - Q(t)]\,[Q(t)]^{-1}\,dt \leqq -k \int_{t_0}^{\infty} v'(t)\,dt = \\ = k[v(t_0) - v(\infty)] < \infty. \end{aligned} \tag{3.5}$$

Analogously

$$\begin{aligned} 0 \leqq \int_{t_0}^{\infty} v(t)\,[Q'(t) - q'(t)]\,[Q(t)]^{-1}\,dt \leqq \\ \leqq v(t_0)\,[Q(t_0)]^{-1} \int_{t_0}^{\infty} [Q'(t) - q'(t)]\,dt < \infty \end{aligned} \tag{3.6}$$

From (3.5) and (3.6) it follows

$$\begin{aligned} \int_{t_0}^{\infty} F(t)\,[Q(t)]^{-1}\,dt = 4 \int_{t_0}^{\infty} v'(t)\,[Q(t) - q(t)]\,[Q(t)]^{-1}\,dt + \\ + 2 \int_{t_0}^{\infty} v(t)\,[Q(t) - q(t)]\,[Q(t)]^{-1}\,dt < \infty \end{aligned}$$

and the condition $\beta' >$ is therefore fulfilled.

Since $\int^{\infty} Q'(t)\,[Q(t)]^{-1}\,dt = [\lg Q(t)]a^{\infty} < \infty$, there is also fulfilled the condition γ').

3. According to Remark 0.1 the equation (3.1) has then a unique solution $z = z(t)$ of class $M_{n,n+2}$. Because of $Q'(\infty) = 0$, there exists, to any constant $c > 0$, a unique solution $z = z(t)$ of (3.1) for which $[z(t) - c] \in M_{n,n+2}$ holds. It follows from the equation (3.1) by introducing a new dependent variable $z(t) - c$. By that the lemma is proved.

Lemma 3.2. *Let the suppositions of Lemma* 3.1 *be satisfied and le further*

$$(3.7) \qquad q'(t) \neq 0, \qquad [q(t) - Q(t)] \neq 0 \quad \textit{for} \quad t \in (0, \infty).$$

Then for each solution $z(t)$ *of* (3.1) *satisfyinq* (3.3) *there holds*

$$(3.8) \qquad (-1)^i z^{(i)}(t) > 0 \qquad i = 0\,1, \ldots, n-1.$$

Proof. Suppose that such a number $\xi > 0$ and such an integer $N \leqq n - 1$ exist that for $z(t)$ satisfying (3.3) $z^{(N)}(\xi) = 0$ holds. Since $0 \leqq z(\infty) = c < \infty$, it holds $z(t) = c$ for $t \in (\xi, \infty)$ (see the proof of Lemma 0.3).

By a substitution to the equation (3.1) we get

$$(3.9) \qquad 2Q'z = -4v'(Q - q) - 2v(Q' - q')$$

which leads to a contradiction because the expression $Q'z$ is non-negative, while the right side of (3.9) is negative for $t \in (0, \infty)$ with regard to Lemma 1.1. This proved the lemma.

Remark 3.1. Conditions (3.7) of Lemma 3.2 may be replaced by any condition, guaranteeing the invalidity of (3.9). If there is, e.g., $q'(t) = 0$ for $t \in \langle \xi, \infty)$, then it is sufficient to suppose $Q'(t) \neq 0$ for $t \in \langle \xi, \infty)$.

We are going to prove now two theorems which, although having been desived because of playing a substantial role in the proof of Theorem 4.1, are not without interest by themselves.

Theorem 3.1. *Let* $q(t)$ *possess a derivative* $q'(t) \in M_{n+1}$ $(n \geqq 1)$, $Q(t)$ *possess a derivative* $Q'(t) \in M_n$, $[q(t) - Q(t)] \in M_{n+1}$, $0 < q(\infty) < \infty$ *and* $c_1 > 0$, $c \geqq 0$ *be arbitrary constants. Then* (0.1) *resp.* (0.2) *has a pair of solutions* $[x(t), y(t)]$, *resp.* $[X(t), Y(t)]$ *such that*

$$(3.10) \qquad v = x^2(t) + y^2(t) \qquad V = X^2(t) + Y^2(t)$$

satisfies

$$[V(t) - v(t) - c] \in M_{n,n+2}, \qquad v(\infty) = c_1$$

$$(3.12) \qquad (-1)^i V^{(i)}(t) \geqq (-1)^i v^{(i)}(t) \geqq 0 \qquad i = 0, 1, \ldots, n,\ t > 0.$$

The pair of solutions (x, y) *of* (0.1), *resp.* (X, Y) *of* (0.2) *is unique up to their replacement by* $(a_1 x + b_1 y, c_1 x + d_1 y)$, *resp.* $(a_2 X + b_2 Y, c_2 X + d_2 Y)$, *where* a_i, b_i, c_i, d_i $(i = 1, 2)$ *are the constants such that* $a_i^2 + c_i^2 = b_i^2 + d_i^2 = 1$, $a_i b_i + c_i d_i = 0$.

Proof. Let $c_1 > 0$, $c_2 = c_1 + c > 0$ are any constants. According to [3], p. 182 (see Lemma 1.1) the equation (0.9) has a unique solution $v(t)$ and the equation

$$V''' + 4Q(t)\, V' + 2Q'(t)\, V = 0 \tag{3.13}$$

a unique solution $V(t)$ such that

$$[v(t) - c_1] \in M_{n+1,n+3} \qquad [V(t) - c_2] \in M_{n,n+2} \tag{3.14}$$

thus $v(\infty) = c_1$, $V(\infty) = c_2$. Moreover, the solutions $v(t)$, $V(t)$ are of the form (3.10) and the pairs (x, y), (X, Y) are unique up to their replacement by the already mentioned linear combination. There exist then pairs (x, y), (X, Y) such that for $v(t)$, $V(t)$ defined in (3.10) there holds (3.14). Denote now

$$z(t) = V(t) - v(t) \tag{3.15}$$

where $V(t)$, resp. $v(t)$, denotes any solution of (3.13), resp. (0.9). The function $z(t)$ defined in (3.15) is a solution of the equation (3.1) and reversely, all solutions of the equation (3.1) are of the form (3.15).

Let us choose now $c_1 > 0$ arbitrarily and $v(t)$ firmly such that the first relation of (3.14) hold. According to Lemma 3.1 there exists, to any constants $c \geqq 0$, a unique solution $z(t)$ such that $[z(t) - c] \in M_{n,n+2}$. Therefore exists a unique solution $V(t)$ of (3.13), for which (3.11) holds true. Because $[v(t) - c_1] \in M_{n+1,n+3}$, it holds $V(\infty) = v(\infty) + c = c_1 + c = c_2 > c \geqq 0$ and at the same time (3.12). Hence it follows that this solution $V(t)$ is of class M_n and with regard to the uniqueness of the solution of (3.13) is $V(t)$ just that solution which fulfils the second of the relations (3.14) and is thus of the form (3.10). By this the theorem is proved.

Remark 3.2. If, in addition to the condition of Theorem 3.1 $q'(t) \neq 0$ and $q(t) \neq Q(t)$ for $t \in (0, \infty)$, then the inequalities (3.12) may be replaced for $i = 0, 1, \ldots, n - 1$ by sharp ones so that

$$(-1)^{(i)}\, V^{(i)}(t) > (-1)^i\, v^{(i)}(t) > 0 \quad i = 0, 1, \ldots, n - 1 \quad t \in (0, \infty) \tag{3.16}$$

holds.

It follows directly from Lemmas 1.1 and 3.2.

Remark 3.3. If in addition to the conditions of Theorem 3.1, we shall suppose only $q'(t) \neq 0$ for $t \in (0, \infty)$, then it is possible to sharpen the inequalities (3.12) only partially so that they will be of the form

$$(-1)^i\, V^{(i)}(t) \geqq (-1)^i\, v^{(i)}(t) > 0 \quad i = 0, 1, \ldots, n, \quad t \in (0\ \infty). \tag{3.17}$$

Remark 3.4. That the inequalities (3.12) may be of the form

(3.18) $(-1)^i V^{(i)}(t) > (-1)^i v^{(i)}(t) \geqq 0, \quad i = 0, 1, \ldots, n-1, \quad t \in (0, \infty)$

it is sufficient to demand—in Theorem 3.1, moreover—any condition guaranteeing the invalidity of (3.9), for example the condition from Remark 3.1.

Theorem 3.2 *Let the function* $q(t)$, *resp.* $Q(t)$, *possess a derivative* $q'(t) \in M_n$ $(n \geqq 1)$, *resp.* $Q'(t) \in M_n$, $[q(t) - Q(t)] \in M_{n+1}$ *and* $0 < q(\infty) = Q(\infty) < \infty$. *Then for functions* $V(t)$, $v(t)$ *defined in* (3.10), *satisfying* (3.14) *for* $c_1 = c_2$, *it holds true that the integrals*

(3.19) $$\int^{\infty} [q(t) - Q(t)]\, dt \qquad \int^{\infty} [V(t) - v(t)]\, dt$$

converge or diverge at the same time.

Proof. For $q'(t) = 0$ the assertion of the theorem is included in Lemma 1.3. Theorem 3.2 may be proved analogously and therefore the proof is given more briefly. Identity (1.9) for the functions $v(t)$, $V(t)$ defined in (3.10) may be written in the form

(3.20) $$qv^2 - w^2 = \frac{1}{4}(vv')' - \frac{3}{4} vv''$$

(3.21) $$QV^2 - W^2 = \frac{1}{4}(VV')' - \frac{3}{4} VV''$$

where $w = xy' - yx'$, resp. $W = XY' - YX'$ denote the Wronskian of solutions x, y, resp. X, Y of (3.10). Consider further only such functions $v(t)$, $V(t)$ which satisfy (3.14) for $c_1 = c_2$, thus $v(\infty) = V(\infty)$.

According to (1.7) there holds

(3.22) $$|w| \cdot [q(\infty)]^{-1/2} = v(\infty) = V(\infty) = |W| [Q(\infty)]^{-1/2}$$

and since $q(\infty) = Q(\infty)$, according to the supposition of the theorem it follows from (3.22)

$$|w| = |W|.$$

$$|w| = |W|.$$

By subtraction of (3.20) and (3.21) we get after forming

$$v^2(q - Q) = (V - v)\left[\frac{3}{4} v'' + Q(v + V)\right] - \frac{3}{4} V(vu'' - V'') +$$

(3.23) $$+ \frac{1}{4}(vv' - VV')'.$$

Let us integrate the identity (3.23) in the interval (t, ∞) where $t > 0$ is sufficiently large. Evidently it holds

$$(3.24)\qquad \left| \int_t^\infty (vv' - VV')' ds \right| = \left| [vv' - VV']_t^\infty \right| < \infty$$

$$(3.25)\qquad \left| \int_t^\infty V(v'' - V'')\, ds \right| \leqq \left| V(t) \int_t^\infty (v'' - V'')\, ds \right| < \infty$$

and further

$$0 < \lim_{t\to\infty} \left[\frac{3}{4} v''(t) + Q(t)\,[v(t) + V(t)] = \right.$$

$$(3.26)\qquad \left. = Q(\infty)\,[v(\infty) + V(\infty)] = 2\,|\,w\,|\,[Q(\infty)] \right]^{1/2} < \infty.$$

Considering the fact that the integrals (3.24) and (3.25) converge, $0 < v^2(\infty) < \infty$ and (3.26) holds, in the same way as in the proof of Lemma 1.3, the assertion of the theorem follows from (3.23).

IV. SEQUENCES OF ZEROS OF THE SOLUTIONS OF EQUATIONS (0.1) AND (0.2)

The following theorem compares the sequences of zeros of the solutions of two differential equations. It describes the sequence of differences of corresponding zeros of the solutions of equations (0.1) and (0.2).

Theorem 4.1. *Let $q(t)$, resp. $Q(t)$ possess a derivative $q'(t) \in M_{n+1}$ $(n \geqq 1)$, resp. $Q'(t) \in M_n$, $0 < q(\infty) = Q(\infty) < \infty$, $[q(t) - Q(t)] \in M_{n+1}$ and $q(t) \neq Q(t)$ for $t \in (0, \infty)$. Let $\{t_k\}$, resp. $\{T_k\}$ denote a sequence of zeros of any solution of* (0.1), *resp.* (0.2).

Then for any fixed choice of $T_0 > 0$ there exists a t_0 sufficiently large such that there holds

$$(4.1)\qquad \begin{gathered} (-1)^i \Delta^i (t_k - T_k) > 0, \qquad (-1)^{n+1} \Delta^{n+1} (t_k - T_k) \geqq 0, \\ i = 0, 1, \dots, n; \; k = 0, 1, 2, \dots \end{gathered}$$

$$(4.2)\qquad (t_k - T_k) \to 0 \text{ for } k \to \infty$$

if and only if integral $\int^\infty [q(t) - Q(t)]\, dt$ converges.

Proof. If $q'(t) \equiv 0$, then (4.1) and (4.2) follow from Theorem 2.1. Theorem 2.1 is consequently included as a special case in the theorem 4.1. We shall further suppose that $q'(t) \neq 0$ for $t \in (0, \infty)$. (The case $q'(t) = 0$ for $t \in \langle \xi, \infty)$, $\xi > 0$ will be observed in the end.) The proof will be performed similarly as in the case of Theorem 2.1.

1. According to Theorem 3.1 and Remark 3.2 the functions $v(t)$, $V(t)$ defined in (3.10) satisfying (3.14), (3.11) and (3.16) exist to any constants $c_1 > 0$, $c_2 > 0$. Let us choose now

$$c_1 = [q(\infty)]^{-1/2} = [Q(\infty)]^{-1/2} = c_2$$

so that

$$|w| = |W| = 1.$$

2. Transformations

$$y(t) = [v(t)]^{1/2} u(s) \qquad t'(s) = v(t) \tag{4.3}$$
$$Y(T) = [V(T)]^{1/2} U(S) \qquad T'(S) = V(T)$$

transform the equations

$$y''(t) + q(t)\, y(t) = 0 \qquad Y''(T) + Q(T)\, Y(T) = 0 \tag{4.4}$$

into the equations

$$u''(s) + u(s) = 0 \qquad U''(S) + U(S) = 0. \tag{4.5}$$

Let us choose the integration constants in (4.3) such that

$$s(t) = \int_{t_0}^{t} \frac{\mathrm{d}r}{v(r)} \qquad S(T) = \int_{T_0}^{T} \frac{\mathrm{d}r}{V(r)} \tag{4.6}$$

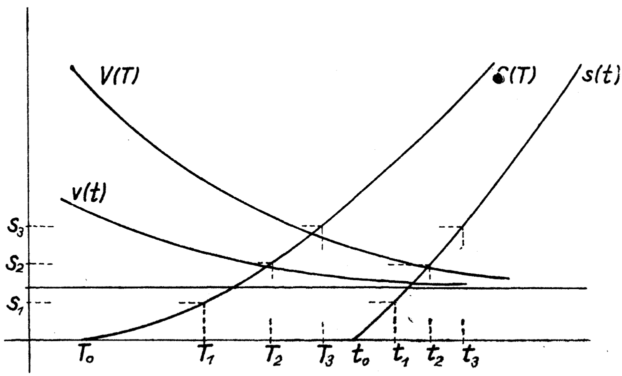

Fig. 2

where $T_0 > 0$ is arbitrary but fixed number and $t_0 > T_0$ is, for the time being, closely not determined. The transformation (4.3) realizes now the one-to-one correspondence between the zeros of $y(t)$ and those of $u(s)$, resp. zeros of $Y(T)$ and those of $U(S)$.

If $\{s_k\}$, resp. $\{S_k\}$ denotes the sequences of zeros of suitable solutions of (4.5) and $\{t_k\}$, $\{T_k\}$ sequences of zeros of solutions of equations (4.4), the principle members of which are t_0, T_0 from (4.6), then

$$S_k = S(T_k) = s(t_k) = s_k \qquad k = 0, 1, \ldots \tag{4.7}$$

Since $V(t) > v(t)$ for $t \in (0, \infty)$ and $t_0 > T_0$, it holds

$$S(t_0) = \int_{T_0}^{t_0} \frac{\mathrm{d}r}{V(r)} > \int_{t_0}^{t_0} \frac{\mathrm{d}r}{v(r)} = s(t_0) = s_0 = 0 \tag{4.8}$$

$$0 < S'(t) = [V(t)]^{-1} \leqq [v(t)]^{-1} = s'(t) < \infty \qquad t \in \langle t_0, \infty\rangle \tag{4.9}$$

where equality occurs only for $t = \infty$. The function $S(t) - s(t)$ is therefore in the interval $\langle t_0, \xi)$ where $\xi > t_0$ is a suitable number or $\xi = \infty$, positive and decreasing, thus it holds $S(t) > s(t) > 0$ for $t \in \langle t_0, \xi)$.

3. Let us suppose now that the integral $\int^{\infty} [q(t) - Q(t)]\, \mathrm{d}t$ converges. Then the integral $\int^{\infty} \{[v(t)]^{-1} - [V(t)]^{-1}\}\, \mathrm{d}t$ converges as well, because it holds analogously as in (2.20)

$$0 < \int^{\infty} \{[v(t)]^{-1} - [V(t)]^{-1}\}\, \mathrm{d}t \leqq q(\infty) \int^{\infty} [V(t) - v(t)]\, \mathrm{d}t$$

and the last integral converges according to the Theorem 3.2.

The function $S(t_0) = \int_{T_0}^{t_0} [V(t)]^{-1}\, \mathrm{d}t$ (see 4.6) is for $t_0 > T_0$, as the function of t_0, continuous, increasing, $S(T_0) = 0$, $S(\infty) = \infty$. Further on, define the function

$$\varphi(t_0) = \int_{t_0}^{\infty} \left\{\frac{1}{v(t)} - \frac{1}{V(t)}\right\} \mathrm{d}t. \tag{4.10}$$

The function $\varphi(t_0)$ is for $t \in \langle T_0, \infty)$ continuous, positive, decreasing and $\varphi(\infty) = 0$. There exists therefore a unique number t_0 such that it holds $S(t_0) = \varphi(t_0)$, thus there exists a unique solution t_0 of the equation

$$\int_{T_0}^{t_0} \frac{\mathrm{d}t}{V(t)} = \int_{t_0}^{\infty} \left\{\frac{1}{v(t)} - \frac{1}{V(t)}\right\} \mathrm{d}t. \tag{4.11}$$

If t_0 is this solution, then it follows

$$(4.12)\qquad \lim_{t\to\infty}\left\{\int_{T_0}^{t}\frac{\mathrm{d}r}{V(r)}-\int_{t_0}^{t}\frac{\mathrm{d}r}{v(r)}\right\}=\lim_{t\to\infty}[S(t)-s(t)]=0.$$

If $\int^{\infty}[q(t)-Q(t)]\,\mathrm{d}t$ diverges, then, according to Theorem 3.2, the integral $\int^{\infty}[V(t)-v(t)]\,\mathrm{d}t$ also diverges and hence the divergence of the integral $\int^{\infty}\{[v(t)]^{-1}-[V(t)]^{-1}\}\,\mathrm{d}t$ follows. In this case the equation (4.11) cannot be therefore satisfied for any finite number t_0.

Thus, we have shown that the number t_0 satisfying (5.12) exists if and only if the integral $\int^{\infty}[q(t)-Q(t)]\,\mathrm{d}t$ converges.

According to Lemma 2.1 it follows from (4.12)

$$(4.13)\qquad [t(s)-T(s)]>0 \text{ for } s\in(0,\infty);\qquad \lim_{S\to\infty}[t(s)-T(s)]=0$$

where $t(s)$, resp. $T(S)$ denotes the inversion function to $s(t)$, resp. $S(T)$. Let us choose such solutions $\bar{Y}(T)$, $\bar{y}(t)$ that for T_0, t_0 from (4.6) there holds $\bar{Y}(T_0)=\bar{y}(t_0)=0$; Let $\{T_k\}$, $\{t_k\}$ further denote the sequences of zeros of these solutions. According to (4.7) it holds with the above mentioned choice $\{T_k\}$, $\{t_k\}$

$$(4.14)\qquad s_k=S_k=k\pi,\qquad t_k=t(s_k),\qquad T_k=T(S_k)\qquad k=0,1,\ldots.$$

Since $S_k\to\infty$ for $k\to\infty$, the assertion (4.2) and (4.1) for $i=0$ follows from (4.13).

4. By using the designation from (2.26) it holds

$$(4.15)\qquad (-1)^i\Delta^i(t_k-T_k)=(-1)^i\Delta^i_{\pi}[t(s_k)-T(s_k)].$$

Suppositions in (2.27) are satisfied for $F(s)=t(s)-T(s)$ and $i=0,1,\ldots,n+1$, and therefore

$$(4.16)\qquad (-1)^i\Delta^i(t_k-T_k)=(-\pi)^i\,[t^{(i)}(s_k+i\pi\Theta)-T^{(i)}(s_k+i\pi\Theta)]$$
$$i=0,1,\ldots,n+1,\quad k=0,1,\ldots,\quad 0<\Theta<1.$$

5. For already defined $v(t)$, $V(T)$ it holds

$$(4.17)\qquad (-1)^i\,V^{(i)}(T)>(-1)^i\,v^{(i)}(T)>0,$$
$$i=0,1,\ldots,n-1,\quad T\in(0,\infty)$$
$$(-1)^n\,V^{(n)}(T)\geqq(-1)^n\,v^{(n)}(T)>0,\qquad (-1)^{n+1}\,v^{(n+1)}(T)\geqq 0$$

The function $(-1)^i v^{(i)}(T)$ is in the interval $(0, \infty)$ positive and for $i = 0, 1, \ldots, n - 1$ decreasing, for $i = n$ non-increasing. Consequently for any numbers T, t, for which $0 < T < t$, it holds

$$(4.18) \qquad (-1)^i v^{(i)}(T) > (-1)^i v^{(i)}(t) > 0 \qquad i = 0, 1, \ldots, n - 1$$
$$(-1)^n v^{(n)}(T) \geqq (-1)^n v^{(n)}(t) > 0.$$

Thus from (4.10) and (4.18) it follows that for arbitrary T, t satisfying $0 < T < t$ it holds

$$(4.19) \qquad (-1)^i V^{(i)}(T) > (-1)^{(i)} v^{(i)}(t) > 0 \qquad i = 0, 1, \ldots, n - 1$$
$$(-1)^n V^{(n)}(T) \geqq (-1)^n v^{(n)}(t) > 0.$$

According to Lemma 0.2 (where $I_{tT} = (t_0, \infty)$, $I_s = (0, \infty)$) and Remark 0.2, it holds on the basis of (4.19) for arbitrary $s > 0$

$$(4.20) \qquad (-1)^i T^{(i)}(s) < (-1)^i t^{(i)}(s) < 0 \qquad i = 0, 1, \ldots, n$$
$$(-1)^{n+1} T^{(n+1)}(S) \leqq (-1)^{n+1} t^{(n+1)}(S) < 0$$

and then with regard to (4.16), the relation (4.1). Thus, in the case of $q'(t) \neq 0$ the proof is finished.

6. Now, it remains to take a notice only of the case $q'(t) \neq 0$ for $t \in (0, \xi)$, $q'(t) \equiv 0$ for $t \in \langle \xi, \infty)$. According to Remark 3.4, there exist the functions $v(t)$, $V(t)$ satisfying (3.18). Thus, (4.19) holds, where in the right side the sign "$>$" is replaced by "$\geqq$". According to Remark 0.2 applied to the interval $\langle \xi, \infty)$ (4.20) holds in the form

$$(4.21) \qquad (-1)^i T^{(i)}(S) < (-1)^i t^{(i)}(S) \leqq 0 \qquad i = 0, 1, \ldots, n$$
$$(-1)^{n+1} T^{n+1}(S) \leqq (-1)^{n+1} t^{(n+1)}(S) \leqq 0$$

and hence it follows, with regard to (4.10), the relation (4.1) and the proof is finished.

Remark 4.1. If in addition to the conditions of Theorem 4.1 $q'(t) \in \in M_{n+2}$, then the inequalities (4.1) hold in the form

$$(4.22) \qquad (-1)^i \Delta^i(t_k - T_k) > 0 \qquad i = 0, 1, \ldots, n + 1, \; k = 0, 1, \ldots .$$

The function $(-1)^i v^{(i)}(t)$ is namely decreasing also for $i = n$ so that (4.18) is in this case of the form

$$(-1)^i v^{(i)}(T) > (-1)^i v^{(i)}(t) > 0 \qquad i = 0, 1, \ldots, n$$

so that in (4.19) the sharp inequality also holds for $i = n$ and therefore (4.20) is of the form

$$(-1)^i T^{(i)}(S) < (-1)^i t^{(i)}(S) < 0 \qquad i = 0, 1, \ldots, n + 1.$$

Hence (4.22) follows.

Theorem 4.2. *Let all the conditions of Theorem 4.1 be satisfied. If* $\{\bar{t}_k\}$ *denotes the sequence of zeros of any solution of* (0.1), *then for any fixed choice of* $T_0 \in (0, \infty)$, $\alpha \in 0, \infty$ *there exists a* t_0 *sufficiently large such that there holds* (4.1) *and*

$$(4.23) \qquad (\bar{t}_k - T_k) \to \alpha \quad \text{for} \quad k \to \infty$$

if and only if $\int^{\infty} [q(t) - Q(t)]\, dt$ *converges.*

At the same time it is possible to admit $q(t) \equiv Q(t) \neq 0$ *for* $t \in \langle\beta, \infty)$, $\beta > 0$.

Proof. Let the conditions of Theorem 4.1 be satisfied and let $0 < \int^{\infty} [q(t) - Q(t)]\, dt < \infty$. Let t_0 denote the solution of (4.11) and $\bar{t}_0$ the solution of equation

$$(4.24) \qquad f(\bar{t}_0) = \alpha[q(\infty)]^{1/2} \quad \text{where} \quad f(\bar{t}_0) = \int_{t_0}^{\bar{t}_0} [v(r)]^{-1}\, dr.$$

Such a number $\bar{t}_0$ exists to any $\alpha \in (0, \infty)$ because $0 < \alpha[q(\infty)]^{1/2} < \infty$ and the function $f(\bar{t}_0)$ as a function of the variable $\bar{t}_0$ is for $\bar{t}_0 \in \langle t_0, \infty)$ continuous, increasing, $f(0) = 0$, $f(\infty) = \infty$ and is unique.

Let now

$$(4.25) \qquad s(t) = \int_{\bar{t}_0}^{t} \frac{dr}{v(r)} = \int_{T_0}^{T} \frac{dr}{V(r)} = S(T).$$

Then regarding to (4.12)

$$(4.26) \qquad 0 = \lim_{t\to\infty} \left\{ \int_{T_0}^{t} \frac{dr}{V(r)} - \int_{t_0}^{\bar{t}_0} \frac{dr}{v(r)} - \int_{\bar{t}_0}^{t} \frac{v(r)}{dr} \right\} =$$

$$= \lim_{t\to\infty} [S(t) - s(t)] - \alpha[q(\infty)]^{1/2}$$

and thus

$$(4.27) \qquad \lim_{t\to\infty} [S(t) - s(t)] = \alpha[q(\infty)]^{1/2}$$

Consequently according to Lemma 2.1

$$(4.28) \qquad \lim_{S\to\infty} [t(s) - T(S)] = \alpha[q(\infty)]^{1/2} [v(\infty)]^{-1} = \alpha$$

and hence (4.23) directly follows.

If $\int^{\infty} [q(t) - Q(t)]\, dt$ diverges, then it may be seen from the proof of Theorem 4.1 (part 3) that in this case the solution of (4.11) does not exist and therefore t_0, satisfying (4.23), does not exist, too.

Now let $q(t) \equiv Q(t)$ for $t \in \langle \beta, \infty)$, $\beta > 0$. Then with regard to uniqueness of the solution of (0.9) $v(t) \equiv V(t)$ holds for $t \in \langle \beta, \infty)$. Because of $\alpha > 0$, with regard to (4.28), $t(s) - T(s) > 0$ holds for $s \in (0, \infty\rangle$ and hence (4.23) follows.

Since $v(t)$ fulfils the inequalities (4.18), (4.19) holds and thus also (4.1) in an unchanged form.

By this the proof is finished.

Remark 4.2. If we put, in Theorem 4.2, $q(t) \equiv Q(t)$ for $t \in (0, \infty)$, then $\{t_k\}$, $\{T_k\}$ denote the sequences of zeros of the same or different solutions of the same differential equation (0.1). Theorem 5.1 and partially Theorem 2.1 of the paper [5] are thus included in the preceding theorem. The part of the proof of Theorem 4.1 is mentioned similarly as the proof of Theorem 5.1 in [5]. This theorem, however, cannot be used because its assertion is weaker and at the same time supposes the existence of integrals $v(t)$, $V(t)$ of desirable properties.

Remark 4.3. Theorem 4.1, resp. 4.2, defines the decomposition of the set $\mathfrak{M}$ of all functions $q(t)$ such that $0 < q(\infty) < \infty$, $q'(t) \in M_n$, $[q_1 \in M_n, q_2 \in M_n] \Rightarrow | q_1' - q_2' | \in M_n$ ($n \geqq 1$, fixed integer) on disjunctive classes so that, if $\mathfrak{M}(q_0)$ denotes the set of all functions $q \in \mathfrak{M}$ equivalent to q_0, then $q \in \mathfrak{M}(q_0)$, if and only if the integral $\int^{\infty} |q(t) - q_0(t)|\, dt$ converges. The inequalities (4.1) and (4.2), resp. (4.23), are then satisfied if and only if the functions $q(t)$, $Q(t)$ are of the same class.

V. THE SEQUENCES OF EXTREMANTS

If $y(t)$ denotes any solution of the (0.1) where $q(t) \in C_2$, $q(t) > 0$ for $t \in (0, \infty)$, then the function $Y(t) = y'(t) \cdot [q(t)]^{-1/2}$ is a solution of the differential equation

$$Y''(t) + Q(t)\, Y(t) = 0 \tag{5.1}$$

where

$$Q(t) = q - \frac{3}{4} \frac{q'}{q} + \frac{1}{2} \frac{q''}{q} \tag{5.2}$$

and conversely if $\bar{Y}(t)$ is any solution of (5.1), then there exists such a solution $\bar{y}(t)$ of (0.1) that $\bar{Y}(t) = [q(t)]^{-1/2} \bar{y}'(t)$ holds (see [1]).

Thus, if T_k is an extremant of the solution $y(t)$ of (0.1), then is zero of the corresponding solution $Y(t)$ and conversely. The sequence $\{T_k\}$

is therefore a sequence of extremants of some solution of the equation (0.1), if and only it is the sequence of zeros of a suitable solution of (5.1).

Lemma 5.1. *Let $q(t)$ possess a derivative $q'(t) \in M_{n+1}$ $(n \geqq 1)$, $q(t) > 0$ for $t \in (0, \infty)$, $q(\infty) < \infty$ and $Q(t)$ be defined by the formula* (5.2).
Then

$$Q'(t) \in M_{n-1}; \qquad [q(t) - Q(t)] \in M_{n0} \tag{5.3}$$

$$0 < \int^{\infty} [q(t) - Q(t)]\, dt < \infty \tag{5.4}$$

holds.

Proof. I have proved the validity (5.3) in the paper [10] (the first part (5.3) follows from Lemma 3, the second part (5.3) follows from the proof of Lemma 3). Because of identically holding

$$\int \frac{q''}{q}\, dt = \frac{q'}{q} - \int \left[\frac{q'}{q}\right]^2 dt$$

we have

$$\int^{\infty} (q - Q)\, dt = \int^{\infty} \left[\frac{3}{4}\left(\frac{q'}{q}\right)^2 - \frac{1}{2}\frac{q''}{q}\right] dt =$$

$$= \frac{5}{4}\int^{\infty} \left(\frac{q'}{q}\right)^2 dt - \frac{1}{2}\left[\frac{q'}{q}\right]^{\infty} \tag{5.5}$$

Since $q'q^{-1} \to 0$ for $t \to \infty$ for sufficiently large t there holds

$$0 < \int_t^{\infty} \left(\frac{q'}{q}\right)^2 dt < \int_t^{\infty} \frac{q'}{q}\, dt = [\lg q]_t^{\infty} < \infty \tag{5.6}$$

and regarding (5.5) the integral $\int^{\infty} (q - Q)\, dt$ converges.

Let the conditions of Lemma 5.1 be satisfied.

Remark 5.1. If there is $q'(t) \neq 0$ for $t \in (0, \infty)$, then for this t

$$(-1)^i Q^{(i+1)}(t) > (-1)^i q^{(i+1)}(t) > 0 \qquad i = 0, 1, \ldots, n-1 \tag{5.7}$$

holds.

It follows directly from Lemmas 0.3 and 5.1.

Theorem 5.1. *Let the function $q(t)$ possess a derivative $q'(t) \in M_{n+1}$ $(n \geqq 2)$, $q(t) > 0$, $q'(t) \neq 0$ for $t \in (0, \infty)$ and $q(\infty) < \infty$. Let $\{t_k\}$ denotes* *the sequence of zeros of any solution $y(t)$ of (0.1) and $\{T_k\}$ the sequence of extremants of the same solution. If there is $t_0 > T_0 > 0$, then*

$$(5.8) \qquad (-1)^i \Delta^i(t_k - T_k) > 0 \qquad i = 0, 1, \ldots, n;\ k = 0, 1, \ldots .$$

Proof. Let the function $Q(t)$ be defined by the formula (5.2). Because $q'(\infty) = q''(\infty) = 0$, it holds $q(\infty) = Q(\infty)$. According to the Lemma 5.1 and Remark 5.1 $[q(t) - Q(t)] \in M_{n,0}$, $Q'(t) \in M_{n-1}$ holds and $q(t) \neq Q(t)$ for $t \in (0, \infty)$. The conditions of Theorem 4.1, resp. 4.2, where n is replaced by $n - 1$, are therefore satisfied. As above mentioned, the sequence $\{T_k\}$ of extremants of any solution of (0.1) is the sequence of zeros of a suitable solution of (5.1) and conversely. According to Lemma 5.1 (5.4) holds. Therefore according to Theorem 4.2 and Remark 4.1 there exists to any number $T_0 > 0$ such a number t_0 sufficiently large that there holds (5.8) and $(t_k - T_k) \to \alpha$ for $k \to \infty$, where $\alpha \geqq 0$ is any, fixed number. It remains to prove that there exists such $\alpha > 0$ that, if T_0 being an extremant of solution $y(t)$ of (0.1), then for t_0 ($\sim \bar{t}_0$ with designation from Theorem 4.2) defined by the equation (4.24) it holds $y(t_0) = 0$ (with the possibility of t_0 being the first zero over T_0).

Since $q(t) > 0$, only one extremant lies between any pair of consecutive zeros of the same solution of (0.1) and conversely only one zero lies between any pair of consecutive extremants of the same solution $y(t)$ of (0.1). If $\{t_k\}$, resp. $\{T_k\}$, denotes the sequence of zeros, resp. of extremants of the same solution, and $t_0 > T_0$, then $t_k > T_k$ holds therefore for all k. By this the proof is finished.

Remark 5.2. It may be seen from the proof of Theorem 4.1 that the convergence of the integral $\int^{\infty} [q(t) - Q(t)]\, dt$ is not necessary to be used to the proof of Theorem 5.1; even on the contrary, the convergence of this integral follows from Theorem 4.1, resp. 4.2.

Remark 5.3. Let the conditions of Theorem 5.1 be satisfied and let $\{t_k\}$ denote now a sequence of zeros either of the same solution whose sequence of extremants being $\{T_k\}$, or any other solution $y(t)$ of the same equation (0.1). If t_0 is greater or is equal to the solution of the equation (4.11) (which is also designated by t_0), then (5.8) remains in validity. Especially (5.21) holds true always, provided t_0 being greater or equal to the first zero over extrement T_0 of the same solution.

Lemma 5.2. *Let $q_\lambda(t) > 0$, $q'_\lambda(t) \in M_{n+1}$ $(n \geqq 1)$ and $[q_1(t) - q_2(t)] \in$ $\in M_{n+2}$ for $\lambda = 1, 2$ and $t > 0$.*

Let further

$$(5.9) \qquad Q_\lambda(t) = q_\lambda - \frac{3}{4}\left(\frac{q'_\lambda}{q_\lambda}\right)^2 + \frac{1}{2}\frac{q''_\lambda}{q_\lambda}.$$

Then the function $Q_1(t) - Q_2(t)$ is of the class M_n.

Proof. According to the suppositions of the lemma it holds for $t > 0$ and $i = 0, 1, \dots, n+1$

$$(5.10) \qquad q_1(t) \geqq q_2(t); \qquad 0 \leqq (-1)^i q_1^{(i+1)}(t) \leqq (-1)^i q_2^{(i+1)}(t)$$

thus it holds true $| q_1^{(i+1)}(t) | \leqq | q_2^{(i+1)}(t) |$. Hence it follows

$$(5.11) \qquad | q_1^{(i+1)}(t)\,[q_1(t)]^{-1} | \leqq | q_2^{(i+1)}(t)\,[q_2(t)]^{-1} \qquad i = 0, 1, \dots, n+1.$$

According to Lemma 2 of [10] it holds for $p < n + 2 - i$

$$(5.12) \qquad \left| \left[\frac{q_\lambda^{(i)}}{q_\lambda} \right]^{(p)} \right| = \sum_{k=1}^{N} \left| C_k \frac{q_\lambda^{(\nu_{1,k})}}{q_\lambda} \cdots \frac{q_\lambda^{(\nu_{l,k})}}{q_\lambda} \right|$$

where C_k, N, l, $\nu_{j,k}$ are suitable numbers the detailed specification of which not being necessary in our case. From a term by term comparison of the expression (5.12) for the functions $q_1(t)$, $q_2(t)$, with regard to (5.11), there follows the inequality

$$(5.13) \qquad \left| \left\{ \frac{q_1^{(i)}}{q_1} \right\}^{(p)} \right| \leqq \left| \left\{ \frac{q_2^{(i)}}{q_2} \right\}^{(p)} \right| .$$

Further on, it holds identically

$$(5.14) \qquad \left\{ \left(\frac{q_\lambda'}{q_\lambda} \right)' \frac{q_\lambda'}{q_\lambda} \right\}^{(p)} = \sum_{i=0}^{p} \binom{p}{i} \left(\frac{q_\lambda'}{q_\lambda} \right)^{(i+i)} \left(\frac{q_\lambda'}{q_\lambda} \right)^{(p-i)} .$$

Since $q'q^{-1} \in M_{n+1}$ (see [10], Lemma 2), all members of the sum on the right side of (5.14) have the same sign, namely $(-1)^{p+1}$. Then with regard to (5.13) it holds

$$(5.15) \qquad \left| \left\{ \left(\frac{q_1'}{q_1} \right)' \frac{q_1'}{q_1} \right\}^{(p)} \right| \leqq \left| \left\{ \left(\frac{q_2'}{q_2} \right)' \frac{q_2'}{q_2} \right\}^{(p)} \right| .$$

For $(i+1)$-th derivative $(i = 0, 1, \dots, n-1)$ of the function Q_λ it holds true

$$(5.16) \qquad Q_\lambda^{(i+1)} = q_\lambda^{(i+1)} - \frac{3}{2} \left\{ \left(\frac{q_\lambda'}{q_\lambda} \right)' \frac{q_\lambda'}{q_\lambda} \right\}^{(i)} + \frac{1}{2} \left\{ \frac{q_\lambda''}{q_\lambda} \right\}^{(i+1)} .$$

By term by term comparing of the right sides of (5.16) for $= 1, 2$ we get with regard to (5.13) and (5.15)

$$(5.17) \qquad | Q_1^{(i+1)}(t) | \leqq | Q_2^{(i+1)}(t) | \qquad i = 0, 1, \dots, n-1$$

and since, according to Lemma 5.1, $Q_\lambda'(t) \in M_{n-1}$, the inequality

$$(5.18) \qquad 0 \leqq (-1)^i Q_1^{(i+1)}(t) \leqq (-1)^i Q_2^{(i+1)}(t) \qquad i = 0, 1, \dots, n-1$$

follows from (5.17).

The inequality $0 \leqq Q_2(t) \leqq Q_1(t)$ follows, with regard to (5.11), directly by term by term comparing of (5.9) for $\lambda = 1, 2$ and the proof is finished.

Lemma 5.3. *Let the conditions of Lemma 5.2 be satisfied and let $q_1'(t) \neq q_2'(t)$ hold for $t \in (0, \infty)$. Then*

$$(5.19) \qquad (-1)^i Q_1^{(i)}(t) > (-1)^i Q_2^{(i)}(t) \qquad i = 0, 1, \ldots, n; \; t \in (0, \infty).$$

Proof. Since $q_1'(t) < q_2'(t)$ for $t \in (0, \infty)$, it holds, with regard to the conditions of Lemma $q_1(t) > q_2(t)$ and according to Lemma 0.3, also $(-1)^i q_1^{(i)}(t) > (-1)^i q_2^{(i)}(t)$ for $i = 2, 3, \ldots, n + 1$, so that it holds

$$(5.20) \quad (-1)^i q_1^{(i)}(t) > (-1)^i q_2^{(i)}(t) \geqq 0 \text{ for } i = 0, 1, \ldots, n + 1; t \in (0, \infty).$$

Therefore non-sharp inequalities in (5.11), (5.13), (5.15), (5.17) and a non-sharp inequality in the right part of (5.18) are possible to be replaced by sharp ones with eventual exception of inequalities for highest derivatives. From (5.16) it may be seen, however, that in (5.17) and in the right part of (5.18) the sharp inequality also remains for $i = n - 1$, thus (5.19) holds for $i = 1, \ldots, n$. The validity of (5.19) for $i = 0$, with regard to (5.11), may be seen directly from (5.9).

Theorem 5.2. *Let the function $q_1(t)$, resp. $q_2(t)$, possess a derivative $q_1'(t) \in M_{n+1}$, resp. $q_2'(t) \in M_n$ $(n \geqq 3)$ and further let $[q_1(t) - q_2(t)] \in \in M_{n+1}$, $q_1(t) > q_2(t) \geqq 0$ hold for $t \in (0, \infty)$ and $q_1(\infty) = q_2(\infty) < \infty$. Let $\{t_{\lambda,k}\}$ for $\lambda = 1, 2$ denote the sequence of zeros of any solution $y_\lambda(t)$ of the equation*

$$(5.21) \qquad y_\lambda'' + q_\lambda(t)\, y_\lambda = 0$$

and $\{T_{\lambda,k}\}$ denote the sequence of extremants of the same or any other solution of (5.21). Then for any fixed choice of $T_{2,0} > 0$ and any numbers $\alpha \in \langle 0, \infty)$, $\beta \in \langle 0, \infty)$ there exist the numbers $T_{1,0}$ and $t_{1,0}$, sufficently large such that there holds

$$(5.22) \qquad (-1)^i \Delta^i(T_{1,k} - T_{2,k}) > 0, \qquad (-1)^{n-1} \Delta^{n-1}(T_{1,k} - T_{2,k}) \geqq 0,$$
$$i = 0, 1, \ldots, n - 2; \; k = 0, 1, \ldots$$

$$(5.23) \qquad (T_{1,k} - T_{2,k}) \to \alpha \quad \text{for} \quad k \to \infty$$

and further on

$$(5.24) \qquad (-1)^i \Delta^i(t_{1,k} - T_{2,k}) > 0 \qquad i = 0, 1, \ldots, n - 1; \; k = 0, 1, \ldots$$

$$(5.25) \qquad (t_{1,k} - T_{2,k}) \to \beta \quad \text{for} \quad k \to \infty$$

if and only if the integral $\int^{\infty} [q_1(t) - q_2(t)]\, \mathrm{d}t$ converges.

Proof. Let the functions $Q_1(t)$, $Q_2(t)$ be defined by the formula of (5.9). According to the suppositions $q_1(\infty) = q_2(\infty) < \infty$. Therefore $q_\lambda'(\infty) =$

$= q''_\lambda(\infty) = 0$ $(\lambda = 1,2)$, and $q_1(\infty) = q_2(\infty) = Q_1(\infty) = Q_2(\infty)$ holds. According to Lemma 5.1 it holds $Q'_1(t) \in M_{n-1}$, $Q'_2(t) \in M_{n-2}$. According to Lemmas 5.3 and 5.1 it holds for $t \in (0, \infty)$ and $i = 0, 1, \ldots, n-1$

$$(5.25) \qquad (-1)^i Q_1^{(i)}(t) > (-1)^i Q_2^{(i)}(t).$$

The conditions of Theorems 4.1 and 4.2 where $Q \sim Q_2$, $q \sim Q_1$ and $n \sim n-2$ are therefore satisfied. Since $\int^\infty [q_\lambda(t) - Q_\lambda(t)]\, dt$ converges always according to Lemma 5.1, the integral $\int^\infty [Q_1(t) - Q_2(t)]\, dt$ converges if and only if the integral $\int^\infty [q_1(t) - q_2(t)]\, dt$ converges. From Theorems 4.1 and 4.2 the assertions (5.22) and (5.23) follow.

The relations (5.24) and (5.25) follow from Theorems 4.1, 4.2 and Remark 4.1 where $q \sim q_1$, $Q \sim Q_2$. The integral $\int^\infty [q_1(t) - Q_2(t)]\, dt$ namely converges with regard to Lemma 5.1 if and only if the integral $\int^\infty [q_1(t) - q_2(t)]\, dt$ does. •

Remark 5.4. If there is $\alpha > 0$, $\beta > 0$, then regarding Theorem 4.2 the identical equality $q_1(t) \equiv q_2(t)$ for $t \in (\gamma, \infty)$ $(\gamma > 0)$ can be admitted, provided $q'_1(t) \neq 0$ for $t \in (0, \infty)$.

Remark 5.5. If in addition to the conditions of Theorem 5.2, $q'_1(t) \in M_{n+2}$, then (5.22) holds in the form

$$(5.26) \qquad (-1)^i \Delta^i(T_{1,k} - T_{2,k}) > 0 \qquad i = 0, 1, \ldots, n-1; \; k = 0, 1, \ldots .$$

It follows from Remark 4.1 because in this case $Q'_1(t) \in M_n$.

VI. APPLICATION OF DERIVED RESULTS ON BESSEL EQUATION

1. The functions $\left(\frac{1}{2}\pi t\right)^{\frac{1}{2}} J_\nu(t)$, $\left(\frac{1}{2}\pi t\right)^{\frac{1}{2}} Y_\nu(t)$ form a fundamental system of solutions of the Bessel equation

$$(6.1)_\nu \qquad y''(t) + \left\{1 - \frac{\nu^2 - \frac{1}{4}}{t^2}\right\} y(t) = 0. \qquad (t > 0).$$

For the Wronskian w of these functions it holds $|w| = 1$ (see [11]). Further let $y_\nu(t)$ denote any non-trivial solution of $(6.1)_\nu$, thus

$$(6.2)_\nu \qquad y_\nu(t) = at^{1/2} J_\nu(t) + bt^{1/2} Y_\nu(t)$$

where a, b are constants.

Further on, let $\{t_{\nu,k}\}_{k=0}^{\infty}$ denote a sequence of zeros of any solutions $y_\nu(t)$ of $(6.1)_\nu$ and $\{T_{\nu,k}\}_{k=0}^{\infty}$ a sequence of extremants of the same or any other solution of $(6.1)_\nu$. The function

$$(6.3)_\nu \qquad q_\nu(t) = 1 - \frac{\nu^2 - \frac{1}{4}}{t^2}$$

evidently possess a completely monotonic derivative $q'(t)$ for $|\nu| > \frac{1}{2}$, i.e. it holds

$$(6.4) \qquad (-1)^i\, q_\nu^{(i+1)}(t) > 0, \qquad i = 0, 1, \ldots, t \in (0, \infty).$$

Further let $\{\bar{t}_{\nu,k}\}_{k=0}^{\infty}$, resp. $\{\overline{T}_{\nu,k}\}_{k=0}^{\infty}$ denote a sequence of zeros, resp. extremants, of the solution $\bar{y}_\nu(t)$ of $(6.1)_\nu$, which can be, but need not be, identical with the above mentioned solution $y_\nu(t)$. Lee Lorch and Peter Szego showed that, if there is $t_{\nu,0} < \bar{t}_{\nu,0}$ then for $|\nu| > \frac{1}{2}$ the sequence $\{\bar{t}_{\nu,k} - t_{\nu,k}\}_{k=0}^{\infty}$ is completely monotonic and it holds $(-1)^i\Delta^i(\bar{t}_{\nu,k} - t_{\nu,k}) > 0$, $(i = 0, 1, \ldots)$ (see [5], p. 63, 72). In paper [10] I showed that for the sequences of differences of extremants there holds also

$$(-1)^i\, \Delta^i(\overline{T}_{\nu,k} - T_{\nu,k}) > 0, \qquad (i, k = 0, 1, \ldots)$$

if there is $\overline{T}_{\nu,0} > T_{\nu,0}$. In paper [5] it is further shown that for $\nu > \mu > \frac{1}{2}$ there exist to any fixed sequences $\{t_{\mu,k}\}_{k=0}^{\infty}$, $\{t_{\nu,k}\}_{k=0}^{\infty}$ integer r so that

$$(6.5) \qquad (-1)^i\, \Delta^i(t_{\mu,k+r} - t_{\nu,k}) > 0 \qquad i, k = 0, 1, \ldots$$

For proving the last assertion it was necessary to use the properties of the functions $J_\nu(t)$, $Y_\nu(t)$ reaching far to the theory of Bessel functions the derivation of which is therefore by no means easy. The inequalities of (6.5) follow, however, directly from Theorem 4.1, because $q_\mu(t) - q_\nu(t) = (\nu^2 - \mu^2)\, t^{-2}$ and consequently for $\nu > \mu \geqq \frac{1}{2}$ it holds

$$(6.6) \qquad [q_\mu(t) - q_\nu(t)] \in M_\infty\,; \qquad 0 < \int^{\infty} [q_\mu(t) - q_\nu(t)]\, \mathrm{d}t < \infty.$$

2. In the following theorem there are contained all properties of Bessel functions (under the notion of Bessel function of order ν we mean any solution $(6.2)_\nu$ of the equation $(6.1)_\nu$) mentioned in paragraph 1, and also all their properties deduced in paper [5] as some special cases.

Theorem 6.1. *Let ν, μ be any numbers satisfying $\nu > \mu \geqq \frac{1}{2}$. Let $\{t_{\nu,k}\}_{k=0}^{\infty}$, resp. $\{t_{\mu,k}\}_{k=0}^{\infty}$, $\{\bar{t}_{\mu,k}\}_{k=0}^{\infty}$ denote a sequence of zeros of any solution $y_\nu(t)$, resp. $y_\mu(t)$, $\bar{y}_\mu(t)$ of $(6.1)_\nu$, resp. $(6.1)_\mu$, and $\{T_{\nu,k}\}_{k=0}^{\infty}$, resp. $\{T_{\mu,k}\}_{k=0}^{\infty}$ a sequence of extremants of the same or any other solution $\bar{y}_\nu(t)$, resp. $\bar{y}_\mu(t)$. Designate $\alpha_\nu = \nu^2 - \frac{1}{4}$, $\alpha_\mu = \mu^2 - \frac{1}{4}$. Then to any pair of numbers $\lambda \in \langle 0, \infty)$ and $t_{\nu,0} \in (0, \infty)$, resp. $T_{\nu,0} \in (\alpha_{\mu,})$ such a number $t_{\mu,0}$, resp. $T_{\mu,0}$, $\bar{t}_{\mu,0}$ exists that it holds*

$$(6.7) \qquad (-1)^i \Delta^i(t_{\mu,k} - t_{\nu,k}) > 0 \qquad i, k = 0, 1, \ldots$$
$$(t_{\mu,k} - t_{\nu,k}) \to \lambda \quad \textit{for} \quad k \to \infty$$

$$(6.8) \qquad (-1)^i \Delta^i(\bar{t}_{\mu,k} - T_{\nu,k}) > 0 \qquad i, k = 0, 1, \ldots$$
$$(\bar{t}_{\mu k} - T_{\nu k}) \to \lambda \quad \textit{for} \quad k \to \infty$$

$$(6.9) \qquad (-1)^i \Delta^i(T_{\mu,k} - T_{\nu,k}) > 0 \qquad i, k = 0, 1, \ldots$$
$$(T_{\mu,k} - T_{\nu,k}) \to \lambda \; \textit{for} \; k \to \infty$$

The numbers $t_{\mu,0}$, $\bar{t}_{\mu,0}$ and $T_{\mu,0}$ are the solutions of corresponding equations (4.24), *respectively* (4.11).

Proof. Since $q_\lambda'(t) \in M\infty$, $q_\mu'(t) \in M_\infty$, $q_\nu(\infty) = q_\mu(\infty) = 1$ and (6.6) holds, the conditions of Theorems 4.1 and 4.2 are satisfied for $n = \infty$ and $q \sim q_\nu$, $Q \sim q_\mu$. Hence (6.7) directly follows. Assertions (6.8) and (6.9) follow analogously from Theorem 5.2 where the interval $(0, \infty)$ of the variable t is replaced by interval (α_μ, ∞).

The function $Q_\nu(t)$ is in the Bessel equation of the form

$$Q_\nu(t) = 1 - \frac{\alpha_\nu}{t^2} - 3\,\frac{\alpha_\nu}{(t^2 - \alpha_\nu)^2}.$$

Remark 6.1. Provided $\mu > \frac{1}{2}$, it is possible to put $\mu = \nu$ in the preceding theorem so that the relation (6.7)—(6.9) are then related to the sequences of zeros, resp. extremants of the same or different solutions of the same Bessel equation. (Except the case $\lambda = 0$ in (6.7) and (6.9) when $\{t_{\nu,k}\}_{k=0}^{\infty} = \{t_{\mu,k}\}_{k=0}^{\infty}$ and $\{T_{\mu,k}\}_{k=0}^{\infty} = \{T_{\nu,k}\}_{k=0}^{\infty}$).

Theorem 6.2. *For $\nu > \frac{1}{2}$ let $\{t_{\nu,k}\}_{k=0}^{\infty}$ denote the sequence of zeros of any solution $y_\nu(t)$ of $(6.1)_\nu$ and $\{T_{\nu,k}\}_{k=0}^{\infty}$ the sequence of extremants of the same solution. If there is $t_0 > T_0 > \alpha_\nu$ $\left(\alpha_\nu = \nu^2 - \frac{1}{4}\right)$ then*

(6.10) $\qquad (-)^i \Delta^i (t_{\nu,k} - T_{\nu,k}) > 0 \qquad i, k = 0,1 \ldots$

Proof. Theorem 6.2 is the direct consequence of Theorem 5.1 for $n = \infty$ and $t \in (\alpha_\nu \ \infty)$ because $q_\nu' (t) \in M_\infty$, $q_\nu(t) > 0$ for $t > \alpha_\nu$ and $q_\nu(\infty) = 1$

3. The results derived in the preceding paragraphs are possible to be used not only to the qualitative but also to quantitative investigation of concrete equations. The general solution of (6.1) is possible, on the basis of transformation (4.3), to be written in the form

$$(6.11) \qquad y_\nu(t) = A[v_\nu(t)]^{1/2} \sin\{ \int_{t_0}^{t} [v_\nu(s)]^{-1} \, ds\}$$

where A, t_0 are integration constants and $v_\nu(t)$ being solution of the equation

$$(6.12) \qquad v_\nu''' + 4\left\{1 - \frac{\nu^2 - \frac{1}{4}}{t^2}\right\} v_\nu' + \frac{4\nu^2 - 1}{t^3} v_\nu = 0$$

satisfying $[v_\nu(t) - 1] \in M_\infty$.

From (6.11) we can see that the k-th zero t_k of (6.11) is a root of the equation

$$(6.13) \qquad \int_{t_0}^{t_k} [v_\nu(s)]^{-1} \, ds = k\pi.$$

The function $\int_{t_0}^{t} [v_\nu(s)]^{-1} \, ds$ is then determining the distribution of zeros of to solution $(6.11)_\nu$.

4. Transformation $t = \frac{1}{x}$ transforms the equation $(6.12)_\nu$ into the equation

$$(6.14)\nu \qquad \dddot{v}_\nu + \frac{6}{x} \ddot{v}_\nu + \left\{\frac{7 - 4\nu^2}{x^2} + \frac{4}{x^4}\right\} \dot{v}_\nu - \frac{4\nu^2 - 1}{x^3} v_\nu = 0$$

where the point denotes the derivative according to x. The equation $(6.14)_\nu$ has an irregular singular point of the second class in the origin (see [4]) having thus maximum one solution, regular in the origin, consequently the solution that is of the form

$$(6.15)_\nu \qquad v_\nu = x^s(a_0 + a_1 x + a_2 x + \ldots) = t^{-s}(a_0 + a_1 t^{-1} + \ldots) \quad a_0 \neq 0.$$

By a direct substitution into the equation $(6.12)_\nu$ or $(6.14)_\nu$ we get $s = 0$, $a_1 = 0$ and for coefficients a_n we get a recurrent formula

(6.16) $$a_{n+2} = \frac{1}{4}\,\frac{n+1}{n+2}\,[4\nu^2 - (n+1)^2]\,a_n \qquad n = 0, 1, \ldots$$

Hence it follows for $k = 0, 1, \ldots, a_{2k+1} = 0$ and

$$a_{2k+2} = \frac{1.3 \ldots (2k-1)}{2^{3k}\,.\,k\,!}\,[4\nu^2 - 1^2]\,[4\nu^2 - 3^2] \ldots$$

(6.17) $$\ldots [4\nu^2 - (2k-1)^2]\,a_0$$

From (6.17) we can instantly see that if there is $\nu = n + \frac{1}{2}$, $n = 0, 1, \ldots$, then for $k = n, n+1, \ldots$ it holds $a_{2k+2} + 0$ and thus the series (6.15) has only finately many non-zero members. If we choose $a_0 = 1$ then the function $v_{n+1/2}(x)$ has for $n = 1, 2, \ldots$ the form

$$v_{n+1/2}(t) = 1 + \sum_{k=1}^{n} \frac{1.3\,.\,(2k-1)}{2^{3k}\,.\,k\,!}\{(2n+1)^2 - 1^2\}\,.\,\{(2n+1)^2 - 3^2\} \ldots$$

(6.18) $$\ldots \{(2n+1)^2 - (2k-1)^2\}\,t^{-2k}$$

and $v_{\frac{1}{2}}(t) = 1$. According to Lemma 1.1 the equation $(6.12)_\nu$ has for $\nu > \frac{1}{2}$ a unique solution $v_\nu(t)$ satisfying the relations $(-1)^i\,v_\nu(t)^{(i)} > 0$ for $i = 0, 1, \ldots$ and $v_\nu(\infty) = 1$. Just this solution, however, is expressed for $\nu = n + \frac{1}{2}$ $(n = 1, 2, \ldots)$ by the formula (6.18) because $a_{2k} > 0$ for $k = 0, 1, \ldots, n$, $a_{2k} = 0$ for $k > n$ and $t^{-2}{}_k \in M_{\infty,0}$.

Therefore there exists such a pair of solutions $(x_{n+1/2}(t), y_{n+1/2}(t))$ of the equation $(6.1)_{n+1/2}$ that it holds for $i = 0, 1, \ldots$

(6.19) $$x^2_{n+1/2} + y^2_{n+1/2} = v_{n+1/2}, \quad (-1)^i\,v^{(i)}_{n+1/2} > 0, \quad v_{n+1/2}(\infty) = 1.$$

For the Wronskian $w_{n+1/2}(x_{n+1/2}, y_{n+1/2})$ of the solutions $x_{n+1/2}, y_{n+1/2}$ from (6.19),

$$|\,W_{n+1/2}(x_{n+1/2}, y_{n+1/2}) = v_{n+1/2}(\infty)\,[q_{n+1/2}(\infty)]^{1/2} = 1$$

then holds according to Lemma 1.2.

From asymptotic formulae for $J_{n+1/2}(t)$, $Y_{n+1/2}(t) = J_{-n-1/2}$ (see e.g. [11], p. 199) it follows that with regard to Lemma 1.2 it is possible to choose $x_{n+1/2} = \left(\frac{1}{2}\pi t\right)^{1/2} J_{n+1/2}(t)$, $y_{n+1/2} = \left(\frac{1}{2}\pi t\right)^{1/2} Y_{n+1/2}(t)$.

5. Let us choose now $\mu = \frac{1}{2}$ in Theorem 6.1. Then evidently $\{t_{\mu,k}\} = \{M + k\pi\}$ where $M = t_{\mu,0} > 0$. Let us calculate now this M for

$\lambda = 0$. M is the root of the equation (2.22) which, in our case, is of the form

$$\int_{t_{n+1/2,0}}^{M} [v_{n+1/2}(t)]^{-1}\, dt = \int_{M}^{\infty} \{1 - [v_{n+1/2}(t)]^{-1}\}\, dt \tag{6.21}$$

For $n = 1, 2, 3, 4, 5$, $v_{n+1/2}(t)$ is of the form

$$v_{3/2} = 1 + \frac{1}{t^2} \qquad v_{5/2} = 1 + \frac{3}{t^2} + \frac{9}{t^4}$$

$$v_{7/2} = 1 + \frac{6}{t^2} + \frac{45}{t^4} + \frac{225}{t^6}$$

$$v_{9/2} = 1 + \frac{10}{t^2} + \frac{135}{t^4} + \frac{1575}{t^6}\ \frac{11025}{t^8}$$

$$v_{11/2} = 1 + \frac{15}{t^2} + \frac{315}{t^4} + \frac{6300}{t^6} + \frac{99225}{t^8} + \frac{893025}{t^{10}}$$

Let us observe now more minutely the equations (6.21) for $n = 1, \ldots, 5$. The concrete calculation of the functions $\int [v_{n+1/2}(t)]^{-1}\, dt$ is thus a little laborious, but the verification of correctness does not make any troubles.

Further on, let

$$-\frac{\pi}{2} < \text{arc tg}\ \varphi(t) \leqq \frac{\pi}{2}$$

always hold for arc tg $\varphi(t)$ where arc tg denotes the principal branche.

6. For $n = 1$ we have (with indefinite integrals we take one of primitive functions)

$$\int \{1 - [v_{3/2}(t)]^{-1}\}\, dt = \int (1 + t^2)^{-1}\, dt = \text{arc tg}\ t$$

and the equation (6.21) is therefore of the form

$$[t - \text{arc tg}\ t]_{t_{3/2,0}}^{M} = [\text{arc tg}\ t]_{M}^{\infty}$$

thus

$$M(t_{3/2,0}) = \frac{\pi}{2} + t_{3/2,0} - \text{arc tg}\ t_{3/2,0} \tag{6.23}$$

The equation (6.13) is in this case of the form

$$t_{3/2,k} - \text{arc tg}\ t_{3/2,k} = k\pi + t_{3/2,0} - \text{arc tg}\ t_{3/2,0} \tag{6.24}$$

The solution $y_{3/2}(t)$ of the equation $(6.1)_{3/2}$, for which it holds $y_{3/2}(t_{3/2,0}) = 0$, is, regarding to (6.11), possible to be written in the form

(6.25) $$y_{3/2}(t) = At^{-1}\sqrt{1+t^2}\sin(t - \operatorname{arc\,tg} t - B)$$

where $B = t_{3/2,0} - \operatorname{arc\,tg} t_{3/2,0}$. If we choose specially $t_{3/2,0} = 0$ (this being possible because such a solution really exists) then $y_{3/2}(t) = At^{1/2}J_{3/2}(t)$ and thus

(6.26) $$J_{3/2}(t) = \sqrt{\frac{2}{\pi}}\, t^{-3/2}\sqrt{1+t^2}\sin(t - \operatorname{arc\,tg} t)$$

and k-th zero $j_{3/2,k}$ is the root of the equation

(6.27) $$t - \operatorname{arc\,tg} t = k\pi \qquad k = 0, 1, \ldots$$

Thus the sequence $\{j_{3/2,k}\}$ is formed by the system of non-negative roots of the equation $t = \operatorname{tg} t$. Analogously the sequence $\{y_{3/2,0}\}$ of zeros of the function

(6.28) $$Y_{3/2}(t) = \sqrt{\frac{2}{\pi}}\, t^{-3/2}\sqrt{1+t^2}\cos(t - \operatorname{arc\,tg} t)$$

is formed by the system of non-negative roots of the equation

$$t - \operatorname{arc\,tg} t = \frac{\pi}{2} + k\pi, \quad k = 0, 1, \ldots,$$

which is equivalent with the equation $t = -\operatorname{cotg} t$.

7. For $n = 2$ we get

$$\int \{1 - [v_{5/2}(t)]^{-1}\}\, dt = \int \frac{3t^2+9}{t^4+3t^2+9}\, dt = \operatorname{arc\,tg} \frac{t^2-3}{3t}$$

The function $\operatorname{arc\,tg} \frac{t^2-3}{3t}$ is for $t \in \langle 0, \infty)$ continuous inclusive all derivatives.

In this case the equation (6.21) has the solution of the form

(6.29) $$M(t_{5/2,0}) = \frac{\pi}{2} + t_{5/2,0} - \operatorname{arc\,tg} \frac{t_{5/2,0}^2 - 3}{3t_{5/2,0}}$$

$J_{5/2}(t)$ and $Y_{5/2}(t)$ are possible to be expressed by the formulae that are analogous to (6.23) and (6.28).

8. For $n = 3$ it holds

$$\int \{1 - [v_{7/2}(t)]^{-1}\}\, dt = \int \frac{6t^4+45t^2+225}{t+6t^4+45t^2+225}\, dt = \operatorname{Arc\,tg} \varphi_{7/2}(t)$$

where $\varphi_{7/2}(t) = \dfrac{t^3-15t}{6t^2-15}$ and

$$\operatorname{Arc\,tg} \varphi_{7/2}(t) = \begin{cases} \operatorname{arc\,tg} \varphi_{7/2}(t) & \text{for} \quad |t| < \left(\frac{5}{2}\right)^{1/2} \\ \frac{\pi}{2} & \text{for} \quad t = \left(\frac{5}{2}\right)^{1/2} \\ \pi + \operatorname{arc\,tg} \varphi_{7/2}(t) & \text{for } t > \left(\frac{5}{2}\right)^{1/2} \end{cases}$$

The function $\operatorname{Arc\,tg} \frac{t^3 - 15t}{6t^2 - 15}$ is continuous for $t \in \langle 0, \infty)$ inclusive of all its derivatives. Because of $\lim_{t \to \infty} \operatorname{Arc\,tg} \frac{t^3 - 15t}{6t^2 - 15} = \frac{3}{2}\pi$, the equation (6.21) has the solution

$$M(t_{7/2,0}) = \frac{3}{2}\pi + t_{7/2,0} - \operatorname{Arc\,tg} \frac{t_{7/2,0}^3 - 15t_{7/2,0}}{6t_{7/2,0}^2 - 15} \tag{6.30}$$

9. For $n = 4$ it holds

$$\int \left\{1 - \frac{1}{v_{9/2}(t)}\right\} dt = \int \frac{10t^6 + 135t^4 + 1575t^2 + 11025}{t^8 + 10t^6 + 135t^4 + 1575t^2 + 11025} dt =$$

$$= \operatorname{Arc\,tg} \varphi_{9/2}(t) \text{ where } \varphi_{9/2}(t) = \frac{t^4 - 45t^2 + 105}{10t^3 - 105t} \text{ and}$$

$$\operatorname{Arc\,tg} \varphi_{9/2}(t) = \begin{cases} \operatorname{arc\,tg} \varphi_{9/2}(t) & \text{for} \quad |t| < \left(\frac{21}{2}\right)^{1/2} \\ \frac{\pi}{2} & \text{for} \quad t = \left(\frac{21}{2}\right)^{1/2} \\ \pi + \operatorname{arc\,tg} \varphi_{9/2}(t) & \text{for} \quad t > \left(\frac{21}{2}\right)^{1/2} \end{cases}$$

The function $\operatorname{Arc\,tg} \varphi_{9/2}(t)$ is continuous for $t \in \langle 0, \infty)$ inclusive of all its derivatives. The equation (6.21) has then the solution

$$M(t_{9/2,0}) = 2\pi + t_{9/2,0} - \operatorname{Arc\,tg} \varphi_{9/2}(t_{9/2,0}). \tag{6.31}$$

10. For $n = 5$ it holds

$$\int \left\{1 - \frac{1}{v_{11/2}(t)}\right\} dt =$$

$$= \int \frac{15t^8 + 315t^6 + 6300t^4 + 99225t^2 + 893025}{t^{10} + 15t^8 + 315t^6 + 6300t^4 + 99225t^2 + 893025} dt =$$

$$= \operatorname{Arc\,tg} \varphi_{11/2}(t) \text{ where } \varphi_{11/2}(t) = \frac{t^5 - 105t^3 + 945t}{15t^4 - 420t^2 + 945} \text{ and}$$

$$\text{Arc tg}\ \varphi_{11/2}(t) = \begin{cases} \text{arc tg}\ \varphi_{11/2}(t) & \text{for} \quad |t| < \beta_1 \\ \dfrac{\pi}{2} & \text{for} \quad t = \beta_1 \\ \pi + \text{arc tg}\ \varphi_{11/2}(t) & \text{for} \quad \beta_1 < t < \beta_2 \\ \dfrac{3\pi}{2} & \text{for} \quad t = \beta_2 \\ 2\pi + \text{arc tg}\ \varphi_{11/2}(t) & \text{for} \quad t > \beta_2 \end{cases}$$

At the same time β_1, β_2 denote the real roots of the equation $15t^4 - 420t + 945 = 0$ $(0 < \beta_1 < \beta_2)$ so that $\beta_{1,2} = (14 \mp \sqrt{133})^{1/2}$. The function Arc tg $\varphi_{11/2}(t)$ is continuous for $t \in \langle 0, \infty)$ inclusive of all its derivatives. In this case the equation (6.21) has then the solution

$$(6.32) \qquad M(t_{11/2,0}) = \frac{5}{2}\pi + t_{11/2,0} - \text{Arc tg}\ \varphi_{11/2}(t_{11/2,0})$$

11. The functions $\int \left\{1 - \dfrac{1}{v_{n+1/2}(t)}\right\} dt$ for $n > 5$ are possible to be expressed in a similar way.

For $\nu \neq n + \dfrac{1}{2}$, $n = 0, 1, \ldots$ $\left(\nu > \dfrac{1}{2}\right)$ the series $v_\nu(t) = \sum\limits_{k=0}^{\infty} a_{2k} t^{-2k}$, where a_{2k} is given by the formula (6.17), diverges for all t. In this case, the series is, however, an asymptotic development of the function $v_\nu(t)$ and therefore it is possible to be used to concrete calculations.

The function $v_{3/2}(t)$ was calculated by F. Neuman in another way in [7].

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