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A NOTE ON THE CRITERIA OF UNIQUENESS OF THE SOLUTION OF EQUATION $y' = f(x, y)$

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To Professor OTAKAR BORŮVKA for his 70th birthday

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1. Say that the function $Q(x, z)$ $(0 < x \leqq a, 0 \leqq z)$ has the property (U), if it is continuous, if $Q(x, 0) = 0$ $(0 < x \leqq a)$, and if the only solution $z = z(x)$ $(0 < x \leqq h)$ of the equation $z' = Q(x, z)$, fulfilling the condition $\lim_{x \to 0} \frac{z(x)}{x} = 0$ is the function $z(x) \equiv 0$. In further considerations we suppose that $f(x, y)$ $(0 \leqq x \leqq a, -\infty < y < \infty)$ is the given real function.

This criterion of uniqueness of the solution of equation $y' = F(x, y)$ is known ([1]):

Theorem 1. Let $| f(x, y_1) - f(x, y_2) | \leqq Q(x, | y_1 - y_2 |)$ $(0 < x \leqq a, -\infty < y_1 < \infty, -\infty < y_2 < \infty)$, where the function $Q(x, z)$ has the property (U). Then the equation $y' = f(x, y)$ has at most one solution $y = y(x)$ $(0 \leqq x \leqq h)$, for which $y(0) = 0$.

There exist functions $Q_1(x, z)$, $Q_2(x, z)$ $(0 < x \leqq a, 0 \leqq z)$ which have not the property (U), but such that the function $Q(x, z) = \min (Q_1(x, z), Q_2(x, z)))$ has the property (U). An example of such a couple is $Q_1(x, z) = Az^\alpha$, $Q_2(x, z) = \frac{k}{x} z$, where $0 < A$, $0 < \alpha < 1$, $1 < k$, $k(1 - \alpha) < 1$. It may be easily proved in such a way:

The solution of equation $z' = Q_1(x, z)$ is $z = ((1 - \alpha) Ax + \text{const.})^{1/(1-\alpha)}$, the solution of equation $z' = Q_2(x, z)$ is $z = \text{const.}\ x^k$. Furthermore there is $Q(x, z) \equiv Q_1(x, z)$ $(Ax < kz^{1-\alpha})$, $Q(x, z) \equiv Q_2(x, z)$ $(Ax \geqq k z^{1-\alpha})$ and from the inequality $k < \frac{1}{1 - \alpha}$ easily follows that every solution $z = z(x)$ of the equation $z' = Q(x, z)$ has the form $z(x) = ((1 - \alpha) Ax + h_1))^{1/(1-\alpha)}$ $(0 < x \leqq h)$, $z(x) = h_2 \cdot x^k (h \leqq x \leqq a)$, where h, h_1, h_2 are the suitable constants. And then, for $h_2 > 0$ there is also $h > 0$. Consequently the function $Q(x, z)$ has the property (U) and the following criterion ([2]) holds:

Theorem 2. Let $| f(x, y) - f(x, y_2) | \leqq A | y_1 - y_2 |^\alpha$, $| f(x, y_1) - f(x, y_2) | \leqq \frac{k}{x} | y_1 - y_2 |$, where $0 < A$, $0 < \alpha < 1$, $1 < k$, $k(1 - \alpha) <$

< 1. Then the equation $y' = f(x, y)$ has at most one solution $y = y(x)$ $(0 \leqq x \leqq h)$, for which $y(0) = 0$.

It is known ([1]) that if the function $f(x, y)$ is continuous and fulfils the supposition of the theorem 1, then the sequence $y_0(x), y_1(x), \ldots,$ where $y_0(x) = 0$, $y_{n+1}(x) = \int_0^x f(x, y_n(x))\, dx$ uniformly converges to the solution of the equation $y' = f(x, y)$. From our considerations there follows the theorem ([2]):

Theorem 3. If the continuous function $f(x, y)$ fulfils the suppositions of theorem 2, then the mentioned sequence $y_0(x), y_1(x), \ldots$ uniformly converges to the solution of the equation $y' = f(x, y)$.

2. It is interesting that the theorem 2 may be likewise deduced from the following criterion:*)

Theorem 4. Let $p(x, y_1, y_2)$, $q(x, y, z)$ $(0 < x \leqq a, -\infty < y_2 \leqq y_1 < \infty, -\infty < y < \infty, -\infty < z < \infty)$ be continuous real functions such that:

a) $p(x, y_1, y_2) > 0 \quad (y_1 > y_2), \quad p(x, y, y) = 0,$

b) inside of its definition domain the function $p(x, y_1, y_2)$ the differential dp exists and there hold $p_x(x, y_1, y_2) + p_{y_1}(x, y_1, y_2)\, f(x, y_1) + p_{y_2}(x, y_1, y_2)\, f(x, y_2) \leqq q(x, y_1, p(x, y_1, y_2))$,

c) for every two solutions $y = y_1(x)$, $y = y_2(x)$ $(0 \leqq x \leqq h)$ of the equation $y' = f(x, y)$ fulfilling the relations $y_1(0) = y_2(0) = 0$, $y_1(x) \geqq y_2(x)$ $(0 \leqq x \leqq h)$ there is $\lim_{x \to 0} p(x, y_1(x), y_2(x)) = 0$.

d) for every solution $y = y(x)$ $(0 \leqq x \leqq h)$ of the equation $y' = f(x, y)$ fulfilling the relation $y(0) = 0$, the function $z(x) \equiv 0$ is the only solution of the equation $z' = q(x, y(x), z)$ defined in the interval of the type $0 < x < h$, for which $\lim_{x \to 0} z(x) = 0$.

Then the equation $y' = f(x, y)$ has at most one solution $y = y(x)$ $(0 \leqq x \leqq h)$, for which $y(0) = 0$.

The theorem is proved in [3], but in a somewhat modified form. Therefore we outline its proof:

Suppose that the assertion of the theorem does not hold and that, consequently, there exist the solutions $y = y_1(x)$, $y = y_2(x)$ $(0 \leqq x \leqq h)$ of the equation $y' = f(x, y)$ such that $y_1(h) \neq y_2(h)$. It may be supposed that $y_1(x) \geqq y_2(x)$ $(0 \leqq x \leqq h)$, thus $p(x, y_1(x), y_2(x)) \geqq 0$, $p(h, y_1(h),$

*) This was also noted by prof. M. Zlámal.

$y_2(h)) > 0$. Consider that according to b) there is $[p(x, y_1(x), y_2(x))]' \leqq \leqq q(x, y_1(x), p(x, y_1(x), y_2(x)))$. Therefore there exists the solution $z = z(x)$ $(0 < x \leqq h)$ of the equation $z' = q(x, y_1(x), z)$ such that $z(h) = = p(h, y_1(h), y_2(h))$, $z(x) \leqq p(x, y_1(x), y_2(x))$. There is $\lim_{x \to 0} z(x) \leqq \lim_{x \to 0} p(x, y_1(x), y_2(x)) = 0$, which is a contradiction with d). By this the theorem is proved.

Show the way of following the theorem 2 from the theorem 4: Choose $p(x, y_1, y_2) = \dfrac{y_1 - y_2}{x^k}$. The claim of a) from theorem 4 then evidently holds. Furthermore choose $q(x, y, z) = 0$. The claim of b) is fulfilled if the function $f(x, y)$ fulfils the inequality $-k \dfrac{y_1 - y_2}{x^{k+1}} + \dfrac{f(x, y_1)}{x^k} - \dfrac{f(x, y_2)}{x^k} \leqq 0$, thus also then if there is $|f(x, y_1) - f(x, y_2)| \leqq \dfrac{k}{x} \cdot (y_1 - y_2)$ $(y_1 > y_2)$. This is one of inequalities of theorem 2. If there hold the inequalities $|f(x, y_1) - f(x, y_2)| \leqq A \mid y_1 - y_2 \mid^\alpha$, $k(1 - \alpha) < 1$, then the claim of c) is fulfilled as well. Really, in this case it may be easily stated that for every two solutions $y = y_1(x)$, $y = y_2(x)$ of the equation $y' = f(x, y)$ fulfilling the relations $y_1(0) = y_2(0) = 0$ there is $|y_1(x) - y_2(x)| \leqq ((1 - \alpha) Ax)^{1/(1-\alpha)}$ and therefore $\lim_{x \to 0} \dfrac{|y_1(x) - y_2(x)|}{x^k} = = 0$. The relation d) is fulfilled trivially.

More generally, we could choose $q(x, y, z) \equiv Cz$. Hence it follows that the theorem 2 remains to hold if there, instead of inequality $|f(x, y_1) - - f(x, y_2)| \leqq \dfrac{k}{x} \mid y_1 - y_2 \mid$, we take the inequality $|f(x, y_1) - f(x, y_2)| \leqq \leqq \dfrac{k}{x} \mid y_1 - y_2 \mid + C$.

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