NOTES ON GEOMETRY

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To Professor Otakar Borůvka at his Seventieth Birthday

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I TWO SHORT PROOFS OF THE FOUR VERTEX THEOREM

The first proof is a slight modification of Herglotz's method and leads to a stronger form of the Four Vertex Theorem due to Hayashi [1].

Lemma: *Let $x(t)$, $0 \leqq t \leqq \omega$, be a closed, convex curve and $f(t)$ a continuous function, periodic of period ω. If*

$$\int_0^\omega f(t)\,dt = \int_0^\omega f(t)\,x(t)\,dt = 0$$

then $f(t)$ changes sign at least four times in $(0, \omega)$ unless $f(t) \equiv 0$.

Let c be a constant vector. Then $\int_0^\omega f(t)\,(x(t) + c)\,dt = 0$. Hence, the conditions are invariant in a translation.

Assume $f(t) \neq 0$. By the first integral conditon, $f(t)$ changes sign at least one in $[0, \omega)$. By the periodicity, it changes sign at least twice. Assume that $f(t)$ changes sign only at t_0 and t_1, $0 \leqq t_0 < t_1 < \omega$. Without loss of generality, assume that $f(t) > 0$ for $t < t < t_1$. The second integral condition becomes

$$\int_{t_0}^{t_1} f(t)\,x(t)\,dt + \int_{t_1}^{t_0+\omega} |f(t)|\,(-x(t))\,dt = 0.$$

Here we may assume that the origin is on the line $x(t\ )\ x(t_1)$. But both integrals are vectors that point into that halfplane of the line $x(t_0)\ x(t_1)$ which contains the arc $x(t)$, $t \in (t\ ,\ t_1)$. Hence the sum cannot be zero. The contradiction proves the lemma.

Four Vertex Theorem: *The radius of curvature of a C^2 closed, convex curve of perimeter L that is not a circle has at least two maxima $> L/2\pi$ and two minima $< L/2\pi$.*

Let Θ be the tangent angle, i.e., the arclength of the tangent image of the curve. The average of the curvature radius is $L/2\pi = (1/2\pi) \int_0^{2\pi} \varrho\,d\Theta$. Put $x(\Theta) = (\cos\Theta, \sin\Theta)$. A function $\varrho(\Theta)$ is the radius of curvature of a closed, convex curve iff $\varrho(\Theta) \neq 0$ and $\int_0^{2\pi} \varrho(\Theta)x(\Theta)\, d\Theta = 0$. The function $f(\Theta) = \varrho(\Theta) - L/2\pi$ satisfies the conditions of the lemma.

The proof shortens considerably the proofs of theorems 1 and 2 in [2].

E. Rembs [3] has shown that the variation δk of the curvature in an infinitesimal deformation of a plane curve $x(s)$ satisfies $\oint \delta k\, ds = 0 = \oint \delta k x\, ds$ and that, therefore, the variation of the curvature changes sign at least four times. He has commented on the similarity of his proof to Herglotz's proof of the Four Vertex Theorem. Our proof shows that Rembs's theorem can be considered the infinitesimal version of the four vertex theorem if the convex curve is looked at as a deformation of the circle of radius $L/2\pi$.

The second proof combines a theorem of Pestov and Ionin [4] and some remarks of Melzi [5]. The Four-Vertex Theorem for Jordan curves with (not necessarily continuous) curvature follows as a corollary from our theorem.

We shall denote by C^{r-} the class of functions that have a derivative of order r everywhere in their domain of definition. G. Peano [6] has shown that these functions admit a Taylor expansion of order r with a remainder which is $0\ (h^{r+1})$.

The theorem of A. Kneser (7, cf. 8, Sec. 3.3] says that in the interior of an arc of monotone curvature the circle of curvature crosses the arc. Therefore, we define a circle of curvature as *minimal* if it touches the curve from within. The minimality property is purely local; a minimal circle of curvature of a Jordan curve may well contain points of the curve in its interior. We have a global statement:

Theorem: *The compact domain of a closed C^{2-} Jordan curve which is not a circle contains at least two minimal circles of curvature of the curve. The two circles are distinct as pointsets.*

Let $x(s)$ be the curve referred to its arclength. The segment of endpoints A, B is denoted by (A, B). By $C(s)$ we denote (a) either the circle of curvature at $x(s)$ if that circle is contained in the compact domain of the curve, or else (b) the maximal circle tangent to the curve at $x(s)$ and contained in the compact domain of the curve.

We may assume that not all $C(s)$ are circles of curvature. If $C(s)$ is not a circle of curvature, we denote by $\hat{x}(s) = x(\hat{s})$ one of the points of contact of $C(s)$ and the curve that is nearest to $x(s)$ in the measure given by the arclength. The center of $C(s)$ is $o(s)$.

Lemma [4]: *If $x(s_1) \notin C(s)$, then* $(x(s), o(s)) \cap (x(s_1), o(s_1)) = \emptyset$.

Assume the lemma to be false and denote the point of intersection by X. We choose the names so that $|X - x(s)| \geqq |X - x(s_1)|$. Then $|o(s) - x(s_1)| < |o(s) - X| + |X - x(s_1)| \leqq |o(s) - X| + |X - x(s)| = |o(s) - x(s)|$.

Hence, $x(s_1)$ is in the closed disc of $C(s)$. This is clearly a contradiction.

Naturally, the lemma remains true for $x(s)$ replaced by $x(\hat{s})$.

Choose s so that $C(s\ )$ is not a circle of curvature. Put $s_1 = \frac{1}{2}(s_0 + \hat{s}_0)$. By the lemma, $\hat{s}_1 \in (s_0, \hat{s}_0)$. Hence, $|\hat{s}_1 - s_1| < \frac{1}{2}|s_0 - \hat{s}_0|$. For $s_i = \frac{1}{2}(s_{i-1} + \hat{s}_{i-1})$ we get by induction

$$|\hat{s}_i - s_i| < (1/2)^i |\hat{s}_0 - s_0|.$$

The sequences s_i and $\hat{s}_i$ converge to a number s^*. By the Blaschke convergence theorem, at least a subsequence of circles $C(s_i)$ converges to a circle C^* tangent to $x(s)$ at $s = s^*$ and contained in the interior of the curve. Let us delete the other circles and rename the arguments so that $s_i \uparrow s^*$, $\hat{s}_i \downarrow s^*$. Put $\Delta s_i = \hat{s}_i - s_i$, $\Delta\Theta_i = \Theta(\hat{s}_i) - \Theta(s_i)$ where $\Theta(s)$ is the angle between the $+x$-axis and the vector $x'(s)$. By Peano's theorem

$$x'(s + h) = x'(s) + hk(s)\, n(s) + 0(h^2),$$

where $k(s)$ is the curvature and $n(s)$ the normal vector of $x(s)$. Under our hypotheses, $n(s)$ is differentiable and $k(s)$ exists everywhere. An easy computation shows that $\Delta\Theta_i = k(s_i)\,\Delta s_i + 0(s_i^2)$. By hypothesis,

$$k(s^*) = \lim \frac{\Theta(s^* + (s_i - s^*) - \Theta(s^* - (s^* - s_i))}{s_i - s_i} = \lim \frac{\Delta\Theta_i}{\Delta s_i}$$

does exist. But this means that $k(s) = \lim k(s_i)$. Since the radius of $C(s_i)$ is $R_i = (1/k(s_i)) + 0(\Delta s_i^2)$ it follows that the radius of $C^* = C(s^*)$ is the radius of curvature of $x(s)$ at $s = s^*$. In fact, $k(s_i) \to 0$ is excluded by the boundedness of the interior of a Jordan curve.

We have now obtained one of the circles whose existence is asserted in the theorem. The second circle is obtained by choosing s_1 as the midpoint of the complementary arc. Let s^{**} be the limit point obtained in this way. If $C(s^*) = C(s^{**})$, repeat the process with $s_1 = \frac{1}{2}(s^* + s^{**})$. In this way, either we end up with a circle of curvature distinct from $C(s^*)$ and contained in the compact domain or we show that the points of $C(s^*)$ are dense on $x(s)$. The second alternative is in contradiction to our assumption that not all $C(s)$ are circles of curvature, i.e., that $x(s)$ is not a circle.

The Four-Vertex Theorem for C^2 Jordan curves follows immediately. For C^{2-} curves, one obtains without difficulty the existence of a minimal circle for any arc bounded by two inflection points and without an inflection point in its interior.

II. AN APPLICATION OF GEOMETRY TO DIFFERENTIAL EQUATIONS

I. We consider differential equations of even order

$$x^{(2k)} + \sum_{i=1}^{k-1} c_i x^{(2i)} + \lambda p(t)\, x = 0, \tag{1}$$

where $p(t)$ is continuous, of bounded variation, non vanishing and periodic of period π, the c_i are constants and λ is a real parameter. In particular, we are interested in equations for which there exists a value $\lambda = \lambda_1 \neq 0$ for which (1) has $2k$ linearly independent solutions of the Liapounoff boundary value problem

$$x(\pi) = -x(0). \tag{2}$$

For $k = 1$ and positive p, (1) is a Hill equation and there are two linearly independent solutions of the problem (2) if and only if a zone of instability of (1) of odd index collapses to a point. Hill equations are easily treated by geometric methods, [9, 10]. If the first zone of instability of a Hill equation collapses, I showed recently (using methods developed for Minkowski geometry) that $p(t)$ must have at least four extrema in one period. In this Note, the result is generalized to $k \geqq 1$.

In a recent letter, O. Borůvka has indicated another proof of the result for $k = 1$. In fact, comparing $x'' + \lambda_1 p(t)\, x = 0$ to $x'' + \lambda_1 x = 0$, he deduces from the theorem on p. 136 of [9] that $p(t)$ takes on the value λ^{-1} at least four times in $(0, \pi)$. This precisation escapes our present method. We may note that all Hill equations with collapsing first zone of instability may be constructed by the formulas of [11].

2. We choose linearly independent solutions $x_1(t), \ldots, x_{2k}(t)$ of (1) for $\lambda = \lambda_1$. If the boundary value problem (2) has $2k$ linearly independent solutions for $\lambda = \lambda_1$, then all solutions of (1) for λ_1 solve (2) and the curve

$$x(t) = (x_1(t), \ldots, x_{2k}(t)) \quad 0 \leqq t \leqq 2\pi \tag{3}$$

in $2k$-space is symmetric with respect to the origin:

$$x(t + \pi) = -x(t). \tag{4}$$

In that case, λ_1 is called a *totally degenerate* eigenvalue.

In the case of a Hill equation, the eigenvalue λ_1 corresponds to the j-th zone of instability if a line intersects the curve in at most $2j$ points. For $j = 1$, the curve is convex. In dimension $2k$, a hyperplane through 0 and $2k - 1$ non-antipodal points on the curve intersects the curve in $2(2k - 1)$ points. Hence, the number $2(2k - 1)$ is the minimum possible for a condition about the maximum number of points of intersection of a hyperplane and the curve. For $k = 1$, the condition of our theorem implies $j = 1$.

Theorem: *If the equation* (1) *has a totally degenerate eigenvalue and no hyperplane through the origin in $2k$-space intersects the curve* (3) *in more than $2(2k-1)$ points, then the function $p(t)$ has at least $4k$ relative extrema in $0 \leqq t < \pi$.*

The intersection condition is affine and, hence, independent of the choice of the x_i.

3. The determinant of $2k$ vectors in $2k$-space is indicated by brackets. Since $[x', x'', \ldots, x^{(2k)}] = \lambda_1 p(t) \neq 0$, the curve $x(t)$ has no inflexion point. We choose $2k-1$ points $y_1, \ldots, y_{2k-1}$ on the arc $x(t)$, $0 \leqq t < \pi$. The hyperplane through 0 and 0 and $y_1, \ldots, y_{2k-1}$ is crossed by the curve at each of the points y_i since otherwise there would be a nearby hyperplane which meets the arc in $2k$ and the whole curve in $4k$ points. By (4),

$$\int_0^{2\pi} x(t)\, \mathrm{d}p(t) = -\int_0^{\pi} p(t) \{x'(t) + x''(t+\pi)\}\, \mathrm{d}t = 0 \tag{5}$$

and, by (1),

$$\begin{aligned} \lambda_1 \int_0^{2\pi} x_i x_j \,\mathrm{d}t &= -\lambda_1 \int_0^{2\pi} p x_i x'_j \,\mathrm{d}t - \lambda_1 \int_0^{2\pi} p x_j x_i \,\mathrm{d}t = \\ &= \int_0^{2\pi} \{x_i^{(2k)} x'_j + x_j^{(2k)} x'_i\} \,\mathrm{d}t + \\ &+ \sum_{s=1}^{k-1} c_s \int_0^{2\pi} \{x_i^{(2s)} x'_j + x_j^{(2s)} x'_i\} \,\mathrm{d}t = 0 \end{aligned} \tag{6}$$

for all $i, j = 1, \ldots, 2k$.

We prove the theorem by showing that the number of extrema cannot be less than $4k$. The number of extrema is even.

Assume that $p(t)$ has exactly $2(2k-1)$ relative extrema in $[0, \pi)$. Assume that the maxima occur at $T_1 < T_2 < \ldots < T_{2k-1}$ and the minima at $t_1 < t_2 < \ldots < t_{2k-1}$. For definiteness, we assume that $T_i < t_i < T_{i+1}$. Let $L_1(x) = 0$ be the equation of the hyperplane through 0 and $x(T_1), \ldots, x(T_{2k-1})$ where the sign is chosen so that $L_1(x(t)) > 0$ for $T_1 < t < T_2$. It follows from the hypothesis of the theorem that 0 and $2k-1$ distinct points on the arc $x(t)$, $0 \leqq t < \pi$, always span a hyperplane. Let $L_2(x) = 0$ be the equation of the hyperplane through 0 and $x(t_i)$, $i = 1, \ldots, 2k-1$, with $L_2(x(t)) > 0$ for $t \in (t_1, t_2)$. Then, for $\mathrm{d}p(t) \neq 0$,

$$\operatorname{sign} \mathrm{d}p(t) = \operatorname{sign} L_1(x(t)\, L_2(x(t))$$

and

$$\int_0^{2\pi} L_1(x(t))\, L_2(x(t))\, \mathrm{d}p\,(t) = 2 \int_0^{\pi} L_1(x(t))\, L_2(x(t))\, \mathrm{d}p\,(t) > 0.$$

But, by (5) and (6), $\int_0^{2\pi} Q(x(t))\, \mathrm{d}p(t) = 0$ for an arbitrary quadratic function $Q(x_1, \ldots, x_{2k})$ of the coordinates of the curve. Contradiction.

The number of extrema cannot be less than $2(2k - 1)$. If the number is $2(2k - 1 - m)$, we ask that the hyperplanes $L_i(x) = 0$, $i = 1, 2$, contain the vectors $x'(T_1), \ldots, x^{(2m)}(T_1)$ for $i = 1$, $x'(t_1), \ldots, x^{(2m)}(t_1)$ for $i = 2$ and, in addition, the points corresponding to the maxima ($i = 1$) or minima ($i = 2$) of $p(t)$. Since the order of contact is even and there are no inflexion points, the curve will still cross the planes at $x(T_1)$ (or $x(t_1)$) and we obtain the same contradiction as before.

REFERENCES

[1] Hayashi, T.: *Some geometrical applications of Fourier series*, Rend. Circ. Mat. Palermo 1, 1926, 96—102.

[2] Guggenheimer, H.: *Geometrical applications of integral calculus*, Lectures on Calculus, ed. by K. O. May, Holden-Day, San Francisco, 1967, 75—96.

[3] Rembs, E.: *Randvorgaben bei infinitesimaler Verbiegung konvexer Flächen*, Arch. Math. 6, 1955, 55—58.

[4] Pestov, G., Ionin, V.: *O naibolšem kruge, vložennom v zamknutuju krivuju* DAN 127, 1959, 1170—1172.

[5] Melzi, G.: *Sulla dimostrazione di un classico teorema riguardante gli ovali nel piano*, Rend. Inst. Lombardo, Cl. di Sc. A 96, 1962, 728—737.

[6] Peano, G.: *Une nouvelle forme du reste dans la formule de Taylor*, Mathesis 9, 1898, 182—183 = Opere Scelte 1, 95—96.

[7] Kneser, A.: *Synthetische Untersuchungen uber die Schmiegungsebenen beliebiger Raumkurven und die Realitätsverhältnisse specieller Kegelschnittsysteme*. Math. Ann. 31, 1888, 507—548.

[8] Guggenheimer, H.: *Differential Geometry*, New York 1963.

[9] Borůvka, O.: *Lineare Differentialtransformationen 2. Ordnung*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1967.

[10] Guggenheimer, H.: *Hill Equations with Coexisting Periodic Solutions II*. Comm. Math. Helv. 44 1969, 381—384.

[11] Guggenheimer, H.: *Pseudo-Minkowskian Differential Geometry*. Annali di mat. (4) 70, 1965, 305—370.

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