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František Fiala

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POLARS IN DISTRIBUTIVE LATTICES WITH THE SMALLEST ELEMENT

František Fiala (Brno)

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Introduction

Let G be a lattice ordered group (l-group). According to [1], two elements a, $b \in G$ are called disjoint if $|a| \wedge |b| = 0$. F. Šik started with the definition in his studies of properties of l-groups (for example in [8]—[12]). The notion of a component in a l-group has a great importance in his considerations. $A \subseteqq G$ is said to be a component in G if there exists $\emptyset \neq B \subseteqq G$, so that

$$A = \{a \in G : |a| \wedge |b| = 0 \text{ for all } b \in B\}.$$

We denote $A = B'$. If $A = \{a\}' = a'$, then we speak about a dual principal component, if $A = (a')' = a''$, then we speak about a principal component. The set $\Gamma(G)$ of all components in G ordered by set-theoretical inclusion is a complete Boolean lattice in which all principal or all dual principal components form sublattices, denoting $\Pi(G)$ or $\Pi'(G)$ [8]. In F. Šik's considerations the notion of ultraantifilter in the lattices $\Gamma(G)$, $\Pi(G)$ and $\Pi'(G)$ shows itself as very useful. Topologies were introduced in [8] into the sets of all ultraantifilters in these lattices and the compactness of the spaces is being investigated in the first place.

Into the set of all standard ultraantifilters in $\Gamma(G)$ (i.e. such ultraantifilters for which the set-theoretical union of all components that belong to them, is not the whole G) a topology was in a certain way introduced and properties of the space were investigated ([3], [4]).

It is shown that a number of the assertions can be expressed in a similar way for an arbitrary distributive lattice L with the smallest element. But it is necessary to find the corresponding notions. It is also possible to transfer many proofs with smaller or greater arrangement on the new assertions. We shall now consider the problems.

In the first part, we define basic notions. Similarly to the definition of the component in an l-group, the notion of polar in a lattice L is introduced in 1.2. We show that the set $P(L)$ of all polars in L is a complete Boolean algebra (1.8) in which the set $H(L)$ or $D(L)$ all principal or all dual principal polars form sublattices.

In the second part, we investigate the topological spaces of all ultraantifilters in $P(L)$, $H(L)$ and $D(L)$. We study compactness (2.9, 2.10) or further properties of the spaces.

The third part deals with the standard ultraantifilters in $P(L)$. On the set of all standard ultraantifilters in $P(L)$ a topology is defined and necessary and sufficient conditions for compactness (3.3), complete regularity (3.17), Hausdorff property (3.19) or normality (3.27) of the space are found.

Notations

If nothing else is said, then we understand by L *a distributive lattice with the smallest element*. We shall not always repeat the assumption. Elements of L will be denoted by small letters from the beginning of the alphabet.

Lattice operations in L are denoted by $\cup$, $\cap$, the order by $\leqq$.

Lattice operations in other lattices are denoted by $\vee$, $\wedge$, or by an index that we shall leave out when it is clear in which lattice the operation is carried out.

Set-theoretical union and intersection are also denoted by $\cup$, $\cap$, inclusion $\subseteqq$. Symbol $\subset$ characterizes a sharp inclusion.

Instead of $\{a\}$ we shall write briefly a.

$U(L)$ = the set of all closed elements of L.
$I(L)$ = the set of all ideals in L.
$P(L)$ = the set of all polars in L.
$H(L)$ = the set of all principal polars in L.
$D(L)$ = the set of all dual principal polars in L.
$\mathfrak{U}(L)$ = the set of all ultraantifilters in L.
$\mathfrak{U}_s(L)$ = the set of all standard ultraantifilters in L.
$\mathfrak{A}(L)$ = the set of all elements that are not dense in L.

1. Boolean algebra $P(L)$ of all polars in L

For the standard definitions and results concerning lattices the reader is referred to [1]. Here we shall introduce only basic notions and some important results.

1.1. *An ideal* in an arbitrary lattice L is a non-empty subset $J \subseteqq L$ with the following properties:

$$a, b \in J \Rightarrow a \cup b \in J. \tag{1}$$

$$a \in J,\ c \in L,\ c \leqq a \Rightarrow c \in J. \tag{2}$$

The set of all ideals in L is denoted by $I(L)$ and ordered by set-theoretical inclusion.

For an arbitrary $a \in L$ the set

$$(a] = \{b: b \in L, b \leqq a\}$$

is obviously an ideal in L. It is called *the principal ideal in L determined by element a*.

An ideal J in L is *a prime ideal* provided that if fulfils the following condition:

(3) $$a \cap b \in J \Rightarrow a \in J \quad \text{or} \quad b \in J.$$

An ideal J in L which fulfils the following condition:

(4) *If the greatest element* 1 *of L exists then* $1 \bar{\in} J$,

is called *an antifilter* in L. The set of all antifilters in L is a subset of $I(L)$ and consequently it is also ordered by set-theoretical inclusion. Its maximal elements will be called *ultraantifilters* in L. [By Zorn's lemma each antifilter in L is contained in an ultraantifilter.] The set of all ultraantifilters in L will be designated by $\mathfrak{U}(L)$. It is evident that $\mathfrak{U}(L) \subseteqq I(L)$.

The following statements hold ([9], 4.1—4.3):

(a) *An antifilter in a Boolean algebra B is an ultraantifilter if and only if it contains strictly one out of each two complementary elements of B.*

(b) *Each ultraantifilter in a distributive lattice is a prime antifilter.*

(c) *Each prime antifilter in a Boolean algebra is an ultraantifilter and viceversa.*

If L is a lattice of subsets of a set A and $J \in I(L)$, then $\cup J$ denotes set-theoretical union of all elements of J. If $J \in \mathfrak{U}(L)$ and $\cup J \neq A$, then J is called *a standard ultraantifilter* in L. The set of all standard ultraantifilters in L is denoted by $\mathfrak{U}_s(L)$.

The term of *filter* is dual to the term of antifilter (with regard to partial ordering of L).

1.2. Let L be an arbitrary lattice with the smallest element 0. Two elements $a, b \in L$ are called *disjoint* if $a \cap b = 0$. Non-empty subsets $A, B \subseteqq L$ are called *disjoint* if $a \cap b = 0$ for each $a \in A$, $b \in B$. We understand by *a disjoint complement* A' of a non-empty subset A of L the following set:

(5) $$A' = \{b \in L: b \cap a = 0 \text{ for each } a \in A\}.$$

$A \subseteqq L$ is *a polar* in L if such a $\emptyset \neq B \subseteqq L$ exists that $A = B'$. The polars of the forms $\{a\}' = a'$ and $(a')' = a''$ will be called *dual principal* and *principal* respectively. We shall denote $P(L)$, $H(L)$ and $D(L)$ the set of all, all principal, all dual principal polars in L respectively.

It follows from (5) that $A' \in I(L)$ and moreover A' is the greatest ideal in L disjoint from A.

1.3. Let L be an arbitrary lattice with the smallest element 0. *The* *pseudo-complement* a^* of an element $a \in L$ is the greatest element disjoint from a, if such an element exists. The defining property of a^* is:

$$a \cap b = 0 \Leftrightarrow b \leqq a^*. \tag{6}$$

If every element $a \in L$ has a pseudo-complement (necessarily unique), then the lattice L is said to be *pseudo-complemented lattice.*

If L is a pseudo-complemented lattice, then the correspondence $a \to a^*$ is a Galois's connection. According to [1] (IX, § 12), the following hold:

$$a \leqq a^{**}, \tag{7}$$

$$a^* = a^{***}, \tag{8}$$

$$a \leqq b \Rightarrow a^* \geqq b^*, \tag{9}$$

$$(a \cup b)^* = a^* \cap b^*, \tag{10}$$

$$(a \cap b)^* \geqq a^* \cup b^*. \tag{11}$$

The elements of L for which $a = a^{**}$ are called *closed.* A closed element a of L is the pseudo-complement of a^*. On the other hand, if $a = b^*$, then $a^{**} = b^{***} = b^* = a$ [according to (8)], consequently a is closed. Therefore *the set $U(L)$ of all closed elements of L consists of all pseudo-complements of L.* Obviously $U(L) \subseteqq L$. $U(L)$ is a lattice with regard to the partial ordering of L (generally not a sublattice!). It holds for $a, b \in U(L)$:

$$a \vee_U b = (a \cup b)^{**} = (a^* \cap b^*)^*, \tag{12}$$

$$a \wedge_U b = a \cap b. \tag{13}$$

Element a^* is the complement of a in the lattice $U(L)$, i.e. it holds:

$$a \wedge_U a^* = 0, \qquad a \vee_U a^* = 1. \tag{14}$$

Moreover

$$(a \cap b)^* = a^* \vee_U b^*. \tag{15}$$

According to [1], IX, § 12, Theorem 16, the following is valid.

1.4. Lemma. *If L is a complete, distributive and pseudo-complemented lattice, then the set $U(L)$ of all closed elements of L with the operations* (12) *and* (13) *forms a complete Boolean algebra.*

1.5. *If L is a distributive lattice with the smallest element, then $I(L)$ is* (with regard to the set-theoretical inclusion) *a complete, distributive and pseudo-complemented lattice* ([1], IX, § 12, Ex. 5a). Let $\emptyset \neq A \subseteqq L$. Then ideal $J(A)$ in L generated by the set A is received in this way:

$$J(A) = \bigvee_{I\, a \in A} (a], \tag{16}$$

i.e. $J(A)$ is formed by all elements of L for which

$$a \leqq a_1 \cup \ldots \cup a_n \tag{17}$$

where a_i $(i = 1, \ldots, n)$ are arbitrary elements of A.

It was said in 1.2 that every polar in L is an ideal in L. Now, we shall investigate the role of polars in L in the lattice $I(L)$.

1.6. Lemma. *For $\emptyset \neq A \subseteqq L$, the following are equivalent:*
1. *A is a polar in L, i.e. $A \in P(L)$.*
2. *A is a pseudo-complement in $I(L)$, i.e. $A \in U[I(L)]$.*

Proof: $1 \Rightarrow 2$. Let $A \in P(L)$ be arbitrary. Then some $\emptyset \neq B \subseteqq L$ exists, so that $A = B'$. We shall prove that $B' = J'(B)$, where $J(B)$ is ideal in L, generated by the set B. Clearly, $B \subseteqq J(B)$. By the definition (1.2) $J'(B) \subseteqq B'$. Conversely, let $a \in B'$, $b \in J(B)$. By (17), there exist $b_1, \ldots, b_n \in B$, so that $b \leqq b_1 \cup \ldots \cup b_n$. It holds $a \cap b_i = 0$ $(i = 1, \ldots, n)$, hence $a \cap b \leqq a \cap (b_1 \cup \ldots \cup b_n) = (a \cap b_1) \cup \ldots \cup (a \cap b_n) = 0$, then $a \cap b = 0$ for an arbitrary $b \in J(B)$, i.e. $a \in J'(B)$. Therefore $B' \subseteqq J'(B)$ and $B' = J'(B)$. By (5), $J'(B)$ is the greatest ideal in L disjoint from $J(B)$, i.e. $J(B) \wedge_I J'(B) = 0$, therefore $J'(B) = J^*(B)$. Hence $A = B' = J'(B)$ is a pseudo-complement of $I(L)$, i.e. $A \in U[I(L)]$.

$2 \Rightarrow 1$. Infimum in $I(L)$ is determined by set-theoretical intersection. For $J \in I(L)$, pseudo-complement $J^* \in I(L)$ is the greatest ideal with the properties: $J^* \wedge_I J = J^* \cap J = 0$, hence

$$J^* = \{b : b \in L, b \cap a = 0 \text{ for every } a \in J\}.$$

Comparing it with (5) we get $J^* = J'$, i.e. $J^* \in P(L)$.

1.7. Remark. *Every polar in L has the form of J^*, where $J \in I(L)$. Especially, for $a \in L$, it is $a' = (a]^*$, $a'' = (a]^{**}$. We shall further denote the polars in this way.*

The following statement follows from the results in 1.3—1.6.

1.8. Theorem. *The set $P(L)$ of all polars in L ordered by set-theoretical inclusion, forms a complete Boolean algebra in which infimum is determined by set-theoretical intersection. Complement of $A \in P(L)$ is A^*.*

1.9. Theorem. *The sets $H(L)$ and $D(L)$ are sublattices of the lattice $P(L)$ and for any $a, b \in L$ the following hold:*

$$(a]^{**} \vee_P (b]^{**} = (a \cup b]^{**}, \quad (a]^{**} \wedge_P (b]^{**} = (a \cap b]^{**}, \tag{18}$$

$$(a]^{*} \vee_P (b]^{*} = (a \cap b]^{*}, \quad (a]^{*} \wedge_P (b]^{*} = (a \cup b]^{*}. \tag{19}$$

Proof: Obviously for any $a, b \in L$ $(a \cup b] = (a] \vee_I (b]$, $(a \cap b] = (a] \wedge_I (b]$. With regard to 1.6, the operations in $P(L)$ are identical with those in $U[I(L)]$. Then $(a \cup b]^* = ((a] \vee_I (b])^* = (a]^* \wedge_P (b]^*$ by

(10). $(a \cup b]^{**} = ((a]^* \wedge_P (b]^*)^* = ((a]^* \wedge_I (b]^*)^* = (a]^{**} \vee_P (b]^{**}$ by (15). Further $(a \cap b]^* = ((a] \wedge_I (b])^* = (a]^* \vee_P (b]^*$ by (15). $(a \cap b]^{**} = ((a]^* \vee_P (b]^*)^*$. We shall der ive that $(a]^{**} \wedge_P (b]^{**}$ is a complement of $(a]^* \vee_P (b]^*$ in $P(L)$. It is $[(a]^{**} \wedge_P (b]^{**}] \wedge_P [(a]^* \bar{\vee}_P (b]^*] = [((a]^{**} \wedge_P (b]^{**}) \wedge_P (a]^*] \vee_P [((a]^{**} \wedge_P (b]^{**}) \wedge_P (b]^*] = 0 \vee_P 0 = 0$, $[(a]^{**} \wedge_P (b]^{**}] \vee_P [(a]^* \vee_P (b]^*] = [(a]^{**} \vee_P ((a]^* \vee_P (b]^*)] \wedge_P \wedge_P [(b]^{**} \vee_P ((a]^* \vee_P (b]^*)] = L \wedge_P L = L$. Hence $(a \cap b]^{**} = (a]^{**} \wedge_P (b]^{**}$. In this way, all equations (18), (19) have been proven.

1.10. Theorem. *Lattice $P(L)$ of all polars in L is $\vee$-generated by sublattice $H(L)$ and $\wedge$-generated by sublattice $D(L)$.*

Proof: Obviously for $A \in P(L)$ the following hold:

$$A = \vee_P \{(a]^{**}: a \in A\},$$

$$A = A^{**} = (\vee_P \{(b]^{**}: b \in A^*\})^* = \wedge_P \{(b]^*: b \in A^*\}.$$

2. Spaces of ultraantifilters in $P(L)$,, $H(L)$, $D(L)$

In this section we shall introduce into the sets $\mathfrak{U}[P(L)]$, $\mathfrak{U}[H(L)]$, $\mathfrak{U}[D(L)]$ of all ultraantifilters in $P(L)$, $H(L)$, $D(L)$ a topology and we shall investigate topological properties of these spaces. Elements of these sets, i.e. ultraantifilters in the particular lattices, will be denoted by small letters of the end of the alphabet.

2.1. Lemma. *For any $x \in \mathfrak{U}[P(L)]$ or any $x \in \mathfrak{U}[D(L)]$ it holds:*

$$a \in \cup x \Leftrightarrow (a]^* \bar{\in} x. \tag{20}$$

Proof: 1. If $a \in \cup x$, then there exists such a polar $A \in x$ that $a \in A$, hence $(a]^{**} \subseteq A$. Moreover, $L = (a]^* \vee (a]^{**} \subseteq (a]^* \vee A$ and so $(a]^* \bar{\in} x$.

2. If $(a]^* \bar{\in} x$, then such a polar $A \in x$ exists that $(a]^* \vee A = L$. Then follows that $A \supseteq (a]^{**}$, $a \in (a]^{**} \subseteq A \subseteq \cup x$, therefore $a \in \cup x$.

2.2. Corollary. *The following are true for any $x \in \mathfrak{U}[P(L)]$:*

$$x \in \mathfrak{U}_s[P(L)] \Leftrightarrow x \cap D(L) \neq \emptyset. \tag{21}$$

$$a \in \cup x \Leftrightarrow (a]^* \bar{\in} x \Leftrightarrow (a]^{**} \in x. \tag{22}$$

Proof: (21) follows immediately from 2.1, (22) follows from 2.1 and 1.1(a).

Let us denote with Δ any set of $\mathfrak{U}[P(L)]$, $\mathfrak{U}_s[P(L)]$, $\mathfrak{U}[D(L)]$.

2.3. Lemma. *Let card $L > 1$, then $\cap \{\cup x: x \in \Delta\} = 0$.*

Proof: Let $0 \neq a \in \cap \{\cup x: x \in \Delta\}$. Then $(a]^* \neq L$, therefore such a $y \in \Delta$ exists that $(a]^* \in y$. By (20) it is $a \bar{\in} \cup y$ and $a \bar{\in} \cap \{\cup x: x \in \Delta\}$. But it is a contradiction to our supposition. Therefore $\cap \{\cup x: x \in \Delta\} = 0$

2.4. An element $a \in L$ is called *dense* in L provided that $(a]^* = 0$. It is evident that the property is equivalent to $(a]^{**} = L$. The dense element is e.g. the greatest element of L, if such an element exists. We shall denote $\mathfrak{A}(L)$ the set of all not dense elements in L.

2.5. Lemma. *Let card* $L > 1$. $\cap \{\cup x : x \in \mathfrak{U}[H(L)]\} = 0$ *if and only if the condition* (23) *holds*:

For any $a \in \mathfrak{A}(L)$, $a \neq 0$, *such a* $b \in \mathfrak{A}(L)$ *exists that*

$$a \cup b \,\overline{\in}\, \mathfrak{A}(L). \tag{23}$$

Proof: Let $a \in \mathfrak{A}(L)$, $a \neq 0$. Then the following holds: For any $b \in \mathfrak{A}(L)$ it is $a \cup b \in \mathfrak{A}(L) \Leftrightarrow$ for any $b \in \mathfrak{A}(L)$ it is $L \neq (a \cup b]^{**} = (a]^{**} \vee_P (b]^{**} \Leftrightarrow (a]^{**}$ is an element of every ultraantifilter in $H(L) \Leftrightarrow 0 \neq a \in \cap \{\cup x : x \in \mathfrak{U}[H(L)]\}$.

2.6. Remark. It follows from (13) that the condition (23) is fulfilled when the lattice L is pseudo-complemented. If the condition (23) is fulfilled, then $H(L)$ contains the greatest element L, it means that L contains a dense element.

2.7. Let L be any lattice with the greatest element 1. We denote

$$\mathfrak{U}a = \{x : x \in \mathfrak{U}(L), a \in x\}$$

for $a \in L$ and

$$\Sigma_L = \{\mathfrak{U}a : a \in L\}.$$

Σ_L is a basis for open sets in $\mathfrak{U}(L)$. By [6], Theorem 1, for the topological space $\mathfrak{U}(L)$ the following are true:

(a) $\mathfrak{U}(L)$ *is a Hausdorff's space.*
(b) *Every set* $\mathfrak{U}a$ *is both open and closed.*
(c) *The space* $\mathfrak{U}(L)$ *is completely regular.*

2.8. Lemma. *If* L *is a distributive lattice with the greatest and smallest elements* 1 *and* 0 *respectively and if* $\cap \{\cup x : x \in \mathfrak{U}(L)\} = 0$, *then the topological space* $\mathfrak{U}(L)$ *is compact if and only if* L *is a Boolean lattice.*
Proof: [8], Theorem 2.

2.9. Theorem. *The topological space* $\mathfrak{U}[P(L)]$ *is compact.*
Proof follows from 2.8, 2.3 and 1.8.

2.10. Theorem. *The following are equivalent*:
1. *The topological space* $\mathfrak{U}[D(L)]$ *is compact.*
2. *The topological space* $\mathfrak{U}[H(L)]$ *is compact and* (23) *is fulfilled.*
3. $D(L)$ *is a Boolean lattice.*
4. $H(L)$ *is a Boolean lattice.*
5. $H(L) = D(L)$.

If one of these conditions stands, then L contains a dense element.

Remark. Theorem 3.5 introduces a further equivalent condition.

Proof: $1 \Rightarrow 3$. First, we shall derive that $0 \in D(L)$. If $\mathfrak{U}[D(L)]$ is compact, then a finite number of elements $a_1, \ldots, a_n \in L$ exists so that

$$\mathfrak{U}[D(L)] = \bigcup_{i=1}^{n} \mathfrak{U}(a_i]^*.$$

If we denote

$$a = a_1 \cup \ldots \cup a_n$$

then

$$(a]^* = (a_1]^* \wedge \ldots \wedge (a_n]^*.$$

Evidently $(a]^* \in x$ for any $x \in \mathfrak{U}[D(L)]$. Hence

$$\cap \{\cup x \colon x \in \mathfrak{U}[D(L)]\} \supseteqq (a]^*.$$

By 2.3 $0 \supseteqq (a]^*$ and obviously $0 \subseteqq (a]^*$, therefore $(a]^* = 0 \in D(L)$. It is the smallest element in $D(L)$. By 2.8, $D(L)$ is a Boolean lattice. The element a is dense in L, therefore the final statement is fulfilled.

$3 \Rightarrow 5$. Firstly, let us remark: If $D(L)$ contains the smallest element, it is 0 of L, because $\bigcap_{a \in L} (a]^* = \bigwedge_{a \in L} {}_P (a]^* = 0$. If $D(L)$ is a Boolean lattice then there exists to any $a \in L$ such a $b \in L$ that $(a]^* \vee (b]^* = L$, $(a]^* \wedge (b]^* = 0$. It follows from the first equation that $(a]^{**} \subseteqq (b]^*$, from the second one $(a]^{**} \supseteqq (b]^*$, therefore $(a]^{**} = (b]^*$. Hence $H(L) \subseteqq D(L)$. Analogically, we derive $(a]^* \subseteqq (b]^{**}$, $(a]^* \supseteqq (b]^{**}$, therefore $(a]^* = (b]^{**}$, i.e. $H(L) \supseteqq D(L)$. Hence $H(L) = D(L)$.

$5 \Rightarrow 2$. If $D(L) = H(L)$, then $D(L)$ is a Boolean lattice because for any $a \in L$ exists such a $b \in L$ that $(a]^* = (b]^{**}$, therefore $(a]^* \vee (b]^* = (a]^* \vee (a]^{**} = L$, $(a]^* \wedge (b]^* = (a]^* \wedge (a]^{**} = 0$. $D(L)$ is a distributive lattice with the greatest and smallest elements, therefore it is also Boolean. By 2.3, 2.5 and 2.8, $\mathfrak{U}[H(L)]$ is compact and (23) is true.

$2 \Rightarrow 4$. It follows from 2.5 and 2.8.

$4 \Rightarrow 1$. If $H(L)$ is a Boolean lattice, then it contains the greatest element, which is L, because the greatest element of $H(L)$ contains all principal polars in L, also the whole L. Further there is for any $a \in L$ such a $b \in L$ that $(a]^{**} \vee (b]^{**} = L$, $(a]^{**} \wedge (b]^{**} = 0$, hence $0 = L^* = (a]^* \wedge (b]^*$, $L = 0^* = (a]^* \vee (b]^*$. Then $D(L)$ is also a Boolean lattice, and $\mathfrak{U}[D(L)]$ is compact by 2.3 and 2.8.

2.11. Lemma. *For any $a \in L$ the following are equivalent:*

1. *$(a]^*$ is a principal ideal in L.*
2. *a^* exists in L.*

Proof: $1 \Rightarrow 2$. If $(a]^*$ is a principal ideal, then there exists $b \in L$ so that $(a]^* = (b]$, since b is the greatest element disjoint from a, i.e. $b = a^*$.

$2 \Rightarrow 1$. If an a^* exists, then $(a]^* = (a^*]$ is a principal ideal.

2.12. Theorem. *The following are equivalent:*

1. *L is a pseudo-complemented lattice.*

2. *Any dual principal polar in L is a principal ideal.*

If one of these conditions holds and if L is a complete lattice, then the conditions 1—5 *of the theorem* 2.10 *hold.*

Proof: $1 \Leftrightarrow 2$. It is seen from 2.11.

If one of the conditions holds, then $(a]^* = (a^*]$ for any $a \in L$, therefore the dual principal polars are determined by all pseudo-complements in L. From the assumption that L is complete, it follows that $D(L)$ is a Boolean lattice (1.4).

2.13. Remark. From the preceding results we can determine further topological properties of the defined topological spaces. First of all, the spaces $\mathfrak{U}[P(L)]$ and $\mathfrak{U}[D(L)]$ are completely regular by 2.7(c). From 2.7(a) and 2.9 it follows that the space $\mathfrak{U}[P(L)]$ is also normal. If any condition of those in the theorem 2.10 is fulfilled, then $\mathfrak{U}[D(L)]$ is also normal. If the lattice L does not contain any dense element, then the lattice $H(L)$ has not the greatest element, and $\mathfrak{U}[H(L)]$ consists of one ultraantifilter, which is the whole lattice $H(L)$. In this case $\mathfrak{U}[H(L)]$ is also a Hausdorffspace, completely regular, normal and compact. If L contains a dense element, then $\mathfrak{U}[H(L)]$ is a Hausdorff's and completely regular space by 2.7(a), (c). If any condition of those in the theorem 2.10 is fulfilled, then $\mathfrak{U}[H(L)]$ is also normal space.

3. Standard ultraantifilters in $P(L)$

The set $\mathfrak{U}_s[P(L)]$ of all standard ultraantifilters in the lattice $P(L)$ will be the matter of our considerations now. We shall introduce to it a topology by an analogical way, as there is in [3], and investigate the properties of the topological space.

3.1. Every standard ultraantifilter in $P(L)$ contains at least a dual principal polar (2.2). Therefore we define

$$\mathfrak{V}(a]^* = \{x : x \in \mathfrak{U}_s[P(L)],\ (a]^* \in x\}$$

for each $(a]^* \in D(L)$ and put

$$\Sigma' = \{\mathfrak{V}(a]^* : (a]^* \in D(L)\}.$$

We could simply show that it is possible, to take Σ' as a basis for open sets in $\mathfrak{U}_s[P(L)]$. The closure in the topological space $\mathfrak{U}_s[P(L)]$ is described as follows:

If $A \subseteqq \mathfrak{U}_s[P(L)]$ then

(24) $\quad \bar{A} = \{x : x \in \mathfrak{U}_s[P(L)],\ (a]^* \in x \Rightarrow \text{there exists } y \in A \text{ so that } (a]^* \in y\},$

that is

(24a) $$\bar{A} = \{x : x \in \mathfrak{U}_s[P(L)];\ x \cap D(L) \subseteqq \bigcup_{y \in A} y\}.$$

Firstly, we shall investigate, when the topological space $\mathfrak{U}_s[P(L)]$ is compact.

3.2. Lemma. *If L is an arbitrary distributive lattice, and if $a, b \in L$, $b \nleqq a$, then a prime antifilter x in L exists, so that $a \in x$, $b \in x$.*

Proof follows from [7], Satz 66.

3.3. Theorem. *The following are equivalent:*

1. *The lattice $D(L)$ contains the smallest element.*
2. *The lattice $H(L)$ contains the greatest element.*
3. *$H(L) \cap D(L) \neq \emptyset$.*
4. *The lattice L contains a dense element.*
5. *The sets $\mathfrak{U}[P(L)]$ and $\mathfrak{U}_s[P(L)]$ are equal to each other.*
6. *The topological space $\mathfrak{U}_s[P(L)]$ is compact.*

Proof: $1 \Rightarrow 2$. If $D(L)$ contains the smallest element, then it is 0 ($=$ one-element set constituted by the smallest element of lattice L). From this follows, that an element $a \in L$ exists such as $(a]^* = 0$. Hence $(a]^{**} = (0]^* = L \in H(L)$ is the greatest element in $H(L)$.

$2 \Rightarrow 3$. If $H(L)$ contains the greatest element, then it is the whole L. It is always $(0]^* = L \in D(L)$, therefore $H(L) \cap D(L) \neq \emptyset$.

$3 \Rightarrow 4$. If L does not contain any dense element, then for each $a \in L$ it is true that $(a]^* \neq 0$, i.e. $(a]^{**} \neq L$. Let x be an ultraantifilter in $P(L)$, which contains as subset, the set $H(L) \subseteqq P(L)$ [by (18) there is such an ultraantifilter], i.e. $H(L) \subseteqq x$. Then $x \cap D(L) = \emptyset$ [by (22)], therefore $H(L) \cap D(L) = \emptyset$ and 3 is not true.

$4 \Rightarrow 5$. If a non-standard ultraantifilter in $P(L)$ exists, and if L contains a dense element a, then an $A \in x$ exists, so that $a \in A$, hence $(a]^{**} = L = A$ and that is in contradiction to the definition of an antifilter (1.1). Therefore L does not contain any dense element and 4 is not true.

$5 \Rightarrow 6$. From the definition of bases Σ' (3.1) and Σ_P (2.7) it follows that $\Sigma' \subseteqq \Sigma_P$. The topological space $\mathfrak{U}[P(L)]$ is compact (2.9), therefore also the space $\mathfrak{U}_s[P(L)]$ is compact.

$6 \Rightarrow 1$. Let us consider the covering of space $\mathfrak{U}_s[P(L)]$ by the system of basic sets Σ'. By assumption, there are finitely many basic sets $\mathfrak{B}(a_1]^*, \mathfrak{B}(a_2]^*, \ldots, \mathfrak{B}(a_n]^*$, which cover the space $\mathfrak{U}_s[P(L)]$. In other words: there are finitely many dual principal polars $(a_1]^*, (a_2]^*, \ldots, (a_n]^*$

for which the following hold: If $x \in \mathfrak{U}_s[P(L)]$ is an arbitrary element, then there exists such an i ($1 \leqq i \leqq n$) that $(a_i]^* \in x$. Let us put

$$(a]^* = \bigwedge_{i=1}^{n} {}_P (a_i]^*.$$

Obviously $(a]^* \in x$ for any $x \in \mathfrak{U}_s[P(L)]$, i.e. $\mathfrak{B}(a]^* = \mathfrak{U}_s[P(L)]$. We shall demonstrate that $(a]^*$ is the smallest element in $D(L)$. Conversely, let $(b]^* \in D(L)$ exist that $(a]^* \nsubseteq (b]^*$. According to 3.2, there exists a prime antifilter y in $P(L)$ so that $(b]^* \in y$, $(a]^* \bar{\in} y$. By 1.1(c) y is an ultraantifilter in $P(L)$ and obviously $y \in \mathfrak{U}_s[P(L)]$ (2.1). Because $(a]^* \bar{\in} y$, we get into a contradiction to the statement that $(a]^*$ is contained in any standard ultraantifilter in $P(L)$. Thus $(a]^*$ is the smallest element in $D(L)$.

3.4. Remark. The conditions of the theorem 3.3 are fulfilled if L has the greatest element, because the greatest element 1 in L is dense in L. This case is for example when L is pseudo-complemented.

3.5. Theorem. *The following are equivalent:*

1. $H(L) = D(L)$.

2. *For any set* $\mathfrak{B}(a]^* \in \Sigma'$ *it holds* $\mathfrak{U}_s[P(L)] \setminus \mathfrak{B}(a]^* \in \Sigma'$. *If one of these two conditions is fulfilled, then the space* $\mathfrak{U}_s[P(L)]$ *is*

(a) *compact,*

(b) *completely regular.*

Proof: $1 \Rightarrow 2$. Let $\mathfrak{B}(a]^* \in \Sigma'$ be arbitrary. According to our assumption, there exists such a $b \in L$, that $(a]^{**} = (b]^*$. By 1.1(a) each ultraantifilter in $P(L)$ contains just one of the two complemented polars $(a]^*$, $(a]^{**}$, and thus for $x \in \mathfrak{U}_s[P(L)]$ $x \bar{\in} \mathfrak{B}(a]^* \Leftrightarrow (a]^* \bar{\in} x \Leftrightarrow (a]^{**} \in x \Leftrightarrow$ $\Leftrightarrow (b]^* \in x \Leftrightarrow x \in \mathfrak{B}(b]^*$.

Consequently,

$$\mathfrak{B}(b]^* = \mathfrak{U}_s[P(L)] \setminus \mathfrak{B}(a]^*.$$

$2 \Rightarrow 1$. It holds $(0]^* = L$, hence $\mathfrak{U}_s[P(L)] \setminus \mathfrak{B}(0]^* = \mathfrak{U}_s[P(L)] \in \Sigma'$. Thus, there exists a dual principal polar $(a]^*$ which belongs to any standard ultraantifilter in $P(L)$. With regard to the proof $6 \Rightarrow 1$ in 3.3, $(a]^*$ is the smallest element in $D(L)$. Therefore every ultraantifilter in $P(L)$ is standard, i.e. $\mathfrak{U}_s[P(L)] = \mathfrak{U}[P(L)]$.

Let $c \in L$ be arbitrary. In accordance with our assumption, there exists to $(c]^* \in D(L)$ a dual principal polar $(d]^* \in D(L)$ so that

$$\mathfrak{B}(d]^* = \mathfrak{U}_s[P(L)] \setminus \mathfrak{B}(c]^*.$$

Consequently,

$$(c]^{**} \in x \Leftrightarrow (d]^* \in x \tag{25}$$

for any $x \in \mathfrak{U}_s[P(L)] = \mathfrak{U}[P(L)]$. We shall prove indirectly that $(c]^{**} = (d]^*$. Let us assume that $(c]^{**} \neq (d]^*$. Because $(a]^{**}, (d]^* \in P(L)$ are different elements, there exists a prime antifilter y in $P(L)$ (= ultra-antifilter), which contains just one of two elements $(c]^{**}$, $(d]^*$ (3.2); for example $(c]^{**} \in y$, $(d]^* \bar{\in} y$. (In the reverse case we should proceed similarly.) For y, (25) is not true, which is a contradiction, thus $(c]^{**} = (d]^*$ and $(c]^* = (d]^{**}$. Consequently, $H(L) = D(L)$.

Proof of the concluding statement (a) follows from the proof $2 \Rightarrow 1$ and the statement (b) is evident.

We must carry out some additional considerations, to find certain necessary and sufficient conditions for the complete regularity of the topological space $\mathfrak{U}_s[P(L)]$ and eventually to investigate some further properties of this space.

3.6. Lemma. *For an ideal J in L, the following are equivalent:*

1. *J contains from any two complemented polars at least one.*
2. *$(a]^* \subseteqq J$ or $(a]^{**} \subseteqq J$ for any $a \in L$.*
3. *$a \in J$ or $(a]^* \subseteqq J$ for any $a \in L$.*

Proof: $1 \Rightarrow 2 \Rightarrow 3$. It is evident.

$3 \Rightarrow 1$. Let us assume that $A \nsubseteqq J$, $A^* \nsubseteqq J$ for some $A \in P(L)$. Then $A^* \subseteqq (a]^*$ for any $a \in A \setminus J$ therefore $a \bar{\in} J$, $(a]^* \subseteqq J$ and 3 is not fulfilled.

3. 7. Lemma. *For any ideal $J \neq L$, the following are equivalent:*

1. *J is a prime ideal in L.*
2. *$L \setminus J$ is a filter in L.*

If one of these conditions is fulfilled, then the statements 1—3 *in* 3.6 *hold.*

Proof: $1 \Rightarrow 2$. Let $a, b \in L \setminus J$ be arbitrary. If $a \cap b \in J$, then $a \in J$ or $b \in J$ and so $a \bar{\in} L \setminus J$ or $b \bar{\in} L \setminus J$ which is a contradiction. Hence $a \cap b \in L \setminus J$. If $a \in L \setminus J$, $b \geqq a$, then $a \bar{\in} J$, thus $b \bar{\in} J$, therefore $b \in L \setminus J$. It is evident that $0 \bar{\in} L \setminus J$ and $L \setminus J \neq \emptyset$. Together $L \setminus J$ is a filter in L.

$2 \Rightarrow 1$. If $a, b \bar{\in} J$, then $a, b \in L \setminus J$, thus $a \cap b \in L \setminus J$, i.e. $a \cap b \bar{\in} J$. J is also a prime ideal in L.

Let J be a prime ideal, $a \bar{\in} J$. As J is non-empty, $0 \in J$, therefore $a \cap b = 0 \in J \Rightarrow b \in J \Rightarrow (a]^* \subseteqq J$ and the condition 3 in 3.6 is fulfilled. The concluding statement is also proven.

3.8. Lemma. *If card $L > 1$ and if x is an ultraantifilter in $P(L)$ or in $D(L)$, then $\bigcup x$ is a prime ideal in L.*

Proof: $a, b \in \bigcup x \Rightarrow$ there exist $A, B \in x$, so that $a \in A$, $b \in B \Rightarrow a \cup b \in A \vee_P B \in x \Rightarrow a \cup b \in \bigcup x$. If $a \in \bigcup x$, $c \leqq a$, then $A \in x$ exists, so that $a \in A$, thus $c \in A$, hence $c \in \bigcup x$. $\bigcup x$ is also an ideal. If

$a, b \bar{\in} \cup x$, then $(a]^* \in x$, $(b]^* \in x$ by (20) $\Rightarrow (a]^* \vee_P (b]^* = (a \cap b]^* \in x \Rightarrow$ $\Rightarrow a \cap b \bar{\in} \cup x$. $\cup x$ is also a prime ideal.

3.9. The notion of a prime ideal in L was defined in 1.1. The set of all prime ideals in L is ordered by set-theoretical inclusion. A minimal element of that set is called *a minimal prime ideal* in L. According to [5], Lemma 2, *every prime ideal in L contains a minimal prime ideal.*

3.10. Theorem. *If card $L > 1$, then the set of all minimal prime ideals in L is $\{\cup x \colon x \in \mathfrak{U}[D(L)]\}$.*

Proof: I. First of all, we shall prove that any two elements of the set $\{\cup x \colon x \in \mathfrak{U}[D(L)]\}$ are incomparable by set-theoretical inclusion. Let x, $y \in \mathfrak{U}[D(L)]$, $x \neq y$, be arbitrary. If $\cup x \subseteqq \cup y$, then $(a]^* \bar{\in} x \Rightarrow$ $\Rightarrow (a]^* \bar{\in} y$ [by (20)], thus $y \subseteqq x$. It follows from maximality of y that $y = x$. Similarly, we can prove $\cup y \subseteqq \cup x \Rightarrow x = y$. We also get in both cases a contradiction to our assumption $y \neq x$. Therefore $\cup x$, $\cup y$ are incomparable by set-theoretical inclusion.

II. Let J be a minimal prime ideal in L, $J \neq L$. According to 3.7, $y = \{(a]^* \colon a \in L \setminus J\}$ is an antifilter in $D(L)$. Let $x \in \mathfrak{U}[D(L)]$ be such an ultraantifilter in $D(L)$ that $y \subseteqq x$. We shall derive $\cup x = J$. $a \in \cup x \Rightarrow$ $\Rightarrow (a]^* \bar{\in} x$ [by (20)] $\Rightarrow (a]^* \bar{\in} y \Rightarrow a \bar{\in} L \setminus J \Rightarrow a \in J$, therefore $\cup x \subseteqq J$. By 3.8, $\cup x$ is a prime ideal, consequently $\cup x = J$ (from minimality of J). Let us now assume that the only minimal prime ideal in L is the whole L. With regard to our assumption card $L > 1$, the set $\{\cup x \colon x \in \mathfrak{U}[D(L)]\}$ is non-empty and by 2.1 it follows $\cup x \neq L$ for any $x \in \mathfrak{U}[D(L)]$. According to 3.8, $\cup x$ is a prime ideal, and obviously $\cup x \subset L$ which is a contradiction to the minimality of the prime ideal L. Thus, L is not minimal.

III. It remains to prove that $\cup x$ is a minimal prime ideal for each $x \in \mathfrak{U}[D(L)]$. According to 3.8, $\cup x$ is a prime ideal. Let J be a minimal prime ideal, so that $J \subseteqq \cup x$. By II. there exists $y \in \mathfrak{U}[D(L)]$, so that $J = \cup y \Rightarrow \cup y \subseteqq \cup x \Rightarrow \cup x = \cup y$ (by I.) and $\cup x$ is a minimal prime ideal.

From the part II. of the last proof it follows.

3.11. Corollary. *If card $L > 1$, then there exists a minimal prime ideal $\neq L$ in L.*

3.12. Lemma. *For ultraantifilters x, $y \in \mathfrak{U}[P(L)]$, the following are equivalent*:

1. $\cup x \subseteqq \cup y$.
2. $x \cap D(L) \supseteqq y \cap D(L)$.

If one of those inclusions is sharp, then the other is sharp, too.

Proof: $1 \Rightarrow 2$. If $\cup x \subseteqq \cup y$ then $(a]^* \bar{\in} x \Rightarrow a \in \cup x \Rightarrow a \in \cup y \Rightarrow$ $\Rightarrow (a]^* \bar{\in} y$ by (20), thus $x \cap D(L) \supseteqq y \cap D(L)$.

$2 \Rightarrow 1$. If $\bigcup x \nsubseteq \bigcup y$, then there exists $a \in \bigcup x$, $a \bar{\in} \bigcup y$. Consequently $(a]^* \bar{\in} x$, $(a]^* \in y$, therefore $y \cap D(L) \nsubseteq x \cap D(L)$ and the condition 2 is not fulfilled.

Proof of the concluding statement:

$\bigcup x \subset \bigcup y \Leftrightarrow$ there exists $a \in \bigcup y$, $a \bar{\in} \bigcup x \Leftrightarrow$ there exists $(a]^* \bar{\in} y$, $(a]^* \in x \Leftrightarrow x \cap D(L) \supset y \cap D(L)$.

3.13. Lemma. *If $x \in \mathfrak{U}[D(L)]$ is arbitrary, then there exists $y \in \mathfrak{U}_s[P(L)]$, so that $\bigcup x = \bigcup y$ and $x = y \cap D(L)$.*

Proof: Let $x \in \mathfrak{U}[D(L)]$ be arbitrary. Let z be an antifilter in $P(L)$ which contains the set x. We shall denote by y an arbitrary ultraantifilter in $P(L)$ which includes z. Obviously, $y \in \mathfrak{U}_s[P(L)]$ and $\bigcup x \subseteq \bigcup y$. If $\bigcup x \neq \bigcup y$, then there exists $a \in \bigcup y$, $a \bar{\in} \bigcup x$. According to (20), $(a]^* \bar{\in} y$, $(a]^* \in x$ which is a contradiction to the definition of ultraantifilter y. Thus $\bigcup x = \bigcup y$. Consequently and according to (20) $x = y \cap \cap D(L)$.

3.14. Corollary. *If card $L > 1$, then all minimal prime ideals of L are included in $\{\bigcup x : x \in \mathfrak{U}_s[P(L)]\}$.*

Proof follows from 3.10 and 3.13.

3.15. Lemma. *For $x \in \mathfrak{U}_s[P(L)]$ the following are equivalent:*

1. $\bigcup y \not\subset \bigcup x$ *holds for each $y \in \mathfrak{U}_s[P(L)]$, $y \neq x$.*
2. $\bigcup x$ *is a minimal prime ideal in L.*
3. $x \cap D(L) \bar{\in} \mathfrak{U}[D(L)]$.
4. *If $(a]^* \bar{\in} x$, then there exists $(b]^* \in x$, so that*

$$\mathfrak{B}(a]^* \cap \mathfrak{B}(b]^* = \emptyset. \tag{26}$$

Proof: $1 \Rightarrow 2$. If $\bigcup y \not\subset \bigcup x$ for each $y \in \mathfrak{U}_s[P(L)]$, then the prime ideal $\bigcup x$ does not sharply contain any minimal prime ideal (according to 3.14), therefore $\bigcup x$ is minimal.

$2 \Rightarrow 3$. From the assumption follows that there exists such an ultra-antifilter z in $D(L)$ that $\bigcup z = \bigcup x$. According to (20), the following holds: $(a]^* \in x \cap D(L) \Leftrightarrow a \bar{\in} \bigcup x \Leftrightarrow a \bar{\in} \bigcup z \Leftrightarrow (a]^* \in z$. Consequently $x \cap D(L) = z \in \mathfrak{U}[D(L)]$.

$3 \Rightarrow 4$. Let $(a]^* \bar{\in} x$ be arbitrary. From the assumption $x \cap D(L) \in \mathfrak{U}[D(L)]$ follows that there exists $(b]^* \in x$, so that $(a]^* \vee (b]^* = L$. Therefore, there exists no ultraantifilter in $P(L)$ which would include both $(a]^*$ and $(b]^*$, thus (26) holds.

$4 \Rightarrow 1$. Let the condition 1 be not true, i.e. there exist ultraantifilters $y \in \mathfrak{U}_s[P(L)]$, so that $\bigcup y \subset \bigcup x$. According to 3.14 among those y's there exists an ultraantifilter, we denote it by z, so that $\bigcup z$ is a minimal prime ideal. By 3.12 it is $z \cap D(L) \supset x \cap D(L)$, i.e. for any $(c]^* \in x$

also $(c]^* \in z$. Moreover there exists $(a]^* \in D(L)$, so that $(a]^* \in z$, $(a]^* \bar{\in} x$. Consequently $z \in \mathfrak{V}(a]^* \cap \mathfrak{V}(b]^*$ for each $(b]^* \in x$, therefore (26) is not true.

3.16. Let $\mathscr{P}$ be a topological space. A non-empty subset $T \subseteqq \mathscr{P}$ is said to be trivial closed set when the following holds for each closed subset $V \subseteqq \mathscr{P}$:

$$\emptyset \neq V \subseteqq T \Rightarrow V = T.$$

3.17. Theorem. *The following are equivalent*:

1. *For any* $x, y \in \mathfrak{U}_s[P(L)]$ *either* $\cup x = \cup y$ *or* $\cup x$, $\cup y$ *are incomparable by set-theoretical inclusion.*
2. *For each* $x \in \mathfrak{U}_s[P(L)]$, $\cup x$ *is a minimal prime ideal in* L.
3. *For each* $x \in \mathfrak{U}_s[P(L)]$, $x \cap D(L)$ *is an ultraantifilter in* $D(L)$.
4. *Each set* $\mathfrak{V}(a]^* \in \Sigma'$ *is both open and closed.*
5. *The topological space* $\mathfrak{U}_s[P(L)]$ *is completely regular.*
6. *In topological space* $\mathfrak{U}_s[P(L)]$ *there exists a decomposition on trivial closed sets.*

Proof: $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$ from 3.15. $4 \Rightarrow 5$. Evident.

$5 \Rightarrow 3$. If the condition 3 is not valid, then there exists at least one ultraantifilter $x \in \mathfrak{U}_s[P(L)]$, so that $x \cap D(L)$ is not an ultraantifilter in $D(L)$, hence there exists $(a]^* \in D(L)$, $(a]^* \bar{\in} x$, so that $(a]^* \vee (b]^* \neq L$ for any $(b]^* \in x$. From 3.15 it follows that $\mathfrak{V}(a]^*$ is not closed. Let y be an arbitrary ultraantifilter in $P(L)$ which includes the set $\{(a]^*, x \cap D(L)\}$. We put

$$A = \mathfrak{U}_s[P(L)] \setminus \mathfrak{V}(a]^*.$$

A is a closed set and $y \bar{\in} A$. Let φ be an arbitrary real function, defined on $\mathfrak{U}_s[P(L)]$, so that $0 \leqq \varphi(t) \leqq 1$ for $t \in \mathfrak{U}_s[P(L)]$, $\varphi(y) = 0$, $\varphi^1(A) = 1$. As $x \in A$, therefore $\varphi(x) = 1$. We shall prove that φ is not continuous in the point x. Let us choose $\varepsilon = 1/2$. To any $1/2$—neighbourhood of the point 1 (denoting $O_{1/2}(1)$) there exists no neighbourhood $\mathfrak{V}(b]^*$ of the point x, so that

$$\varphi^1(\mathfrak{V}(b]^*) \subseteqq O_{1/2}(1)$$

because from the definition of ultraantifilter y it follows that $y \in \mathfrak{V}(b]^*$ for any $(b]^* \in x$. Consequently, $\varphi(y) = 0 \in \varphi^1(\mathfrak{V}(b]^*) \nsubseteqq O_{1/2}(1)$. Thus the function φ is not continuos in the point x and $\mathfrak{U}_s[P(L)]$ is not completely regular. Therefore the condition 5 is not valid.

$1 \Rightarrow 6$. We defined a decomposition on $\mathfrak{U}_s[P(L)]$ in the following way: $x, y \in \mathfrak{A}_S[P(L)]$ belong to the same class of decomposition if $\bar{x} = \bar{y}$. We shall prove that this decomposition is the one on trivial closed sets. Let x be arbitrary. Let $\emptyset \neq V \subseteqq \mathfrak{A}_s[P(L)]$ be a closed set, so that

$V \subseteqq \bar{x}$. If $z \in V$ is an arbitrary element, then $z \in \bar{x} \Rightarrow z \cap D(L) \subseteqq$ $\subseteqq x \cap D(L)$ (by (24a)) $\Rightarrow \cup x \subseteqq \cup z$ (by 3.12). According to the assumption $\cup x = \cup z$, therefore $z \cap D(L) = x \cap D(L)$ (by 3.12) and with regard to (24a) $\bar{x} = \bar{z} \subseteqq V \subseteqq \bar{x}$, hence $V = \bar{x}$. Thus $\bar{x}$ is a trivial closed set.

$6 \Rightarrow 1$. Let $x, y \in \mathfrak{U}_s[P(L)]$ be arbitrary. If $\mathscr{R}$ is a decomposition on trivial closed sets in $\mathfrak{U}_s[P(L)]$, $T \in \mathscr{R}$, $x, y \in T$, then $\bar{x} = T = \bar{y}$ (because $x \in T \Rightarrow \bar{x} \subseteqq T \Rightarrow \bar{x} = T$). The following is true

(27) $\bar{x} = \bar{y} \Leftrightarrow x \cap D(L) = y \cap D(L)$ (by (24a)) $\Leftrightarrow \cup x = \cup y$ (by 3.12).

If T, $V \in \mathscr{R}$, $T \neq V$, $x \in T$, $y \in V$, then $\bar{x} \neq \bar{y} \Rightarrow \cup x \neq \cup y$ by (27) If $\cup x \subset \cup y$, then $x \cap D(L) \supset y \cap D(L)$, and thus $y \in \bar{x} \subseteqq T$ by (24a) which is a contradiction to the choice of y. Similarly, we shall get a contradiction, if we assume that $\cup x \supset \cup y$. Consequently $\cup x$, $\cup y$ are incomparable by set-theoretical inclusion and the condition 1 is true.

3.18. Lemma. *Let $x \in \mathfrak{U}_s[P(L)]$ be an arbitrary fixed element. Then the following are equivalent:*

1. *$x \in \bar{y}$ for each $y \in \mathfrak{U}_s[P(L)]$, $y \neq x$.*
2. *$\cup y \nsubseteqq \cup x$ for each $y \in \mathfrak{U}_s[P(L)]$, $y \neq x$.*
3. *$x \cap D(L)$ is an ultraantifilter in $D(L)$ and x is the only ultraantifilter in $P(L)$ which contains $x \cap D(L)$.*

Proof: $1 \Leftrightarrow 2$. Let $y \in \mathfrak{U}_s[P(L)]$ be arbitrary. It holds:
$x \bar{\in} \bar{y} \Leftrightarrow x \cap D(L) \nsubseteqq y \cap D(L)$ (by (24a)) $\Leftrightarrow \cup y \nsubseteqq \cup x$ (by 3.12). $2 \Rightarrow 3$. Let the condition 2 be true. From this assumption it follows (according to 3.15) that $x \cap D(L)$ is an ultraantifilter in $D(L)$. With regard to 3.12, $x \cap D(L) \nsubseteqq y \cap D(L)$ for each ultraantifilter $y \in \mathfrak{U}_s[P(L)]$, $y \neq x$. Hence x is the only ultraantifilter in $P(L)$ which includes $x \cap D(L)$.

$3 \Rightarrow 2$. According to 3.15, $\cup y \not\subset \cup x$ for any $y \in \mathfrak{U}_S[P(L)]$, $y \neq x$. If $\cup y = \cup x$ for some $y \in \mathfrak{U}_s[P(L)]$, $y \neq x$, then with regard to 3.12 $y \cap D(L) = x \cap D(L)$, hence y includes $x \cap D(L)$ what is in contradiction to the assumption that x is the only ultraantifilter in $P(L)$, which includes $x \cap D(L)$. Consequently, $\cup x \neq \cup y$ and the condition 2 is true.

3.19. Theorem. *If card $L > 1$, then the following are equivalent:*

1. *$\bar{x} = \{x\}$ for each $x \in \mathfrak{U}_s[P(L)]$, i.e. $\mathfrak{U}_s[P(L)]$ is T_1-space.*
2. *$\cup x$, $\cup y$ are incomparable by set-theoretical inclusion for arbitrary $x, y \in \mathfrak{U}_s[P(L)]$, $y \neq x$.*
3. *$\{\cup x : x \in \mathfrak{U}_s[P(L)]\}$ is the set of all minimal prime ideals in L (i.e. $\cup x \neq \cup y$ for $x \neq y$).*
4. *$x \cap D(L) \in \mathfrak{U}[D(L)]$ for any $x \in \mathfrak{U}_S[P(L)]$ and x is the only ultraantifilter in $P(L)$ which contains $x \cap D(L)$.*
5. *Topological spaces $\mathfrak{U}_s[P(L)]$ and $\mathfrak{U}[D(L)]$ are homeomorph.*
6. *Topological space $\mathfrak{U}_s[P(L)]$ is Hausdorff.*

If one of these conditions holds, then the conditions 1—6 *of* **3.17** *hold, too.*

Proof: $1 \Leftrightarrow 2 \Leftrightarrow 4$ follows from 3.18.
$2 \Leftrightarrow 3$. The assumption card $L > 1$ guarantees that L is not a minimal prime ideal (3.11). Further, it is evident (3.8; 3.14).

$4 \Rightarrow 5$. Let us define a mapping φ from $\mathfrak{U}_s[P(L)]$ into $\mathfrak{U}[D(L)]$ as follows

$$\varphi(x) = x \cap D(L) \text{ for } x \in \mathfrak{U}_s[P(L)].$$

From the condition 4 it follows that φ is injective and from 3.12 that φ is surjective, i.e. φ is an isomorphism between the sets $\mathfrak{U}_s[P(L)]$ and $\mathfrak{U}[D(L)]$. We shall prove that φ is a homeomorphism of topological spaces $\mathfrak{U}_s[P(L)]$ and $\mathfrak{U}[D(L)]$. It is sufficient to prove that $\varphi^1(\mathfrak{V}(a]^*) = \mathfrak{U}(a]^*$ for any $(a]^* \in D(L)$. Then the assertion follows from the fact that the open basis Σ' in $\mathfrak{U}_s[P(L)]$ corresponds with the open basis Σ_D in $\mathfrak{U}[D(L)]$. $x \in \mathfrak{V}(a]^* \Leftrightarrow (a]^* \in x \Leftrightarrow (a]^* \in x \cap D(L) \Leftrightarrow x \cap D(L) \in \mathfrak{U}(a]^* \Leftrightarrow \varphi(x) \in \mathfrak{U}(a]^*$.

$5 \Rightarrow 6$. According to 2.13, $\mathfrak{U}[D(L)]$ is a Hausdorff's space, therefore $\mathfrak{U}_s[P(L)]$ is also a Hausdorff's space.

$6 \Rightarrow 1$ is evident.

The concluding assertion follows, for example, from the condition 4.

3.20. Corollary. *If one of the conditions* 1—6 *of* 3.19 *holds, and if the lattice L contains a dense element, then*

(a) $\mathfrak{U}[D(L)]$ *is compact (i.e. the conditions* 1—5 *of* 2.10 *are satisfied) and normal.*

(b) $\mathfrak{U}_s[P(L)]$ *is normal.*

Proof: According to the condition 5 of 3.19, the spaces $\mathfrak{U}_s[P(L)]$ and $\mathfrak{U}[D(L)]$ are homeomorph. If L contains a dense element, then $\mathfrak{U}_s[P(L)]$ is compact (3.3). Each topological space, which is Hausdorff's and compact, is also normal.

3.21. We say that *condition (p) is satisfied* for an ultraantifilter x in $P(L)$ if there exists $(a]^* \in D(L)$ for any $A \in x$, so that $(a]^* \in x$, $A \subseteqq (a]^*$.

If the condition (p) is satisfied for $x \in \mathfrak{U}[P(L)]$, then $x \in \mathfrak{U}_s[P(L)]$.

3.22. Theorem. *For any $x \in \mathfrak{U}_s[P(L)]$ the following are equivalent*:

1. *x fulfils the condition (p).*
2. *There exists no $A \in P(L)$, so that $A \cup A^* \subseteqq \cup x$.*
3. *$A \in x \Leftrightarrow A \subseteqq \cup x$.*
4. *For each $A \in P(L)$, $A \bar{\in} x$, there exists $(b]^* \in x$, so that $A \vee_P (b]^* = L$.*

If one of these conditions holds, then the conditions 1—3 *of* 3.18 *hold, too*.

Proof: $1 \Rightarrow 2$. Let us assume that for a polar $A \in P(L)$, $A \subseteqq \cup x$, $A^* \subseteqq \cup x$. According to 1.1(a), x contains exactly one of two complemented polars A, A^*. For example $A \in x$. We shall prove that $A \nsubseteqq (a]^*$

for any dual principal polar $(a]^* \in x$. On the contrary, let us assume that $A \subseteqq (a]^*$ holds for some $(a]^* \in x$. Then $a \in (a]^{**} \subseteqq A^* \subseteqq \bigcup x$. But according to (20), $(a]^* \in x \Rightarrow a \,\bar{\in}\, \bigcup x$ which is a contradiction. The condition 1 is not true.

$2 \Rightarrow 3$. It is evident $A \in x \Rightarrow A \subseteqq \bigcup x$. Let us assume that $A \,\bar{\in}\, x$. With regard to 1.1(a), $A^* \in x$. According to the assumption $A \cup A^* \nsubseteq \bigcup x$, and since $A^* \subseteqq \bigcup x$ it follows that $A \nsubseteq \bigcup x$. Therefore the condition 3 is satisfied.

$3 \Rightarrow 4$. Do not let the condition 4 hold. Then there exists $A \,\bar{\in}\, x$, so that $A \vee_P (a]^* \neq L$ for each $(a]^* \in x$. For the complemented polar A^* it holds $A^* \in x$, $A \vee_P A^* = L$ and for each $(b]^* \in D(L)$ with the property $A^* \subseteqq (b]^*$ it holds $(b]^* \,\bar{\in}\, x$. We shall prove that $A \subseteqq \bigcup x$. For any $c \in A$ it is $A^* \subseteqq (c]^*$, therefore $(c]^* \,\bar{\in}\, x$ and with regard to (20) $c \in \bigcup x$. Consequently, $A \subseteqq \bigcup x$ and moreover $A \,\bar{\in}\, x$, hence the condition 3 is not true.

$4 \Rightarrow 1$. If the condition 1 is not true, then there exists $A \in x$, so that $(a]^* \,\bar{\in}\, x$ for every $(a]^* \supseteqq A$. For the complemented polar A^* the following hold: $A^* \,\bar{\in}\, x$, $A \vee_P A^* = L$. If $A^* \vee_P (b]^* = L$ for some $(b]^* \in D(L)$, then $A \subseteqq (b]^*$ and according to the assumption, it is $(b]^* \,\bar{\in}\, x$. Thus 4 is not true.

Proof of the concluding assertion:

With regard to the condition 4, there exists to each $(a]^* \,\bar{\in}\, x$ such a $(b]^* \in x$ that $(a]^* \vee_P (b]^* = L$ which means that $x \cap D(L)$ is an ultraantifilter in $D(L)$. By (20) it is $\bigcup (x \cap D(L)) = \bigcup x$. If y is an arbitrary ultraantifilter in $P(L)$, which contains $x \cap D(L)$, then $\bigcup x = \bigcup y$. According to the condition 3

$$A \in y \Rightarrow A \subseteqq \bigcup y \Rightarrow A \subseteqq \bigcup x \Rightarrow A \in x.$$

Hence $y \subseteqq x$. From the maximality of y it follows $y = x$. Therefore, the condition 3 of 3.18 is true.

3.23. Corollary. *Let the condition (p) hold for each $y \in \mathfrak{U}_s[P(L)]$. Then the conditions 1—6 of the theorem 3.19 hold, too.*

3.24. Corollary. *If L contains a dense element and if the condition (p) holds for each $x \in \mathfrak{U}_s[P(L)]$, then $\mathfrak{U}_s[P(L)]$ and $\mathfrak{U}[D(L)]$ are normal.*

Proof is evident.

Further we shall determine a necessary and sufficient condition for the normality of the space $\mathfrak{U}_s[P(L)]$. First the evident assertion.

3.25. Lemma. *For arbitrary $(a]^*$, $(b]^* \in D(L)$ the following are equivalent:*

1. $\mathfrak{B}(a]^* \cap \mathfrak{B}(b]^* = \emptyset$.
2. $(a]^* \vee (b]^* = L$.

3.26. Theorem. *The following are equivalent*:

1. *$\mathfrak{U}_s[P(L)]$ is a normal topological space.*

2. *For any two closed subsets $A, B \subseteqq \mathfrak{U}_s[P(L)]$, $A \cap B = \emptyset$, there exist subsets $\mathscr{D}(A), \mathscr{D}(B) \subseteqq D(L)$ with the following properties*:

a)
$$\mathscr{D}(A) \subseteqq \bigcup_{x \in A} x, \qquad \mathscr{D}(B) \subseteqq \bigcup_{y \in B} y.$$

b) *For any $x \in A$, $y \in B$ it holds*
$$x \cap \mathscr{D}(A) \neq \emptyset, \qquad y \cap \mathscr{D}(B) \neq \emptyset.$$

c) *For any $(a]^* \in \mathscr{D}(A)$, $(b]^* \in \mathscr{D}(B)$ it holds $(a]^* \vee (b]^* = L$.*

Proof: $1 \Rightarrow 2$. If $\mathfrak{U}_s[P(L)]$ is normal, then there exist to any two closed sets $A, B \subseteqq \mathfrak{U}_s[P(L)]$, $A \cap B = \emptyset$, such open sets $A_1, B_1 \subseteqq \subseteqq \mathfrak{U}_s[P(L)]$ that $A \subseteqq A_1$, $B \subseteqq B_1$, $A_1 \cap B_1 = \emptyset$. Moreover, to each open set $C \subseteqq \mathfrak{U}_s[P(L)]$ there exists $\mathscr{D}(C) \subseteqq D(L)$, so that
$$C = \bigcup_{(c]^* \in \mathscr{D}(C)} \mathfrak{B}(c]^*.$$
Thus holds
$$A_1 = \bigcup_{(a]^* \in \mathscr{D}(A_1)} \mathfrak{B}(a]^*, \qquad B_1 = \bigcup_{(b]^* \in \mathscr{D}(B_1)} \mathfrak{B}(b]^*.$$
We denote
$$\mathscr{D}(A) = \mathscr{D}(A_1) \cap (\bigcup_{x \in A} x), \qquad \mathscr{D}(B) = \mathscr{D}(B_1) \cap (\bigcup_{y \in B} y).$$
The property a) is fulfilled for $\mathscr{D}(A)$, $\mathscr{D}(B)$. Let $x \in A$ be arbitrary. Then $x \in A_1$ and hence there exists $(a_1]^* \in \mathscr{D}(A_1)$, so that $x \in \mathfrak{B}(a_1]^*$, i.e. $(a_1]^* \in x$. It is evident that $(a_1]^* \in \mathscr{D}(A)$, then $(a_1]^* \in \mathscr{D}(A) \cap x$, that is $\mathscr{D}(A) \cap x \neq \emptyset$. Similarly, it will be shown that $y \cap \mathscr{D}(B) \neq \emptyset$ for any $y \in B$, consequently the property b) is fulfilled. Further, let $(a]^* \in \mathscr{D}(A)$, $(b]^* \in \mathscr{D}(B)$ be arbitrary. Then $\mathfrak{B}(a]^* \subseteqq A_1$, $\mathfrak{B}(b]^* \subseteqq B_1$, thus $\mathfrak{B}(a]^* \cap \cap \mathfrak{B}(b]^* = 0$. According to 3.25 holds $(a]^* \vee (b]^* = L$, consequently the property c) is fulfilled.

$2 \Rightarrow 1$. Let $A, B \subseteqq \mathfrak{U}_s[P(L)]$ be arbitrary closed sets, for which $A \cap B = \emptyset$. Let us denote
$$A_1 = \bigcup_{(a]^* \in \mathscr{D}(A)} \mathfrak{B}(a]^*, \qquad B_1 = \bigcup_{(b]^* \in \mathscr{D}(B)} \mathfrak{B}(b]^*.$$
A_1, B_1 are open and $A \subseteqq A_1$, $B \subseteqq B_1$ (it follows from the property b)). Moreover $A_1 \cap B_1 = \emptyset$ (from c)). Consequently $\mathfrak{U}_s[P(L)]$ is normal.

3.27. Remarks. Let us consider an l-group G. We have already said in the introduction, how the notion of a component in G is defined: $A \subseteqq G$ is a component in G if there exists $\emptyset \neq B \subseteqq G$, so that $A = B'$, where

$$B' = \{a \in G: \mid a \mid \wedge \mid b \mid = 0 \text{ for each } b \in B\}.$$

In the definition of a component only absolute values of element are used, i.e. elements $\geqq 0$. Consequently, any component in G is determined by a set of positive elements in G. In fact, if A is a component in G, then there exists $\emptyset \neq B \subseteqq G$, so that $A = B'$. Let us denote

$$\mid B \mid = \{\mid b \mid: b \in B\}.$$

Evidently, $\mid B \mid \subseteqq G^+$ and $A = \mid B \mid'$.

The set $\Gamma(G)$ of all components in G is ordered by set-theoretical inclusion.

Let us put $L = G^+$. L is a distributive lattice with the smallest element 0 ($=$ zero of G). It was shown in 1.8 that the set $P(L)$ of all polars in L ordered by set-theoretical inclusion, forms a complete Boolean algebra, in which infimum is determined by intersection.

Let us consider the relation between the sets $\Gamma(G)$ and $P(L)$.

From the definition of polar in L and of component in G it follows: If $K \in \Gamma(G)$, then $K^+ \in P(L)$. We put for $A \in P(L)$

$$K = \{a \in G: \mid a \mid \in A\}.$$

Evidently $K \in \Gamma(G)$ and $K^+ = A$. The correspondence

$$K \in \Gamma(G) \Leftrightarrow K^+ \in P(L)$$

is also a bijective mapping from the (non ordered) set $\Gamma(G)$ on the (non ordered) set $P(L)$. Further, it is easily seen that the mapping preserves ordering in both directions. Thus, it is an isomorphism of ordered sets $\Gamma(G)$ and $P(L)$. Consequently, the set $\Gamma(G)$ ordered by set-theoretical inclusion forms a complete Boolean algebra (see [13], Teorema 1).

Similarly, we can determine relations between $H(L)$ and $\Pi(G)$ and between $D(L)$ and $\Pi'(G)$. From 1.9 follows the theorem 1 in [8], but we must leave out in the formulae (1) and (2) the part that contains group operation.

From what we said, further connections follow between our results and those known for l-groups. For example, some results of the second part correspond to the one in [8]. In the considerations, the notion of a weak unit of l-group had to be replaced by the notion of a dense element of lattice.

Likewise, by study of standard ultraantifilters in $P(L)$, we get similar results as in [3], [4]. But it was necessary to replace the notion of a prime subgroup by a notion of a prime ideal. However, some assertions are specific only for lattices, and they can not be applied to l-groups.

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Department of Applied Mathematics
Technical University at Brno
Brno, Hilleho 6