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# Simultaneous nondeterministic games. I

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# SIMULTANEOUS NONDETERMINISTIC GAMES (I)

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## § 0. INTRODUCTION

In this paper so-called *simultaneous nondeterministic games* (SN-games, in § 0 often only "games") are introduced and investigated. They can be non-formally characterized as games with the following properties:

*a* in every game there are only two kinds of positions: *final positions* (at which play necessarily ends), and *non-final positions* (from which play necessarily goes on);

*b* at each non-final position all players move (play) mutually independently (i.e. "*simultaneously*");

*c* at each non-final position the common move of all players need not determine the following position uniquely (the *nondeterminateness* of game);

*d* in the notion of position the "preceding course (of play)" need not be included.

Further it is necessary to call special attention to:

— the cardinal number of the set of all positions of a game, and the cardinal number of the set of all players of a game are not bounded a priori (only it is supposed that these sets are non-empty);

— play may be infinite;

— the *pay-off function* to a player of a game is a real function on the set of all plays of the game;

— there is not designated an initial position (i.e. game is *non-initial*);

— the nondeterminateness of game is "essential" (e.g. it cannot be weakened by means of any experiments, compare § **1.3**);

— the nondeterminateness of game is not "diminished" e.g. by defining some probabilistic concepts; *notions corresponding to the notions of mixed or behaviour strategy are not considered.*

In this paper we study mainly questions about *possibilities of isolated player* to control (partially) the course of play. (From this point of view, every game can be considered as a certain derived one-player game at which the other players are "included in the new nondeterminateness".) We shall suppose that at each position of every play each *player*

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*) Final version (arisen at reading the galley-proof) received March 5, 1970.

*knows the preceding course of the play*, including the (momentary) position, but there will be studied also the case at which a player uses only a poorer information, especially when he always knows only the (momentary) position. (For several reasons we prefer this conception, although it would be possible to use the more usual way, i.e. to construct to a given game the new game in which positions are initial parts of plays of the given game. Compare also § 7.) In the latter case, which is characterized from the formal point of view by using so-called *plain strategies*, among others a very important problem is studied, namely, (roughly speaking) when player can use a suitable plain strategy for enforcing some aims from each position of a certain set of positions (§ 3).

As it is shown in the following, some questions usually studied e.g. at two-player games with perfect information (e.g. values of games, saddle points, etc.) can be studied also at *two-player SN-games*, even if only the separate possibilities of players are considered (compare §§ 9, 4, et al). Even also in investigations relating to only one player it is often advantageous "to add" (in a suitable way, see § 4) one more player, at which strategic properties derived at certain situations for one (so-called *active*) of the players often enable to construct suitable strategies for the other (so-called *passive*) player.

In the introductory part of this paper considerable attention is devoted to the exact formalization of basic concepts. The notion of SN-game — in the above mentioned sense — can be easily formalized: an SN-game is a *nondeterministic automaton* with multiple input (each player controls one "component" of the input), together with a system of pay-off functions (each player receives one pay-off function). (This way is realized in § 1.) Here the term "(nondeterministic) automaton" is used mainly because the usual idea of the working of automaton (by affecting the input some process is partly controlled) corresponds to the dynamics of SN-game; nevertheless, our definition of nondeterministic automaton (in § 1) and its role in our considerations are rather unusual from the point of view of the Automata Theory.

It is easy to see that that formalization of SN-game is too "rich" from the point of view of the study of the above mentioned problems (which can be exactly characterized by means of "tactic predicates" and graphs, see § 1.) Therefore, so-called *reduced SN-games* (RSN-games), and the notion of *reduction* of SN-game are introduced in § 2, of course in such a way that

— the reduction of every SN-game is an RSN-game,

— the reduction of SN-game maintains the informations essential from the above mentioned point of view,

— every RSN-game is the reduction of an SN-game.

The author took it for necessary to devote relatively considerable attention to the choice of elementary concepts (which appear besides others in the definition of RSN-game): there were several ways, e.g. to use expressing in terms of so-called general topologies (i.e. mappings of the set exp $P$ *) into itself (where $P$ is interpreted as the set of positions, see § 2.8), which would be advantageous especially in § 6, opposite this to characterize some local properties mappings of $P$ into exp exp $P$ would be suitable. But expressing in terms of *correspondences* between $P$ and exp $P$ (by which the corresponding general topologies, and also mappings of $P$ into exp exp $P$ can be naturally expressed) was preferred, for their advantageous formal properties. (See § 2*a*.)

Let us note that the connection between RSN-games and *general topologies* gives a very interesting possibility to interprete some general topologies (and concepts defined to them, e.g. certain *modifications*, may they be considered as extreme solutions of certain fixpoint problems, or as limits of transfinite sequences of "successive approximations") from the game point of view; this can be used e.g. for proving some equalities (with general topologies and modifications of them) by means of "game methods". (See §§ 5, 6, et al.)

Results of this paper can be applied to *games with perfect information* (*without chance influences*). Namely, these games can be considered as special (R)SN-games, in which at each non-final position there is always a player (depending on the position) who determines the following position (and thus, the other players play at this position "emptily"). Especially, some Berge's results (of chapters 1, 2 of [1]), and also results belonging to the so-called descriptive theory of games (see e.g. [11], [12]) follow from our theorems, too; the same may be said about Pears' results on topological games ([8]). At these considerations so-called *complete games* (or derived objects) have an important role; complete games are special two-player (R)SN-games in which for each non-final position $x$, for each player, and for every set $A$ of positions there holds: either the player can enforce that the following position will belong to $A$, or the other player can enforce that the following position will not belong to $A$. (Clearly, this is satisfied especially for two-player games with perfect information.) This condition can be considered as a certain local version of the "strict determinateness" (see e.g. [12]).

It is clear that, besides the formalizations of the notion of simultaneous nondeterministic game which are introduced in this paper, it is possible to use some others, especially in certain special cases, e.g. if the number

*) Instead of the usual denotation $2^A$ for the set of all subsets of a set $A$ we use the symbol exp $A$. $A^B$ means the set of all mappings of $B$ into $A$.

of players of a game is finite (then by defining auxiliary positions it is possible to reach that at each non-final position only one player moves; then the connections with games with perfect information can appear in another light, see § 10). But we may be of the opinion that the formalizations introduced in this paper are most adequate to the given problems, and they make easy orientation possible (e.g. many investigations of two-player games with perfect information simplify if these games are considered as a special case of more general complete games; conveniently (R)SN-games with the same strategic properties but with distinct graphs are investigated; etc.), apart from this that the used formalizations allow also the mentioned applications to general topologies, and "*back-applications*" to games (§ 7). Our conception has also some methodic priority: we proceed from a game which is given (as RSN-game) by its rules, not by some more abstract mathematical model.

With respect to the considerable extent of this work it is necessary to publish it in parts. Research Memoranda [4], [5], which were cyclostyled as materials of the Seminar of Mathematical Theory of Political Decisions, (UJEP, Brno) can be considered as a preliminary version of some parts of this paper. The author would like to thank especially doc. dr. Václav Polák, CSc., the head of this seminar, to whom he is indebted for the numerous important suggestions, advice, and critical remarks, which were used for this work.

## § 1. NONDETERMINISTIC AUTOMATA AND SN-GAMES

**1.** Let $(P, P_0)$ be a pair of sets such that $P_0 \subset P \neq \emptyset$; we shall call such a pair a *type*. By a *variant* of the type $(P, P_0)$ (or shortly: by a variant) we mean every sequence $\mathbf{x} = (x_k)_{0 \leqq k < 1 + l(\mathbf{x})}$ of elements of $P$ such that $0 \leqq l(\mathbf{x}) \leqq \omega_0$ (where $\omega_0$ is the first infinite ordinal number, thus $1 + \omega_0 = \omega_0$), and $x_k \in P - P_0 \Leftrightarrow k < l(\mathbf{x})$ (for $0 \leqq k < 1 + l(\mathbf{x})$). If we deal with a variant $\mathbf{x}$, then the symbols $k$, $k_j$ and similarly will denote nonnegative integers being smaller than $1 + l(\mathbf{x})$. Instead of the symbols $\mathbf{x}$, $x_k$ we shall sometimes use the symbols $\mathbf{y}$, $y_k$; $\mathbf{x}^j$, $x^j_k$; $\mathbf{y}^j$, $y^j_k$, respectively. $\mathbf{P}$ (or $\mathbf{P}_{(P, P_0)}$) denotes the set of all variants (of the type $(P, P_0)$). We call $l(\mathbf{x})$ the *length* of $\mathbf{x} \in \mathbf{P}$; $x_k$ is the $k$th element of $\mathbf{x}$. $\mathbf{L}_{(P, P_0)}\, x = \mathbf{L}x := \{\mathbf{x} \mid \mathbf{x} \in \mathbf{P},\ x_0 = x\}$*) is the set of all variants

*) The denotations of the type "$\mathscr{A} := \mathscr{B}$" are used in three distinct meanings in this paper: "the symbol $\mathscr{A}$ will mean $\mathscr{B}$" *(definition)*, "as $\mathscr{A}$ we now choose $\mathscr{B}$" *(substitution)*, "$\mathscr{A}$ is now replaced by $\mathscr{B}$" *(replacement)*.

which *start from* $x$. Evidently, $\{\mathbf{L}x \mid x \in P\}$ is a decomposition on the set $\mathbf{P}$, $(\mathbf{L}x)_{x\in P}$ is a partition*) of $\mathbf{P}$.

Elements of exp $\mathbf{P}$ will be called *aims* (of the type $(P, P_0)$) . $\mathbf{A}$ will always be an aim.

2. Let $Z$ be a set. By a $Z$-segment (or shortly: by a *segment*) we mean every finite sequence $\mathbf{z} = (z_k)_{0\leq k\leq l(\mathbf{z})}$ of elements of $Z$. $l(\mathbf{z})$ is the length of $\mathbf{z}$. (We shall denote segments in the analogous manner as variants.) $\mathbf{Z}$ (or $\mathbf{Z}_Z$) will be the set of all segments.

If $\mathbf{a} = (a_0, \ldots, a_n)$ is an arbitrary finite sequence, then we put $\varkappa(\mathbf{a}) := a_n$ (i.e. $\varkappa(\mathbf{a})$ is the *last element* of $\mathbf{a}$). We say that $\sim$ is a *memory relation* (at $Z$) iff $\sim$ is an equivalence relation on $\mathbf{Z}$ such that there holds: if $\mathbf{z}^1 \sim \mathbf{z}^2$, then $\varkappa(\mathbf{z}^1) = \varkappa(\mathbf{z}^2)$. (Therefore, the relation $\overset{\circ}{\sim}$ on $\mathbf{Z}$ defined by: $\mathbf{z}^1 \overset{\circ}{\sim} \mathbf{z}^2 \Leftrightarrow \varkappa(\mathbf{z}^1) = \varkappa(\mathbf{z}^2)$ is the greatest memory relation, and the equality on $\mathbf{Z}$ is the smallest memory relation.)

3. By a ***(nondeterministic) automaton*** we shall understand a triple $\mathcal{R} = (P, R, \varrho)$, where $P$ is a non-empty set, $R$ is a mapping of $P$ such that $R(x)$ is a set for all $x \in P$, and $\varrho$ is a mapping of the set $D := \{(x, r) \mid x \in P,\ r \in R(x)\}$ into $(\exp P) - \{\emptyset\}$. We put $P_0 := \{x \mid x \in P, R(x) = \emptyset\}$, $Z := P - P_0$. $P$, $P_0$, $Z$ is the set of all *states*, final states, non-final states (of $\mathcal{R}$), respectively. $R(x)$ is the set of all *inputs* at the state $x$. If at a (non-final) state $x$ some $r \in R(x)$ occurs on the "input place" of $\mathcal{R}$, then after $x$ an arbitrary state $y \in \varrho(x, r)$ may immediately follow (and some state of $\varrho(x, r)$ must immediately follow after $x$), but— as $\mathcal{R}$ is considered as nondeterministic — there is not given any information about the "internal device (of $\mathcal{R}$)" which chooses this state $y$. (And e.g. if the input $r$ occurs at this $x$ in some two cases, then the states $y$ may be in these two cases distinct.) We say that $\mathcal{R}$ is *deterministic* iff $\varrho(x, r)$ is a one-element set for all $(x, r) \in D$.

4. Let $\mathcal{R}$ be an automaton (with the same denotations as above). $(P, P_0)$ we call the type of $\mathcal{R}$, and at $\mathcal{R}$ we define $\mathbf{P}$ to $(P, P_0)$, and $\mathbf{Z}$ to $Z$. By *$\mathcal{R}$-tactics* we mean a mapping $\boldsymbol{\tau}$ of $\mathbf{Z}$ such that $\boldsymbol{\tau}\,\mathbf{z} \in R(\varkappa(\mathbf{z}))$ for all $\mathbf{z} \in \mathbf{Z}$. We say that $\mathcal{R}$-tactics $\boldsymbol{\tau}$ are *$\sim$-acceptable* iff $\boldsymbol{\tau}\,\mathbf{z}^1 = \boldsymbol{\tau}\,\mathbf{z}^2$ holds for all segments such that $\mathbf{z}^1 \sim \mathbf{z}^2$. The set of all $\sim$-acceptable $\mathcal{R}$-tactics is non-empty.

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*) By a *collection* we mean a set such that each of its elements is a set (the existence of a set which is not a collection depends on the choice of an axiomatic set theory). A *decomposition* on a set $P$ is a collection of mutually disjoint non-empty sets *(classes)* such that $P$ is the set-theoretical union of them. A *partition* of a set $P$ is a system $(P_j)_{j\in J}$ (i.e. a mapping of $J$) such that $P_{j_1} \cap P_{j_2} = \emptyset$ for all distinct $j_1, j_2, \in J$, and $\bigcup_{j\in J} P_j = P$.

For $\mathscr{R}$-tactics $\boldsymbol{\tau}$ we define $\mathrm{t}_{\mathscr{R}}(\boldsymbol{\tau}) := \{\mathbf{x} \mid \mathbf{x} \in \mathbf{P}, x_{k+1} \in \varrho(x_k, \boldsymbol{\tau}(x_0, \ldots, x_k))$ for all $k < l(x)\}$; $\mathrm{t}_{\mathscr{R}}(\boldsymbol{\tau})$ is the set of all variants which *comply with* $\boldsymbol{\tau}$ (at $\mathscr{R}$). Further we denote $\mathrm{t}_{\mathscr{R}}(x, \boldsymbol{\tau}) := \mathrm{t}_{\mathscr{R}}(\boldsymbol{\tau}) \cap \mathbf{L}x$ for $x \in P$. Clearly, $(\mathrm{t}_{\mathscr{R}}(x, \boldsymbol{\tau}))_{x \in P}$ is a partition of $\mathrm{t}_{\mathscr{R}}(\boldsymbol{\tau})$. $\mathrm{t}_{\mathscr{R}}(x, \boldsymbol{\tau}) \subset \boldsymbol{A}$ can be interpreted in such a way: $\boldsymbol{\tau}$ *enforces* the aim $\boldsymbol{A}$ from $x$. Consequently, $\mathrm{t}_{\mathscr{R}}(\boldsymbol{\tau}) \subset \boldsymbol{A}$ means that $\boldsymbol{\tau}$ enforces $\boldsymbol{A}$ from each position. Clearly, $\mathrm{t}_{\mathscr{R}}(\boldsymbol{\tau}) \subset \boldsymbol{A}$ iff*) $\mathrm{t}_{\mathscr{R}}(x, \boldsymbol{\tau}) \subset \boldsymbol{A} \cap \mathbf{L}x$ for each $x \in P$. On the other hand, if $\boldsymbol{\alpha}$ is a mapping of $P$ into $\exp \mathbf{P}$, then $\mathrm{t}_{\mathscr{R}}(x, \boldsymbol{\tau}) \subset \boldsymbol{\alpha}(x)$ for each $x \in P$ iff $\mathrm{t}_{\mathscr{R}}(\boldsymbol{\tau}) \subset \bigcup_{x \in P} \boldsymbol{\alpha}(x) \cap \mathbf{L}x$. Further $\mathrm{t}_{\mathscr{R}}(x^0, \boldsymbol{\tau}) \subset \boldsymbol{A}$ (for some $x^0$) iff $\mathrm{t}_{\mathscr{R}}(\boldsymbol{\tau}) \subset \boldsymbol{A} \cup (P - \mathbf{L}x^0)$.

**5.** At $\mathscr{R}$ we define the *tactic-predicate* $\mathbf{T}_{\mathscr{R}}$ on the cartesian product of the set of all memory relations and the set of all aims: $\mathbf{T}_{\mathscr{R}}(\sim, \boldsymbol{A})$ means the assertion: there exists $\sim$-acceptable $\mathscr{R}$-tactics $\boldsymbol{\tau}$ such that $\mathrm{t}_{\mathscr{R}}(\boldsymbol{\tau}) \subset \boldsymbol{A}$ for all $x \in P$. The simple remarks mentioned in 4 have obvious corollaries in terms of tactic predicates.

**6.1.** In the following in § 1 let $J$ be a non-empty set.

We say that an automaton $\mathscr{R} = (P, R, \varrho)$ is a *J-automaton* iff for all $x \in P$ the set $R(x)$ is a cartesian product of a system (of sets) having $J$ as the set of all indices.

Let $\mathscr{R} = (P, R, \varrho)$ be a $J$-automaton; then the conditions

$$R(x) = \mathop{\mathsf{X}}_{j \in J} R_j(x) \text{ **)}$$

$$R(x) = \emptyset \Rightarrow R_j(x) = \emptyset$$

$(x \in P,\ j \in J)$ determine the sets $R_j(x)$ uniquely. Let us put $D_j := \{(x, r) \mid x \in P,\ r \in R_j(x)\}$; then we define $\varrho_j : D_j \to (\exp P) - \{\emptyset\}$ by

$$\varrho_j(x, r^0) := \bigcup_{\substack{r \in R(x) \\ \mathrm{pr}_j r = r^0}} \varrho(x, r).$$

Then

$$\mathscr{R}_j := (P, R_j, \varrho_j)$$

is an automaton of the type $(P, P_0)$; we call it the *j-pseudocomponent* of $\mathscr{R}$. It can be simply proved that there holds

$$\text{(1)} \qquad \bigcup_{r^0 \in R_j(x)} \varrho_j(x, r^0) = \bigcup_{r \in R(x)} \varrho(x, r) \qquad (x \in P, \quad j \in J),$$

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*) Instead of "*if and only if*" one says shortly "*iff*".

**) $\mathsf{X}$ is the symbol of the *cartesian product*, $\times$ denotes the binary cartesian product, $\mathrm{pr}_j\, r$ is the denotation of the $j$th *projection* of $r$ (i.e. the image of $j$ under the mapping $r$ which is considered as an element of a cartesian product).

$$(2) \qquad \varrho(x, r) \subset \bigcap_{j \in J} \varrho_j(x, \mathrm{pr}_j\, r) \qquad (x \in P, \quad r \in R(x)).$$

A $J$-automaton $\mathscr{R}$ need not be determined by the system $(\mathscr{R}_j)_{j \in J}$ of its pseudocomponents, as it follows from the following lemma:

**6.2. Lemma.** *The following assertions are equivalent:*

(A) $\qquad P \neq P_0 \wedge \operatorname{card} J \geqq 2 \wedge \operatorname{card} P \geqq 2.$*)

(B) *There exist $J$-automata $\mathscr{R}^n = (P, R^n, \varrho^n)$ $(n = 1, 2)$ of the type $(P, P_0)$ such that $\mathscr{R}_j^1 = \mathscr{R}_j^2$ for all $j \in J$, but $\mathscr{R}^1 \neq \mathscr{R}^2$.*

(C) *There exist $J$-automata $\mathscr{R}^n = (P, R, \varrho^n)$ $(n = 1, 2)$ of the type $(P, P_0)$ such that $\mathscr{R}_j^1 = \mathscr{R}_j^2$ for all $j \in J$, but $\mathscr{R}^1$ is deterministic while $\mathscr{R}^2$ is not deterministic.*

*Proof. 1.* Let $\mathscr{R}^n = (P, R^n, \varrho^n)$ $(n = 1, 2)$ be arbitrary $J$-automata of the type $(P, P_0)$ such that $\mathscr{R}_j^1 = \mathscr{R}_j^2$ for all $j \in J$. Then $R^1 = R^2$. If $P = P_0$, then for $n = 1, 2$ $\{(x, r) \mid x \in P,\ r \in R^n(x)\} = \emptyset$, and thus $\varrho^1 = \varrho^2$ (because there exists exactly one mapping of the empty set into $(\exp P) - \{\emptyset\}$). If $J = \{j\}$, then $\varrho^n(x, (r)) = \varrho_j^n(x, r)$, where $\mathrm{pr}_j\,(r) = r$ (for $r \in R_j(x)$, i.e. $\mathrm{pr}_j$ can be considered as the canonical mapping of $R_j(x)$ onto $R(x)$), hence $\varrho^1 = \varrho^2$ (because $\varrho_j^1 = \varrho_j^2$). If $P = \{x\}$, then either $P = P_0$, then $\varrho^1 = \varrho^2$, or $P \neq P_0$, then $R^1(x) = R^2(x) \neq \emptyset$, $\varrho^n(x, r) = \{x\}$ for $n = 1, 2$, $r \in R^n(x)$, i.e. $\varrho^1 = \varrho^2$.

Thus, if (A) does not hold, then $\mathscr{R}^1 = \mathscr{R}^2$, i.e. both (B) and (C) do not hold.

*2.* Let (A) hold. Let $x^0 \in P - P_0$, $y^0 \in P - \{x^0\}$ be arbitrary. We define

$$R_j(x) := \begin{cases} \emptyset & \text{if } x \in P_0 \\ \{x^0, y^0\} & \text{if } x \in P - P_0 \end{cases}, \quad j \in J, \quad R(x) := \mathop{\times}_{j \in J} R_j(x).$$

The mappings $\varrho^n : \{(x, r) \mid x \in P,\ r \in R(x)\} \to (\exp P) - \{\emptyset\}$ are defined in the following way:

$$\varrho^1(x, r) = \begin{cases} \{x^0\} & \text{if } x = x^0, \text{ and all } \mathrm{pr}_j\, r \text{ are equal} \\ \{y^0\} & \text{otherwise} \end{cases},$$

$$\varrho^2(x, r) = \begin{cases} \{x^0, y^0\} & \text{if } x = x^0. \\ \{x^0\} & \text{otherwise} \end{cases}.$$

Now $\mathscr{R}^n = (P, R, \varrho^n)$ $(n = 1, 2)$ are $J$-automata of the type $(P, P_0)$, $\mathscr{R}^1$ is deterministic, $\mathscr{R}^2$ is not deterministic, but $\varrho_j^1 = \varrho_j^2$ (since $\varrho_j^n(x, r) = \{y^0\}$ for $n = 1, 2$, $x \in P - (P_0 \cup \{x^0\})$, $j \in J$, $r \in R_j(x)$, $\varrho_j^n(x^0, r) = \{x^0, y^0\}$ for $n = 1, 2$, $j \in J$, $r \in R_j(x)$). Hence (B), (C) hold.

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*) card $A$ is the cardinal number of a set $A$. $\wedge$ means "*and*", $\vee$ means "*or*".

Q. E. D.

**7.** Let $\mathscr{R} = (P, R, \varrho)$ be an automaton of the type $(P, P_0)$. Then we define the set $\mathbf{X}_{\mathscr{R}}$ of all variants which can occur at $\mathscr{R}$:

$$\mathbf{X}_{\mathscr{R}} := \{\mathbf{x} \mid \mathbf{x} \in \mathbf{P},\ x_{k+1} \in \bigcup_{r \in R(x_k)} \varrho(x_k, r) \text{ for all } k < l(\mathbf{x})\}.$$

From (1) we conclude

(3) If $\mathscr{R}$ is a $J$-automaton, then $\mathbf{X}_{\mathscr{R}} = \mathbf{X}_{\mathscr{R}_j}$ for all $j \in J$.

**8.1.** By a ***simultaneous nondeterministic game*** (SN-game) of the type $(P, P_0)$ we shall understand a pair

$$\mathscr{G} = (\mathscr{R}, (f_j)_{j \in J}),$$

where $\mathscr{R}$ is a $J$-automaton of the type $(P, P_0)$, and $f_j$ is a real function on $\mathbf{X}_{\mathscr{R}}$ for all $j \in J$. The *interpretation* is natural: the automaton $\mathscr{R}$ describes the "*dynamics*" of the game, $P$ is the set of all *positions* of $\mathscr{G}$, $P_0$ is the set of all *final positions*, $\mathbf{X}_{\mathscr{R}}$ is the set of all *plays* of $\mathscr{G}$, $J$ is the set of all *players*, and $f_j$ is the *pay-off function* of the player $j$ in $\mathscr{G}$. A player $j$ controls the $j$'s "input place" of $\mathscr{R}$: we assume that at every moment of a play the player $j$ knows the preceding course of the play including the momentary position $x$ (nevertheless, we often consider the case in which the "memory of the player $j$" is bounded by a memory relation $\sim$, and then the player $j$ knows only that class of the decomposition on $\mathbf{Z}$ ($Z := P - P_0$) determined by $\sim$ which contains the segment which characterizes the preceding course of the play; here we suppose that the momentary position is non-final), and at $x$ $j$ *plays* such that he chooses some input $r_j \in R_j(x)$ (if $x$ is non-final), and all players play at $x$ "simultaneously" (i.e. mutually independently); the position $y$ which immediately follows after $x$ in this play is given by $r = (r_j)_{j \in J} \in R(x)$ in the sense mentioned in 3. $\mathbf{T}_{\mathscr{R}_j}$ is called the *player $j$'s tactic predicate* (at the game).

**8.2.** *In this paper only the questions which can be expressed in terms of the tactic predicates of players and sometimes questions which deal with the set of all plays, will be studied.* For these purposes it is sufficient to know (at a game $\mathscr{G}$) e. g. only $(\mathscr{R}_j)_{j \in J}$ instead of $\mathscr{R}$, but we shall show that it is possible to study stronger "*reductions*" of $\mathscr{G}$ than $((\mathscr{R}_j)_{j \in J}, (f_j)_{j \in J})$.

**8.3.** Let us note that our conception of tactics, segments, memory relations and strategies (see § 2c) is in a certain sense "*redundant*", e.g. $\boldsymbol{\tau}\mathbf{z}$ must be defined also for such a segment $\mathbf{z}$ which at an automaton $\mathscr{R}$ ($\boldsymbol{\tau}$ is an $\mathscr{R}$-tactics) cannot "occur", but it is clear that this redundance does not matter, and on the contrary, it is useful, as it facilitates expression. Similarly, we judge that the introduction of the notion of variant is well justifiable, especially for the study of games with the same "strategic properties" but with distinct sets of plays (compare e.g. with § 2.26.3).

## § 2. BASIC CONCEPTS

### a) *Collections. Correspondences. General topologies*

**1.** The letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ will always designate *collections* (see p. 33). For a collection $\mathfrak{A}$ we define its *norm* $\|\mathfrak{A}\| := \bigcup_{A \in \mathfrak{A}} A$. We say that $\mathfrak{A}$ is a collection *in* a set $Q$ iff $\|\mathfrak{A}\| \subset Q$. So, $\exp \exp Q$ is the set of all collections in $Q$. We say that $\mathfrak{A}$ is a collection *on* $Q$ iff $\|\mathfrak{A}\| = Q$ or $\mathfrak{A} = \emptyset$. Hence every collection on $Q$ is a collection in $Q$. Collections in $Q$ we shall also call $(Q)$-collections.

**2.** In the following let $E$, $F$, $G$, $P$, $Q$, $J$ be sets. We define

$$\operatorname{Corr}(E, F) := \{(\mu, E, F) \mid \mu \subset E \times F\};$$

elements of Corr $(E, F)$ we call *correspondences* (between $E$ and $F$). Our definition is the same as Bourbaki's definition of the "correspondences" ([2], II, § 3.1). The other basic notions which we use without defining them (e.g. the *composition* of two correspondences; *mappings* (functions) as a certain kind of correspondences etc.) we understand also in the same sense as Bourbaki ([2]). By the letters $u$, $v$, $w$ we shall always denote correspondences.

**3.** For $u = (\mu, E, F) \in \operatorname{Corr}(E, F)$ and for any $x$, $y$ we write

$$xuy$$

iff $(x, y) \in \mu$; further we put

$$\begin{aligned} xu &:= \{y \mid y \in F, \quad xuy\}, \\ uy &:= \{x \mid x \in E, \quad xuy\}. \end{aligned}$$

Thus for all $x$, $y$ there holds

$$xuy \Leftrightarrow xu \ni y \Leftrightarrow x \in uy, \tag{1}$$

and for $v \in \operatorname{Corr}(E, F)$

$$u = v \Leftrightarrow \text{for all } x \in E,\ y \in F \text{ there holds: } xuy \text{ iff } xvy. \tag{2}$$

4. We can define certain one-to-one mappings among the sets $\exp(E \times F)$, $\mathrm{Corr}(E, F)$, $(\exp E)^F$, $(\exp F)^E$ : the *trivial mapping* of $\exp(E \times F)$ onto $\mathrm{Corr}(E, F)$ under which $(\mu, E, F)$ is the image of $\mu \subset E \times F$, and the *canonical mappings* of $\mathrm{Corr}(E, F)$ onto $(\exp E)^F$ and onto $(\exp F)^E$ under which $\vec{u} \in (\exp E)^F$ and $\overleftarrow{u} \in (\exp F)^E$ are the images of $u \in \mathrm{Corr}(E, F)$ if and only if

$$xuy \Leftrightarrow \overleftarrow{u}(x) \in y \Leftrightarrow x \in \vec{u}(y)$$

holds for all $x \in E$, $y \in F$. We shall always use the denotations $\vec{u} \in (\exp E)^F$, and $\overleftarrow{u} \in (\exp F)^E$ for the images of $u \in \mathrm{Corr}(E, F)$ under the canonical mappings. Thus there holds

$$xu = \overleftarrow{u}(x), \qquad uy = \vec{u}(y) \qquad (\text{for all } x \in E,\ y \in F). \tag{3}$$

5.1. Let $\omega$ be an operation of the type $J$ in a set $M$, $\lambda$ be a relation of the type $J$ in $M$, i.e. $\omega : M^J \to M$, $\lambda \in \exp M^J$. (How it is often done, we shall not distinguish between (unary) operations and relations in $M$ on the one side, and operations and relations of the type $\{1, 2\}$ ($\{1\}$) in $M$ on the other side. Further, properties defined for elements of $M$ we consider as unary relations in $M$ or directly as elements of $\exp M$.) If $M = \exp(E \times F)$, then we define the *induced operation* $\overleftrightarrow{\omega}$ and the *induced relation* $\overleftrightarrow{\lambda}$ of the type $J$ in $\mathrm{Corr}(E, F)$ by the following conditions:

$$\overleftrightarrow{\omega}((u_j)_{j \in J}) = (\omega((\mu_j)_{j \in J}), E, F),$$
$$(u_j)_{j \in J} \in \overleftrightarrow{\lambda} \Leftrightarrow (\mu_j)_{j \in J} \in \lambda,$$

where $(u_j)_{j \in J} \in (\mathrm{Corr}(E, F))^J$, $u_j = (\mu_j, E, F)$ $(j \in J)$. If $M = \exp E$, then we define the *induced operation* $\vec{\omega}$ and the *induced relation* $\vec{\lambda}$ of the type $J$ in $\mathrm{Corr}(E, F)$ by the conditions

$$\vec{\omega}((u_j)_{j \in J})\, y = \omega((u_j y)_{j \in J}) \text{ for all } y \in F,$$
$$(u_j)_{j \in J} \in \vec{\lambda} \Leftrightarrow (u_j y)_{j \in J} \in \lambda \text{ for all } y \in F.$$

If $M = \exp F$, then we define the *induced operation* $\overleftarrow{\omega}$ and the *induced relation* $\overleftarrow{\lambda}$ of the type $J$ in $\mathrm{Corr}(E, F)$ by the conditions

$$x\, \overleftarrow{\omega}((u_j)_{j \in J}) = \omega((x u_j)_{j \in J}) \text{ for all } x \in E,$$
$$(u_j)_{j \in J} \in \overleftarrow{\lambda} \Leftrightarrow (x u_j)_{j \in J} \in \lambda \text{ for all } x \in E.$$

(The before-mentioned construction of induced operations and relations are well-known in other terms in general algebra.)

5.2. Let us consider the set-theoretical operations $\bigcap_{j \in J}$ $(J \neq \emptyset)$, $\bigcup_{j \in J}$, — (the set-theoretical difference) as operations, and the inclusion $\subset$ and the equality $=$ as relations in a set $M$. For $M := \exp(E \times F)$, $M := \exp E$, $M := \exp F$ we get three kinds of induced operations (relations) in Corr $(E, F)$ (according to 5.1), but it is easily seen that for each of those operations (relations) all these three induced operations (relations) are equal. (Moreover, the "induced equality" is equal to the equality considered only on elements of Corr $(E, F)$ as certain ordered triples.) Since we shall not apply the above mentioned set-theoretical operations and the inclusion to correspondences (which can be introduced as certain sets), *we shall use the symbols* $\bigcap_{j \in J}$, $\bigcup_{j \in J}$, — *for the induced operations*, and $\subset$ *for the "induced inclusion"* (without regard to $E$, $F$).

5.3. Let $\omega_1$, $\omega_2$ be unary operations in $\exp F$; we can take (for $j = 1, 2$) $\omega_j : \exp F \to \exp F$ (instead of $\omega_j : (\exp F)^{\{1\}} \to \exp F$), and $\overleftarrow{\omega_j} : \mathrm{Corr}\,(E, F) \to \mathrm{Corr}\,(E, F)$. From 5.1 there immediately follows

$$(4) \qquad \overleftarrow{\omega_2 \circ \omega_1} = \overleftarrow{\omega_2} \circ \overleftarrow{\omega_1},$$

where $\circ$ denotes the composition of mappings.

6.1. For $u \in \mathrm{Corr}\,(E, F)$ and $v \in \mathrm{Corr}\,(G, \exp E)$ we define their *product* $v \,.\, u$ (or $vu$) as the element of Corr $(G, F)$ which is determined by the condition $\overrightarrow{v \,.\, u} = \vec{v} \circ \vec{u}$; this definition is correct because $\vec{u} \in (\exp E)^F$, $\vec{v} \in (\exp G)^{\exp E}$ $\vec{v} \circ \vec{u} \in (\exp G)^F$. We can characterize $v \,.\, u$ also by the condition

$$(5) \qquad (v \,.\, u)\, a = v(ua) \text{ for all } a \in F;$$

we shall write $vuA$ instead of $(v \,.\, u)\, A$ (or $v(uA)$), too.

6.2. If $\omega$ is one of the operations $\bigcap_{j \in J}$, $\bigcup_{j \in J}$, —(see 5.2), then there holds a certain *distributive law:*

$$(6) \qquad \omega((v_j)_{j \in J}) \,.\, u = \omega(v_j \,.\, u)_{j \in J}),$$

where $J$ is the type of $\omega$, $u \in \mathrm{Corr}\,(E, F)$, $v_j \in \mathrm{Corr}\,(G, \exp E)$ for all $j \in J$. Further, for all $v_1, v_2 \in \mathrm{Corr}\,(G, \exp E)$ there holds

$$(7) \qquad v_1 \subset v_2 \Leftrightarrow v_1 \,.\, u \subset v_2 \,.\, u \text{ for all } u \in \mathrm{Corr}\,(E, F).$$

7. Let $E = P$, $F = \exp Q$, $u, v \in \mathrm{Corr}\,(E, F) = \mathrm{Corr}\,(P, \exp Q)$. Then $\exp F = \exp \exp Q$ is the set of all $(Q)$-collections; according to 5.1 we define the induced operations and relations (including properties) to operations and relations in the set of all $(Q)$-collections. Since we denote correspondences and collections in different ways, *we shall use the same*

*symbols (or terms) for the induced operations and relations as for the corresponding operations and relations in* exp exp $Q$. E.g. "$u$ is an R-correspondence" means the same as "$xu$ is an R-collection for all $x \in P$", $u \sqcap v$ is the element of Corr ($P$, exp $Q$) for which $x(u \sqcap v) = (xu) \sqcap (xv)$ for all $x \in P$ (see part *b* of this paragraph), and similarly. According to 5.3 the composition of induced unary operations is the same as the operation induced by the composition of those unary operations. E.g. $\sim$ is the composition of certain unary operations $'$, $\bar{\ }$, which are defined in the set of all ($Q$)-collections (see § 4.2), hence $\overline{(u')} = \tilde{u}$; we may use denotations as $u''$ ($(u)^{('')} = (u')'$), $\tilde{u}'$, etc.

**8.** $(\exp P)^{\exp P}$ is known as the set of all general topologies (on $P$). (In the following we shall often say "*topology*" instead of "general topology".) Concepts dealing with general topologies (on $P$) can be expressed in terms of Corr ($P$, exp $P$), or $P^{\exp \exp P}$, or exp ($P \times \exp P$) by means of the canonical mappings and the trivial mapping (see 4 with $E := P$, $F := \exp P$); cf. also J. Schmidt's approach in [9]. E.g. the composition of the trivial mapping and the canonical mapping $^{\rightarrow}$ is an isomorphism of (exp ($P \times \exp P$), $\subset$) (as a partially ordered set, where $\subset$ is the set-theoretical inclusion) onto $((\exp P)^{\exp P}, \leqq)$, where $\leqq$ is the usual partial ordering of the set of all general topologies on $P$ (see [6], [10]), and the canonical mapping $^{\rightarrow}$ is an isomorphism of (Corr ($P$, exp $P$), $\subset$) (where $\subset$ is the "induced inclusion") onto $((\exp P)^{\exp P}, \leqq)$. Further, the canonical mapping $^{\rightarrow}$ is an isomorphism of Corr ($P$, exp $P$) as a semigroup with $\cdot$ (as the binary operation) onto the semigroup of all topologies on $P$ (as mappings of exp $P$ into itself) with the composition as operation( see 6.1 with $E := G := P$, $F := \exp P$). In the following *we shall sometimes not distinguish between a (general) topology and its original under the canonical mapping* $^{\rightarrow}$ *of* Corr ($P$, exp $P$) *onto* $(\exp P)^{\exp P}$.

b) *Special collections and correspondences. Some operations. Graphs*

**9.** In part *b* let $\mathfrak{A}$, $\mathfrak{B}$ be ($Q$)-collections, $(\mathfrak{A}_j)_{j \in J}$ be a system of ($Q$)-collections, $u, v \in$ Corr ($P$, exp $Q$), $(u_j)_{j \in J}$ be a system of elements of Corr ($P$, exp $Q$), $P_0 \subset P \neq \emptyset$. We shall consider with every operation (relation, property) in exp exp $Q$ the corresponding *induced* operation (relation, property) in Corr ($P$, exp $Q$) (according to 7).

**10.** A collection $\mathfrak{C}$ we call *regular (singular)* or shortly an R-collection (S-collection) iff $\emptyset \notin \mathfrak{C}$ ($\emptyset \in \mathfrak{C}$). For a collection $\mathfrak{C}$ we define

$\mathfrak{C}^{\#} := \begin{cases} \mathfrak{C} & \text{if } \emptyset \notin \mathfrak{C} \\ \emptyset & \text{if } \emptyset \in \mathfrak{C} \end{cases}$. Thus $\mathfrak{C}^{\#}$ is regular; $\mathfrak{C} = \mathfrak{C}^{\#}$ iff $\mathfrak{C}$ is regular. Hence $\mathfrak{C}^{\#\#} = \mathfrak{C}^{\#}$.

Evidently $u\emptyset$ is the set of all $x \in P$ such that $xu$ is an S -collection, hence

(8) $u$ is an R-correspondence (S-correspondence) $\Leftrightarrow u\emptyset = \emptyset$ $(u\emptyset = P)$.

Further $x \in u^{\#} A \Leftrightarrow (xu^{\#}) \ni A \Leftrightarrow (xu)^{\#} \ni A \Leftrightarrow (xu) \ni A \wedge (xu) \not\ni \emptyset \Leftrightarrow x \in uA \wedge x \notin u\emptyset \Leftrightarrow x \in uA - u\emptyset$, i.e.

$$u^{\#}A = uA - u\emptyset \text{ for all } A \subset Q. \tag{9}$$

**11.** For $X \subset P$ let $\underline{X}$ be the element of Corr $(P, \exp Q)$ for which $\underline{X}A = X$ for all $A \subset Q$. Now we can e.g. the equality (9) express in the form $u^{\#} = u - \underline{u\emptyset}$.

**12.** We put

$$[\mathfrak{A}]_Q := \{B \mid B \subset Q,\ \mathfrak{A} \cap \exp B \neq \emptyset\} \ ^{*)}$$

it is easy to see that the operation $[\ ]_Q$ (see § 1.5.2) has the following properties*)

$$\begin{gathered} [\emptyset]_Q = \emptyset \\ \mathfrak{A} \subset [\mathfrak{A}]_Q \\ \mathfrak{A} \subset \mathfrak{B} \Rightarrow [\mathfrak{A}]_Q \subset [\mathfrak{B}]_Q \\ [[\mathfrak{A}]_Q]_Q = [\mathfrak{A}]_Q, \end{gathered} \tag{10}$$

moreover there holds

$$[\bigcup_{j \in J} \mathfrak{A}_j]_Q = \bigcup_{j \in J} [\mathfrak{A}_j]_Q; \tag{11}$$

therefore, $\exp Q$ and the operation $[\ ]_Q$ form a Kuratowski topological space (see e.g. [7], p. 44).

We say that $\mathfrak{A}$ is an M($Q$)-collection (or shortly: an M-*collection*) iff the conditon

$$A \subset B \subset Q, \qquad A \in \mathfrak{A} \Rightarrow B \in \mathfrak{A}$$

is satisfied. Every M-collection is a collection on $Q$, because if it is non-empty, then it contains $Q$. Since

$$\mathfrak{A} = [\mathfrak{A}]_Q \Leftrightarrow \mathfrak{A} \text{ is an M}(Q)\text{-collection} \tag{12}$$

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*) namely, $[\ ]_Q$ in $\exp \exp Q$ is a particular case of that transformation which at a partially ordered set $(M, \leq)$ transforms $A \subset M$ to the smallest end containing $A$ (i.e. $\{m \mid m \in M,\ a \leq m$ for some $a \in A\}$); here $M = \exp Q$ and $\leq$ is the (restricted) inclusion.

holds, M-collections are exactly closed sets of that topological space. Hence the set of all M-collections is closed under the general intersection, but it is closed under the general union (according to (11)), too, i.e. the set of all M-collections is a complete lattice with respect to $\subset, \bigcup, \bigcap$. $\exp Q$ $(\emptyset)$ is the greatest (the smallest) element of this lattice.

$\exp Q$ is the only singular M-collection (SM-collection); $(\exp Q) - \{\emptyset\}$ is the greatest regular M-collection (RM-collection); $\{Q\}$ is the smallest non-empty M-collections. Clearly the set of all RM-collections is a complete lattice with respect to $\subset, \bigcup, \bigcap$.

We shall not present in particular the results which follow from the above for the induced concepts, i.e. the induced operation $[\ ]_Q$ in Corr $(P, \exp Q)$, and $\mathrm{M}(Q)$-correspondences — or shortly M-*correspondences* (M — monotony, cf. [9], p. 313). Let us only note that

(13) $u$ is an M-correspondence $\Leftrightarrow$ if $A \subset B \subset Q$, then $uA \subset uB$

(14) $$[u]_Q A = \bigcup_{B \subset A} uB \text{ for all } A \subset Q$$

hold. (To prove it is simple, e.g. for (14): $x \in [u]_Q A \Leftrightarrow x[u]_Q \ni A \Leftrightarrow$ $\Leftrightarrow [xu]_Q \ni A \Leftrightarrow$ there exists $B \subset A$ such that $xu \ni B \Leftrightarrow$ there exists $B \subset A$ such that $x \in uB \Leftrightarrow x \in \bigcup_{B \subset A} uB$.)

**13.** By the *type* of a correspondence $u$ we understand $(P, \{x \mid x \in P, xu = \emptyset\})$. If $u$ is an M-correspondence, then for every $x \in P$ $xu$ is an M-collection, thus $xu \neq \emptyset$ iff $xu \ni Q$, hence

(15) $u$ is an M-correspondence $\Rightarrow (P, P - uQ)$ is the type of $u$.

**14.** We say that collections $\mathfrak{A}, \mathfrak{B}$ are M-*equivalent* (or $\mathrm{M}(Q)$-equivalent) — and we write $\mathfrak{A} \approx \mathfrak{B}$ — iff $[\mathfrak{A}]_Q = [\mathfrak{B}]_Q$. That decomposition on $\exp \exp Q$ which is determined by the equivalence relation $\approx$ will be called the M-*decomposition* or $\mathrm{M}(Q)$-decomposition (on $\exp \exp Q$). Clearly $\{\emptyset\}$ is a class of the M-decomposition, hence each of the other classes of this decomposition contains only non-empty collections. Further, the set of all singular $(Q)$-collections is a class of the M-decomposition, hence each of the other classes of this decomposition contains only regular collections.

Each class of the M-decomposition contains exactly one M-collection (because $[[\mathfrak{A}]_Q]_Q = [\mathfrak{A}]_Q$), thus M-collections are certain natural "*representatives*" *of classes of the* M-*decomposition* on $\exp \exp Q$.

Let us mention the following corollaries which deal with the corresponding concepts for correspondences — with the induced equivalence relation $\approx$ (on Corr $(P, \exp Q)$), and with the M-*decomposition* (or $\mathrm{M}(Q)$-decomposition, i.e. the decomposition on Corr $(P, \exp Q)$ which is determined by this $\approx$):

(16) $$u \approx v \Rightarrow u \text{ and } v \text{ have the same type;}$$

(17) $$u \approx v,\ u \text{ is regular} \Rightarrow v \text{ is regular;}$$

each class of the M-decomposition on Corr $(P, \exp Q)$ contains exactly one M-correspondence, i.e. M-correspondences are certain natural *"representatives" of classes of the M-decomposition* on Corr $(P, \exp Q)$.

**15.** The operation $\sqcap$ is defined by

$$\bigsqcap_{j \in J} \mathfrak{C}_j := \{ \bigcap_{j \in J} C_j \mid (C_j)_{j \in J} \in \bigtimes_{j \in J} \mathfrak{C}_j \}$$

for every non-empty system $(\mathfrak{C}_j)_{j \in J}$ of collections. Naturally we denote

$$\mathfrak{C}_1 \sqcap \mathfrak{C}_2 := \{ C_1 \cap C_2 \mid C_1 \in \mathfrak{C}_1, C_2 \in \mathfrak{C}_2 \}.$$

Clearly there holds

(18) $$[\bigsqcap_{j \in J} \mathfrak{A}_j]_Q = \bigsqcap_{j \in J} [\mathfrak{A}_j]_Q \qquad (\text{for } J \neq \emptyset)$$

(similarly for the induced operation $\sqcap$ in Corr $(P, \exp Q)$). Clearly

(19) $$\bigsqcap_{j \in J} u_j \text{ is regular} \Leftrightarrow \text{if } (A_j)_{j \in J} \in (\exp Q)^J,\ \bigcap_{j \in J} A_j = \emptyset, \text{ then } \bigcap_{j \in J} u_j A_j = \emptyset \qquad (\text{for } J \neq \emptyset)$$

We say that a system $(\mathfrak{A}_j)_{j \in J}$ $((u_j)_{j \in J})$ with $J \neq \emptyset$ is *conjugate* iff $\bigsqcap_{j \in J} \mathfrak{A}_j$ $(\bigsqcap_{j \in J} u_j)$ is regular. From 14, and from (18) we conclude

(20) $(\mathfrak{A}_j)_{j \in J}$ $((u_j)_{j \in J})$ is conjugate $\Leftrightarrow$ $([\mathfrak{A}_j]_Q)_{j \in J}$ $([u_j]_Q)_{j \in J})$ is conjugate (where $J \neq \emptyset$).

**16.** The norm $\| \ \|$ applied only to $(Q)$-collections can be considered as a mapping of $\exp \exp Q$ into $\exp Q$. Now we introduce the corresponding notion for Corr $(P, \exp Q)$ : we define to $u \in$ Corr $(P, \exp Q)$ *its generalized graph* $\Gamma_u$ as the element of Corr $(Q, P)$ which satisfies the condition

$$\Gamma_u x = \| xu \| \qquad \text{for all} \quad x \in P.$$

In the following elements of Corr $(Q, P)$ are sometimes called *generalized graphs*, and $\Gamma$ denotes a generalized graph. We say that $\Gamma$ is *$P_0$-ended* iff $\{x \mid x \in P,\ \Gamma x = \emptyset\} = P_0$. Every $P_0$-ended generalized graph $\Gamma$ is that of some correspondence $u$ having the type $(P, P_0)$; and if $u$ is a correspondence having the type $(P, P_0)$, then the generalized graph of $u$ is $P_0$-ended. Let us put $\mathbf{L} := (Q \times (P - P_0), Q, P)$; clearly, $\mathbf{L}$ is the *greatest* (under $\subset$ in Corr $(Q, P)$) $P_0$-ended generalized graph:

$$\mathbf{L}x = \begin{cases} Q \\ \emptyset \end{cases} \quad \text{for} \begin{cases} x \in P - P_0 \\ x \in P_0 \end{cases}.$$ Hence, if $u$ is an $\mathbf{M}(Q)$-correspondence of the type $(P, P_0)$, then $\mathbf{L}$ is its generalized graph.

**17.1.** In the following let $Y$, $Y_j \subset Q$. We say that $\mathfrak{A}$ is *$Y$-generable* iff there exists $\mathfrak{B}$ such that $\| \mathfrak{B} \| = Y$, $[\mathfrak{B}]_Q = \mathfrak{A}$. Hence, if $\mathfrak{A}$ is $Y$-generable, then $\mathfrak{A}$ is an M-collection; and if $\mathfrak{A}$ is some M-collection, then $\mathfrak{A}$ is $Q$-generable for $\mathfrak{A} \neq \emptyset$, and it is $\emptyset$-generable for $\mathfrak{A} = \emptyset$ ($\mathfrak{A} = [\mathfrak{A}]_Q$). Clearly, an M-collection $\mathfrak{A}$ is $Y$-generable iff the class of the M-decomposition which is represented by $\mathfrak{A}$ (see 14) contains a collection having the norm $Y$. Further there holds:

(21) $\mathfrak{A}$ is an M-collection $\Rightarrow$ ($\mathfrak{A}$ is $Q$-generable $\Leftrightarrow \mathfrak{A} \neq \emptyset \vee Q = \emptyset$)

(22) $\mathfrak{A}$ is an M-collection $\Rightarrow$ ($\mathfrak{A}$ is $\emptyset$-generable $\Leftrightarrow \mathfrak{A} = \emptyset \vee \mathfrak{A} = \exp Q$)

(23) $\emptyset$ is $Y$-generable $\Leftrightarrow Y = \emptyset$

**17.2. Lemma.** *A $(Q)$-collection $\mathfrak{A}$ is $Y$-generable if and only if*

$$\mathfrak{A} \textit{ is an } \text{M-}\textit{collection} \wedge \mathfrak{A} \sqcap \{Y\} \subset \mathfrak{A} \wedge (\mathfrak{A} = \emptyset \Rightarrow Y = \emptyset).$$

*Proof.* It follows from (23) that the lemma is valid for $\mathfrak{A} = \emptyset$. Now let $\mathfrak{A} \neq \emptyset$. If $\| \mathfrak{B} \| = Y$, $[\mathfrak{B}]_Q = \mathfrak{A}$, then $\mathfrak{A}$ is an M-collection, $\mathfrak{A} \sqcap \{Y\} \subset [\mathfrak{B}]_Q \sqcap [\{Y\}]_Q = [\mathfrak{B} \sqcap \{Y\}]_Q = [\mathfrak{B}]_Q = \mathfrak{A}$ (see (18)). On the other hand, if $\mathfrak{A}$ is an M-collection, $\mathfrak{A} \sqcap \{Y\} \subset \mathfrak{A}$, then $Q \in \mathfrak{A}$, and thus $\mathfrak{A} = [\mathfrak{A}]_Q \supset [\mathfrak{A} \sqcap \{Y\}]_Q = [\mathfrak{A}]_Q \sqcap [\{Y\}]_Q \supset \mathfrak{A} \sqcap \{Q\} = \mathfrak{A}$ (see (10), (18)), $Y \supset \| \mathfrak{A} \sqcap \{Y\} \| \supset \| \{Q\} \sqcap \{Y\} \| = Y$. Q.E.D.

**17.3.** From lemma 17.2 there follow some corollaries, e.g.

(24) $\mathfrak{A}$ is $Y_0$-generable, $\mathfrak{A} \neq \emptyset$, $Y_0 \subset Y \Rightarrow \mathfrak{A}$ is $Y$-generable,

(25) $\mathfrak{A}$ is $Y_j$-generable for $j = 1, 2 \Rightarrow \mathfrak{A}$ is $(Y_1 \cap Y_2)$-generable

(where $Y_j \subset Q$). (E.g. from $\mathfrak{A} \sqcap \{Y_j\} \subset \mathfrak{A}$ ($j = 1, 2$) there follows $\mathfrak{A} \sqcap \{Y_1 \cap Y_2\} = (\mathfrak{A} \sqcap \{Y_1\}) \sqcap \{Y_2\} \subset \mathfrak{A} \sqcap \{Y_2\} \subset \mathfrak{A}$, etc.) But the analogue of (25) with infinitely many $j$ does not hold:

**17.4. Example.** Let $Q = \{1, 2, \ldots\}$, $\mathfrak{A}$ be the collection of all infinite subsets of $Q$, $Y_j = \{j, j+1, j+2, \ldots\}$ ($j = 1, 2, \ldots$). Then $\mathfrak{A}$ is $Y_j$-generable for $j = 1, 2, \ldots$, but $\bigcap_{j=1}^{\infty} Y_j = \emptyset$, and $\mathfrak{A}$ is not $\emptyset$-generable.

**17.5. Lemma.**

(i) $$A \in \mathfrak{A} \Rightarrow A \cap Y \in \mathfrak{A}$$

(ii) $$\mathfrak{A} \sqcap \{Y\} \subset \mathfrak{A}$$

(iii) $$\mathfrak{A} \cap \exp Y = \mathfrak{A} \sqcap \{Y\}$$

*are equivalent assertions. If* $\mathfrak{A}$ *is an* M-*collection, then each of the following conditions is equivalent to* (*i*) — (*iii*):

(iv) $$\mathfrak{A} = [\mathfrak{A} \sqcap \{Y\}]_Q \ (= \mathfrak{A} \sqcap [\{Y\}]_Q)$$

(v) $$A \cup (Q - Y) \in \mathfrak{A} \Rightarrow A \in \mathfrak{A}$$

(vi) $$A \cup (Q - Y) \in \mathfrak{A} \Rightarrow A \cap Y \in \mathfrak{A}.$$

*Proof*. Clearly (i) and (ii) are equivalent, from (iii) (ii) follows, and (iii) follows from (ii) by means of $\mathfrak{A} \cap \exp Y \subset \mathfrak{A} \sqcap \{Y\} \subset \exp Y$. Now let $\mathfrak{A}$ be an M-collection. Then from (iv) (ii) follows ($\mathfrak{A} = [\mathfrak{A} \sqcap \{Y\}]_Q \supset \supset \mathfrak{A} \sqcap \{Y\}$), and (iv) follows from (ii) ($\mathfrak{A} = [\mathfrak{A}]_Q \supset [\mathfrak{A} \sqcap \{Y\}]_Q = = [\mathfrak{A}]_Q \sqcap [\{Y\}]_Q \supset \mathfrak{A} \sqcap \{Q\} = \mathfrak{A}$). From (vi) there immediately follow (i), (v) (as $\mathfrak{A}$ is an M-collection); if we substitute $A := A \cup (Q - - Y)$ ($A := A \cap Y$) in (i) ((v), respectively), then we get (vi). Q.E.D.

**17.6.** We say that $u$ is $\Gamma$-*generable* iff for each $x \in P$ the collection $xu$ is ($\Gamma x$)-generable (i.e. — in other words — iff there exists $v$ such that $\Gamma$ is the generalized graph of $v$, and $[v]_Q = u$). From 17.1—17.5 there follow e.g. the following corollaries:

(26) $u$ is $\Gamma$-generable $\Leftrightarrow$ ($x \in uA \Rightarrow x \in u(A \cap \Gamma x)$ (for all $x \in P$, $A \subset Q$)),

(27) $u$ is $\Gamma_0$-generable, $\Gamma_0 \subset \Gamma \Rightarrow u$ is $\Gamma$-generable,

(28) $u$ is $\Gamma_j$-generable for $j = 1, 2 \Rightarrow u$ is ($\Gamma_1 \cap \Gamma_2$)-generable,

where $u$ is an M-correspondence of the type ($P$, $P_0$), and $\Gamma$, $\Gamma_0$, $\Gamma_1$, $\Gamma_2$ are $P_0$-ended generalized graphs.

**18.0.** In 18.1—18.3 we shall deal with the *case* $P = Q$, especially with correspondences of Corr ($P$, $\exp P$); according to 8 we call them (general) *topologies* (on $P$). $u$, $v \in$ Corr ($P$, $\exp P$) in 18.1—18.3.

**18.1.** We define 1, (—1) $\in$ Corr ($P$, $\exp P$) by the conditions

$$1A = A \text{ for all } A \subset P,$$
$$(-1)\, A = P - A \text{ for all } A \subset P$$

(hence the identical mapping of $\exp P$ onto itself is the image of 1 under the canonical mapping $\rightarrow$; (—1) is considered as one symbol). There holds

(29) $$1 . u = u . 1 = u \text{ (for all } u)$$

(30) 1 is an M-correspondence

$$(31) \qquad (-1) \cdot (-1) = 1$$

**18.2.** We say that $u$ is a *Čech topology* (see [9], 2.1, or [3]) iff

$$(\mathrm{R}) \qquad u\emptyset = \emptyset$$

$$(\mathrm{M}) \qquad A \subset B \subset P \Rightarrow uA \subset uB$$

$$(\mathrm{E}) \qquad 1 \subset u$$

holds, i.e. iff $u$ is an RM($P$)-correspondence such that $A \subset uA$ for all $A \subset P$. E.g. $1$ is a Čech topology.

One says that $u$ is a *Čech derivative* iff $u$ is an RM-correspondence satisfying the condition

$$(\mathrm{D}) \qquad x \in uA \Rightarrow x \in u(A - \{x\}) \qquad (\text{for all } x \in P,\ A \subset P).$$

It is known that the mapping of the set of all Čech derivatives under which $1 \cup u$ is the image of $u$ is a one-to-one mapping onto the set of all Čech topologies (see [3], pp. D 253—254).

The following lemma deals with Čech derivatives and with the notion of generability. By *graphs* elements of the set Corr $(P, P)$ are meant (i.e. generalized graphs in the case $P = Q$). A graph *without loops* is a graph $\Gamma$ such that $x \notin \Gamma x$ for all $x \in P$.

**18.3. Lemma.** *Let $u$ be an* RM-*correspondence. Then there holds: $u$ is a Čech derivative if and only if there exists a graph $\Gamma$ without loops such that $u$ is $\Gamma$-generable.*

*Proof.* If $v$ is a $\Gamma$-generable RM-correspondence and $\Gamma$ is a graph without loops, then according to (26), (13) $x \in vA \Rightarrow x \in v(A \cap \Gamma x) \subset \subset v(A - \{x\})$, i.e. $v$ is a Čech derivative. On the other hand, let $u$ be a Čech derivative. We define a graph $\Gamma_0$ without loops by the condition

$$\Gamma_0 x = \begin{cases} P - \{x\} & \text{if } x \in uP \\ \emptyset & \text{if } x \in P - uP \end{cases}$$

; $\Gamma_0$ is $(P - uP)$-ended: if $x \in uP$, then $x \in u(P - \{x\})$, hence $P - \{x\} \neq \emptyset$ (as $u\emptyset = \emptyset$), $(P, P - uP)$ is the type of $u$ (see (15)), and if $x \in uA$, then $x \in u(A - \{x\}) = u(A \cap \Gamma_0 x)$; hence $u$ is $\Gamma_0$-generable ((26)). Q.E.D.

**19.** Now let us mention some properties of the product of correspondences. Let

$$u, u_1, u_2, u_j \in \mathrm{Corr}\,(E, F),\ v, v_j \in \mathrm{Corr}\,(G, \exp E)$$

for $j \in J$ $(J \neq \emptyset)$. (Hence e.g. "$v$ is an M-correspondence" means "$v$ is an M($E$)-correspondence", and similarly.) Clearly, there holds

(32) $v$ is an M-correspondence $\Leftrightarrow (u_1 \subset u_2 \Leftrightarrow v \,.\, u_1 \subset v \,.\, u_2)$ for each $u_1, u_2$

Further let

$$F = \exp Q.$$

Then

(33) $u, v$ are R-correspondences $\Rightarrow v \,.\, u$ is an R-correspondence;

(34) $u, v$ are M-correspondences $\Rightarrow v \,.\, u$ is an M-correspondence;

35) $(u_j)_{j \in J}$, $(v_j)_{j \in J}$ are conjugate systems $\Rightarrow (v_j \,.\, u_j)_{j \in J}$ is a conjugate system.

((32)—(34) are trivial, (35) follows from (19).)

c) *Strategies and automata*

**20.** All notions introduced in this part $c$ are defined for a fixedly chosen type $(P, P_0)$. We put $Z := P - P_0$. Elements of $P$, $P_0$, $Z$ are sometimes called *positions*, final positions, non-final positions, respectively. $\boldsymbol{P}$ is the set of all variants of the type $(P, P_0)$ (§ 1.1). We shall denote by $\mathbf{L}$ the generalized graph L of 16 in the case $Q := \boldsymbol{P}$; hence the denotation $\mathbf{L}x$ (see 16; $x \in P$) has the same sense as in § 1.1. The symbols $\boldsymbol{Z}$, $\sim$, $\overset{\circ}{\sim}$ have the senses introduced in § 1. Further in part $c$ $u, v \in$ $\in \operatorname{Corr}(P, \exp P)$.

**21.1.** We put

$$\boldsymbol{S} := ((\exp P) - \{\emptyset\})^{\boldsymbol{Z}};$$

elements of $\boldsymbol{S}$ are called ***strategies*** (or: *free strategies* of the type $(P, P_0)$). The letter $\boldsymbol{\sigma}$ always means a strategy, $\boldsymbol{\sigma z}$ is the image of $\boldsymbol{z} \in \boldsymbol{Z}$ under $\boldsymbol{\sigma}$. Further we define the set of all strategies which are $\sim$ *-acceptable*:

$$\sim \boldsymbol{S} := \{\boldsymbol{\sigma} \mid \boldsymbol{\sigma} \in \boldsymbol{S}, \text{ if } \boldsymbol{z}^1, \boldsymbol{z}^2 \in \boldsymbol{Z}, \boldsymbol{z}^1 \sim \boldsymbol{z}^2, \text{ then } \boldsymbol{\sigma z}^1 = \boldsymbol{\sigma z}^2\}$$

$\sim \boldsymbol{S}$ is considered as one symbol). Instead of $\overset{\circ}{\sim} \boldsymbol{S}$ we write $\overset{\circ}{\boldsymbol{S}}$, too; $\overset{\circ}{\boldsymbol{S}}$ is the set of all *plain strategies.*

**21.2.** We denote

$$\overset{\circ}{S} := ((\exp P) - \{\emptyset\})^{Z},$$

$\sigma$ will always be an element of $S$. We can define a one-to-one mapping of $\overset{\circ}{S}$ onto $\overset{\circ}{\boldsymbol{S}}$ such that the image of $\sigma$ is the element $\boldsymbol{\sigma} \in \boldsymbol{S}$ (we denote $\sigma \longleftrightarrow \boldsymbol{\sigma}$) for which $\boldsymbol{\sigma z} = \sigma\varkappa(\boldsymbol{z})$ for all $\boldsymbol{z} \in \boldsymbol{Z}$. Often we shall not distinguish between an element $\sigma \in \overset{\circ}{S}$ and its image $\boldsymbol{\sigma}$ (e.g. $\mathrm{s}(\sigma)$ is the same as $\mathrm{s}(\boldsymbol{\sigma})$

for $\sigma \longleftrightarrow \boldsymbol{\sigma}$, and similarly), and elements of $\overset{\circ}{S}$ we shall call ***plain strategies***, too.

**21.3.** Now we introduce

$$\mathrm{s}(\boldsymbol{\sigma}) := \{\mathbf{x} \mid \mathbf{x} \in \mathbf{P},\, x_{k+1} \in \boldsymbol{\sigma}(x_0, \ldots, x_k) \text{ for all } k < l(\mathbf{x})\};$$

hence

$$\mathrm{s}(\sigma) = \{\mathbf{x} \mid \mathbf{x} \in \mathbf{P},\, x_{k+1} \in \sigma x_k \text{ for all } k < l(\mathbf{x})\}.$$

$\mathrm{s}(\boldsymbol{\sigma})$ is the set of all variants which *comply with* $\boldsymbol{\sigma}$. The set of all variants which comply with $\boldsymbol{\sigma}$ and which begin in a position $x$ is denoted by $\mathrm{s}(x, \boldsymbol{\sigma})$, i.e.

$$\mathrm{s}(x, \boldsymbol{\sigma}) = \mathrm{s}(\boldsymbol{\sigma}) \cap \mathbf{L}x;$$

$(s\,(x, \boldsymbol{\sigma}))_{x \in P}$ is a partition of $s(\boldsymbol{\sigma})$ (cf. § 1.4). It is easy to prove that always $\mathrm{s}(x, \boldsymbol{\sigma}) \neq \emptyset$, cf. lemma 21.4.

We say that a non-empty system $(\boldsymbol{\sigma}_j)_{j \in J}$ of strategies is *conjugate* iff $\bigcap_{j \in J} \boldsymbol{\sigma}_j \mathbf{z} \neq \emptyset$ for all $\mathbf{z} \in \mathbf{Z}$. (E.g., if $(u_j)_{j \in J}$ is a game system (29.1) and $\boldsymbol{\sigma}_j \in S(u_j)$ (23.1) for all $j \in J$, then $(\boldsymbol{\sigma}_j)_{j \in J}$ is a conjugate system of strategies.) By an induction a variant belonging to $\bigcap_{j \in J} \mathrm{s}(x, \boldsymbol{\sigma}_j)$ can be constructed if $(\boldsymbol{\sigma}_j)_{j \in J}$ is conjugate:

**21.4. Lemma.** *If $(\boldsymbol{\sigma}_j)_{j \in J}$ is a (non-empty) conjugate system of strategies, then*

$$\bigcap_{j \in J} \mathrm{s}(x, \boldsymbol{\sigma}_j) \neq \emptyset$$

*for all $x \in P$. Especially $s(x, \boldsymbol{\sigma}) \neq \emptyset$ for all strategies $\boldsymbol{\sigma}$.*

**22.** Let $\mathscr{R} = (P, R, \varrho)$ be an automaton of the type $(P, P_0)$. We define the mapping $\boldsymbol{\chi}_{\mathscr{R}}$ which maps the set of all $\mathscr{R}$-tactics into $S$ by the condition

$$\boldsymbol{\chi}_{\mathscr{R}}(\boldsymbol{\tau})\, \mathbf{z} = \varrho(\varkappa(\mathbf{z}), \boldsymbol{\tau}\mathbf{z})$$

for all $\mathbf{z} \in \mathbf{Z}$. Evidently there holds

(36) $\boldsymbol{\tau}$ is $\sim$-acceptable $\Rightarrow \boldsymbol{\chi}_{\mathscr{R}}(\boldsymbol{\tau})$ is $\sim$-acceptable

(37) $\mathrm{t}_{\mathscr{R}}(x, \boldsymbol{\tau}) = \mathrm{s}(x, \boldsymbol{\chi}_{\mathscr{R}}(\boldsymbol{\tau}))$ , $\mathrm{t}_{\mathscr{R}}(\boldsymbol{\tau}) = \mathrm{s}(\boldsymbol{\chi}_{\mathscr{R}}(\boldsymbol{\tau}))$
for all $\mathscr{R}$-tactics $\boldsymbol{\tau}$ $(x \in P)$.

Now it is natural to introduce the following notions.

**23.1.** Let $u$ be a regular correspondence of the type $(P, P_0)$. We call

$$S(u) := \{\boldsymbol{\sigma} \mid \boldsymbol{\sigma} \in S, \quad \boldsymbol{\sigma}\mathbf{z} \in \varkappa(\mathbf{z})\, u \text{ for all } \mathbf{z} \in \mathbf{Z}\}$$

the set of all *u-strategies*. Then

$$\sim S(u) := (\sim S) \cap S(u)$$

is the set of all $\sim$-acceptable $u$-strategies; analogically to the "free case" we define

$$\mathring{S}(u) := \mathring{\sim} S(u),$$

$$\mathring{S}(u) := \{\sigma \mid \sigma \in \mathring{S}, \qquad \sigma z \in zu \text{ for all } z \in Z\},$$

henceforth, if $\sigma \leftarrow\rightarrow \boldsymbol{\sigma}$, then $\sigma \in \mathring{S}(u)$ iff $\boldsymbol{\sigma} \in \mathring{S}(u)$. There holds (for all memory relations $\sim$ ) $\sim S(u) \neq \emptyset$.

23.2. Let $u$ be a regular correspondence of the type $(P, P_0)$. At $u$ we define the *strategic-predicate* $\boldsymbol{\Sigma}_u$ on the cartesian product of the set of all memory relations and the set of all aims: $\boldsymbol{\Sigma}_u(\sim, \mathbf{A})$ means the assertion: there exists a $\sim$-acceptable $u$-strategy $\boldsymbol{\sigma}$ such that $\mathrm{s}(\boldsymbol{\sigma}) \subset \mathbf{A}$. Cf. § 1:4—5.

24.1. Let $\mathscr{R} = (P, R, \varrho)$ be an automaton of the type $(P, P_0)$. Under the *correspondence of* $\mathscr{R}$ there is meant the correspondence $u_{\mathscr{R}} \in$ $\in \operatorname{Corr}(P, \exp P)$ such that

$$xu_{\mathscr{R}}A \Leftrightarrow \text{there exists } r \in R(x) \text{ such that } \varrho(x, r) = A$$

for all $x \in P$, $A \subset P$. Clearly, $u_{\mathscr{R}}$ is a regular correspondence of the type $(P, P_0)$, and evidently there holds

24.2. **Lemma.** *The restriction* *) *of* $\chi_{\mathscr{R}}$ *on the set of all* $\sim$*-acceptable* $\mathscr{R}$*-tactics is a mapping onto* $\sim \mathbf{S}(u_{\mathscr{R}})$. *Especially,* $\chi_{\mathscr{R}}$ *is a mapping of the set of all* $\mathscr{R}$*-tactics onto* $\mathbf{S}(u_{\mathscr{R}})$.

24.3. (A **corollary** of (37) and lemma 24.2.)

$$\mathbf{T}_{\mathscr{R}} \equiv \boldsymbol{\Sigma}_{u_{\mathscr{R}}} \tag{38a}$$

for every automaton $\mathscr{R}$ of the type $(P, P_0)$, where $\equiv$ means that $(\mathbf{T}_{\mathscr{R}}(\sim, \mathbf{A}) \Leftrightarrow \boldsymbol{\Sigma}_{u_{\mathscr{R}}}(\sim, \mathbf{A}))$ for every memory relation $\sim$ and each $\mathbf{A} \subset \mathbf{P}$.

25. Let $u$ be a regular correspondence of the type $(P, P_0)$. We construct the *automaton* $\mathscr{R}_u = (P, R_u, \varrho_u)$ in this way: $R_u(x) = xu$, $\varrho_u(x, r) = r$ for all $x \in P$, $r \in R_u(x)$. Evidently, $\mathscr{R}_u$ is an automaton of the type $(P, P_0)$ such that

$$u = u_{\mathscr{R}_u}.$$

From this and from (38a) there follows

$$\boldsymbol{\Sigma}_u \equiv \mathbf{T}_{\mathscr{R}_u} \tag{38b}$$

---

*) By the *restriction* of a mapping $f: A \to B$ on $C \subset A$ the mapping $f \mid C : C \to B$ such that $(f \mid C)(x) = f(x)$ (for all $x \in C$) is meant.

**26.0.** For $x \in P$, $A \subset P$ we define

$$\boldsymbol{A}(x, A) := \{\mathbf{x} \mid \mathbf{x} \in \boldsymbol{P},\ x_0 = x \Rightarrow (l(\mathbf{x}) > 0, \quad x_1 \in A)\}.$$

**26.1. Theorem.** *Let both $u$ and $v$ be regular and have the type $(P, P_0)$. Then*

(i) $$x[u]PA \Leftrightarrow \boldsymbol{\Sigma}_u(\sim, \boldsymbol{A}(x, A))$$

*(for all $x \in P$, $A \subset P$ and every $\sim$),*

(ii) $$\boldsymbol{\sigma}^1 \in \sim \boldsymbol{S}(u),\ u \approx v \Rightarrow \text{ there exists } \boldsymbol{\sigma}^2 \in \sim \boldsymbol{S}(v) \text{ such that } \boldsymbol{\sigma}^2\mathbf{z} \subset \boldsymbol{\sigma}^1\mathbf{z} \text{ for each } \mathbf{z} \in \boldsymbol{Z}$$

*(for every $\sim$ ),*

(iii) $$u \approx v \Leftrightarrow \boldsymbol{\Sigma}_u \equiv \boldsymbol{\Sigma}_v$$

*Proof.* (i) and (ii) are simple. — If $u \approx v$ and e. g. $\boldsymbol{\Sigma}_u(\sim, \boldsymbol{A})$ is valid for some $\boldsymbol{A}$, then $s(\boldsymbol{\sigma}^1) \subset \boldsymbol{A}$ for a suitable $\boldsymbol{\sigma}^1 \in \sim \boldsymbol{S}(u)$, hence $s(\boldsymbol{\sigma}^2) \subset s(\boldsymbol{\sigma}^1) \subset \boldsymbol{A}$ for a suitable $\boldsymbol{\sigma}^2 \in \sim \boldsymbol{S}(v)$ (cf. (ii)), therefore $\boldsymbol{\Sigma}_v(\sim, \boldsymbol{A})$ is valid. By exchanging $u := v$, $v := u$ we obtain $\boldsymbol{\Sigma}_v(\sim, \boldsymbol{A}) \Rightarrow \Rightarrow \boldsymbol{\Sigma}_u(\sim, \boldsymbol{A})$. Thus $u \approx v$ implies $\boldsymbol{\Sigma}_u \equiv \boldsymbol{\Sigma}_v$. On the other hand, if $\boldsymbol{\Sigma}_u \equiv \boldsymbol{\Sigma}_v$, then from this and (i) $[u]_P = [v]_P$ follows, i. e. $u \approx v$. Therefore, (iii) holds. Q.E.D.

**26.2.** Let $\Gamma \in \operatorname{Corr}(P, P)$ be a $P_0$-ended graph; we put

$$\boldsymbol{X}_\Gamma := \{\mathbf{x} \mid \mathbf{x} \in \boldsymbol{P},\quad x_{k+1} \in \Gamma x_k \text{ for all } k < l(\mathbf{x})\}$$

(if $\Gamma$ is fixedly chosen, then we often use the denotation $\boldsymbol{X}$ instead of $\boldsymbol{X}_\Gamma$). Evidently for arbitrary graphs $\Gamma_1$, $\Gamma_2$ there holds

(39) $$\Gamma_1 = \Gamma_2 \Leftrightarrow \boldsymbol{X}_{\Gamma_1} = \boldsymbol{X}_{\Gamma_2}.$$

By the *graph $\Gamma_{\mathscr{R}}$ of an automaton $\mathscr{R}$* the graph of $u_{\mathscr{R}}$ is meant. Clearly (see 7 of § 1)

(40) $$\boldsymbol{X}_{\mathscr{R}} = \boldsymbol{X}_{\Gamma_{\mathscr{R}}}$$

for every automaton $\mathscr{R}$.

**26.3.** If at an automaton $\mathscr{R}$ only questions which can be expressed in terms of $\mathbf{T}_{\mathscr{R}}$ and $\Gamma_{\mathscr{R}}$ are studied (compare with 8.1—8.2 in § 1), then it is sufficient to know only the correspondence $u_{\mathscr{R}}$ of $\mathscr{R}$, since $\mathbf{T}_{\mathscr{R}} \equiv \equiv \boldsymbol{\Sigma}_{u_{\mathscr{R}}}$, $\Gamma_{\mathscr{R}} = \Gamma_{u_{\mathscr{R}}}$. Moreover, from that point of view it is sufficient to know only $\Gamma_{\mathscr{R}}$ and the class of the M-decomposition (on $\operatorname{Corr}(P, \exp P)$) which contains $u_{\mathscr{R}}$, since all correspondences of that class are regular (and they have the same type), and the strategic predicates of them are equal (theorem 26.1). Instead of "regular" classes of the M-decomposition we can take RM-correspondences as their natural representatives (see 14). We shall call RM-correspondences (of $\operatorname{Corr}(P, \exp P)$) ***game***

***correspondences***. From the above (especially from 24.1, 25) there immediately follows

**26.4. Lemma.** *Let $v$ be a correspondence, $\Gamma$ be a graph. Then $v$ is a $\Gamma$-generable game correspondence if and only if there exists an automaton $\mathscr{R} = (P, R, \varrho)$ such that $v = [u_{\mathscr{R}}]_P$, $\Gamma = \Gamma_{\mathscr{R}}$.*

d) *Regular systems. RSN-games*

**27.1.** In 27.1 let $\mathscr{C} = (\mathfrak{C}_j)_{j \in J}$ be a non-empty *system* of collections. If $\mathscr{C}$ is conjugate, then either some $\mathfrak{C}_{j_0}$ is empty or all $\mathfrak{C}_j$ are non-empty and regular; and if moreover all $\mathfrak{C}_j$ have the same norm $C$, then either $C \neq \emptyset$, hence all $\mathfrak{C}_j$ are non-empty and regular, or $C = \emptyset$, hence $\mathscr{C}_j \in \{\emptyset, \{\emptyset\}\}$ for all $j \in J$. We say that $\mathscr{C}$ is *regular* iff it is conjugate, all $\mathfrak{C}_j$ have the same norm (we call this norm the *norm* of $\mathscr{C}$), and all $\mathfrak{C}_j$ are regular; from the above we get

(41) $\mathscr{C}$ is regular $\Leftrightarrow \mathscr{C}$ is conjugate, all $\mathfrak{C}_j$ have the same norm, and if $C = \emptyset$, then $\mathfrak{C}_j = \emptyset$ for all $j \in J$

Further, clearly there holds

(42) $$J = \{j\} \Rightarrow (\mathscr{C} \text{ is regular} \Leftrightarrow \mathfrak{C}_j \text{ is regular})$$

(43) $$\mathscr{C} \text{ is regular} \Rightarrow \mathfrak{C}_j \neq \emptyset \text{ for all } j \in J \vee \mathfrak{C}_j = \emptyset \text{ for all } j \in J$$

(44) $$\mathscr{C} \text{ is regular}, \emptyset \neq J_1 \subset J \Rightarrow (\mathfrak{C}_j)_{j \in J_1} \text{ is regular}$$

**27.2. Lemma.** *Let $\mathscr{A} = (\mathfrak{A}_j)_{j \in J}$ be a non-empty system of $(Q)$-collections, let $J_2$ be a set. Then there holds*

(45) *If $\mathscr{A}$ is regular, $J \subset J_2$, then there exists a regular system $(\mathfrak{B}_j)_{j \in J_2}$ of $(Q)$-collections such that $\mathfrak{A}_j = \mathfrak{B}_j$ for all $j \in J$*

(46) $$\mathscr{A} \textit{ is regular} \Rightarrow ([\mathfrak{A}_j]_Q)_{j \in J} \textit{ is regular}$$

*Proof.* Clearly (46) holds (see (20) etc.). Under the suppositions of (45) we put $\mathfrak{B}_j := \emptyset$ for all $j \in J$ if the norm $A$ of $\mathscr{A}$ is empty, and we put $\mathfrak{B}_j := \{A\}$ for $j \in J_2 - J$ (and $\mathfrak{B}_j := \mathfrak{A}_j$ for $j \in J$) if $A \neq \emptyset$. Clearly (45) holds. Q.E.D.

**27.3.** The property "to be regular" for systems of $(Q)$-collections with a (non-empty) set $J$ of all indices can be considered as a relation of the type $J$ in exp exp $Q$. According to 7 the induced property (i.e. "to be *regular*" for *systems* of elements of Corr $(P, \exp Q)$ with the set $J$ of all indices) can be introduced. Let in 27.3 $\mathscr{U} = (u_j)_{j \in J}$ be a non-empty system of elements of Corr $(P, \exp Q)$. Then

$$\text{(47)} \qquad \mathscr{U} \text{ is regular} \Leftrightarrow \mathscr{U} \text{ is conjugate, all } u_j \text{ are regular and have the same graph}$$

Thus, if $\mathscr{U}$ is regular, then all $u_j$ have the same type, and the (common) type of all correspondences of $\mathscr{U}$ is called the *type* of $\mathscr{U}$.

Evidently, the other properties (42), (44) — (46) hold analogously for systems of correspondences (instead of systems of collections).

**28. Theorem.** *Let* $(P, P_0)$ *be a type,* $\mathscr{U} = (u_j)_{j \in J}$ *be a non-empty system of elements of* Corr $(P, \exp P)$. *Then the following assertions are equivalent:*

(A) *There exists a* $J$*-automaton* $\mathscr{R} = (P, R, \varrho)$ *of the type* $(P, P_0)$ *such that* $u_j = u_{\mathscr{R}_j}$ *for all* $j \in J$.

(B) $\mathscr{U}$ *is a regular system of the type* $(P, P_0)$.

*Proof.*

*1.* Let (A) hold. Then all $u_j (= u_{\mathscr{R}_j})$ are regular (24.1), and all $u_j$ have the same graph, namely the graph of $\mathscr{R}$ (26.2; (1) of § 1). Now let $(A_j)_{j \in J} \in (\exp P)^J$, $\bigcap_{j \in J} A_j = \emptyset$, $x \in \bigcap_{j \in J} u_j A_j$. Then there exists $r = (r_j)_{j \in J} \in R(x)$ such that $A_j = \varrho_j(x, r_j)$ for all $j \in J$ (24.1; 6.1 of § 1), hence $\emptyset = \bigcap_{j \in J} A_j = \bigcap_{j \in J} \varrho_j(x, r_j) \supset \varrho(x, r) \neq \emptyset$ ((2) of § 1); from this contradiction and from (19) it follows that the system $\mathscr{U}$ is conjugate. Consequently, $\mathscr{U}$ is regular, i.e. (B) holds.

*2.* Let (B) hold. If $A_j \in xu_j$ for all $j \in J$ and for some $x \in P$, then $\bigcap_{j \in J} A_j \neq \emptyset$, since $\mathscr{U}$ is conjugate (see (19)). Now we may define the mapping $\varrho : D \to (\exp P) - \{\emptyset\}$, where $D := \{(x, r) \mid x \in P,\ r \in \times_{j \in J} xu_j\}$, by $\varrho(x, r) := \bigcap_{j \in J} \mathrm{pr}_j r$. We put $R(x) := \times_{j \in J} xu_j$ $(x \in P)$, then $\mathscr{R} := (P, R, \varrho)$ is a $J$-automaton of the type $(P, P_0)$. According to (42) $R_j(x) = xu_j$ for $j \in J$ (see 6.1 of § 1).

Let $x \in P$, $i \in J$, $A \in R_i(x)$. Then $A \supset \bigcup_{\substack{r \in R(x) \\ \mathrm{pr}_i r = A}} \bigcap_{j \in J} \mathrm{pr}_j r = \varrho_i(x, A)$. If $y \in A$, then $y \in \| xu_j \|$ for all $j \in J$, and hence $r \in \times_{j \in J} xu_j = R(x)$ exists such that $\mathrm{pr}_i r = A$, $\mathrm{pr}_j r \ni y$ for all $j \in J$, and thus $y \in \bigcap_{j \in J} \mathrm{pr}_j r \subset \varrho_i(x, A)$. Consequently $\varrho_i(x, A) = A$ for all $x \in P$, $j \in J$,

$A \in R_j(x)$. Now $xu_{\mathscr{R}_j} = \{\varrho(x, A) \mid A \in R_j(x)\} = \{A \mid A \in xu_j\} = xu_j$ for all $x \in P$, $j \in P$, i.e. $u_{\mathscr{R}_j} = u_j$ for all $j \in J$. Thus (A) holds.

Q.E.D.

**29.0.** We say that a non-empty system of collections (correspondences) is an RM-*system* iff this system is regular and all its elements are M-collections (M-correspondences).

We say that a system of correspondences is $\Gamma$-*generable* iff all elements of this system are $\Gamma$-generable.

**29.1.** If at a $J$-automaton $\mathscr{R}$ there are studied only questions which can be expressed in terms of $(\mathbf{T}_{\mathscr{R}_j})_{j \in J}$, $\Gamma_{\mathscr{R}}$, then it is sufficient to know only $(u_{\mathscr{R}_j})_{j \in J}$ (since $\mathbf{T}_{\mathscr{R}_j} \equiv \boldsymbol{\Sigma}_{u_{\mathscr{R}_j}}$, $\Gamma_{\mathscr{R}} = \Gamma_{\mathscr{R}_j} = \Gamma_{u_{\mathscr{R}_j}}$); moreover, it is even sufficient to know only $([u_{\mathscr{R}_j}])_P)_{j \in J}$, $\Gamma_{\mathscr{R}}$ (see theorem 26.1). We shall call RM-systems of elements of Corr $(P, \exp P)$ ***game systems***. (Compare with 26.3.) Clearly, all elements of a game system are game correspondences. By means of theorem 28 we get

**29.2. Lemma.** *Let $\mathscr{V} = (v_j)_{j \in J}$ be a non-empty system of elements of* Corr $(P, \exp P)$, $\Gamma$ *be a graph. Then $\mathscr{V}$ is $\Gamma$-generable game system if and only if there exists a $J$-automaton $\mathscr{R} = (P, R, \varrho)$ such that $v_j = [u_{\mathscr{R}_j}]_P$ (for all $j \in J$), $\Gamma = \Gamma_{\mathscr{R}}$.*

**30.1.** By the *reduction* of an SN-game $\mathscr{G} = (\mathscr{R}, (f_j)_{j \in J})$ (see 8.1 of § 1) the triple

$$\mathscr{H}(\mathscr{G}) = (([u_{\mathscr{R}_j}]_P)_{j \in J},\ \Gamma_{\mathscr{R}},\ (f_j)_{j \in J})$$

is meant. From 29.1 it follows that from the point of view of § 1.8.1 it is possible to consider only $\mathscr{H}(\mathscr{G})$ instead of $\mathscr{G}$.

**30.2.** By a ***reduced simultaneous nondeterministic game*** (RSN-game) of the type $(P, P_0)$ we shall understand a triple

$$\mathscr{H} = ((u_j)_{j \in J},\ \Gamma,\ (f_j)_{j \in J}),$$

where $\Gamma \in$ Corr $(P, P)$ is a $P_0$-ended graph, $(u_j)_{j \in J}$ is a $\Gamma$-generable game system, and all $f_j$ are real functions on $\mathbf{X}_\Gamma$. From lemma 29.2 it immediately follows that *a triple is an* RSN-*game if and only if it is the reduction of some* SN-*game*. From this we get the natural *interpretation* of that RSN-game $\mathscr{H}$: $J$ is the set of all players, $u_j$ is the game correspondence of the player $j$ ($xu_jA$ means: at $x$ the player $j$ can *immediately enforce* the set $A$), $\Gamma$ is the graph of $\mathscr{H}$, $\mathbf{X}_\Gamma$ is the set of all plays of $\mathscr{H}$, $f_j$ is the pay-off function of the player $j$ in $\mathscr{H}$.

**30.3.** A $J$-automaton (of the type $(P, P_0)$) will be also called an *SN game structure* (of the type $(P, P_0)$); under an *RSN game structure* (of the type $(P, P_0)$) there will be meant a pair $(\mathscr{U}, \Gamma)$ where $\Gamma \in$ $\in$ Corr $(P,P)$ is a $P_0$-ended graph and $\mathscr{U}$ is a $\Gamma$-generable game system. The interpretations in both the cases are natural, cf. § 1.8.1, § 2.30.2. (Thus a ***game structure*** is a "game without pay-off functions".)

## § 3. $\sim$-STRATEGIC CORRESPONDENCES. "ABSOLUTE" STRATEGIES

**0.** In § 3 we suppose the same as in § 2.20, especially $u, v \in$ $\in$ Corr $(P, \exp P)$ are regular correspondences of the type $(P, P_0)$. $\Gamma$ will be a graph. For $\Gamma$ we define the *corresponding generalized graph* $\boldsymbol{\Gamma}_\Gamma$ (or only $\boldsymbol{\Gamma}$) by

$$\boldsymbol{\Gamma} \in \text{Corr}\,(\mathbf{P}, P)$$
$$\boldsymbol{\Gamma} x = \mathbf{X} \cap \mathbf{L}x \text{ for all } x \in P,$$

where $\mathbf{X} := \mathbf{X}_\Gamma$. (Hence $\boldsymbol{\Gamma} x$ can be considered as the set of all plays which begin from $x$, if $\Gamma$ is the graph of a game under consideration.)

### a) $\sim$-*strategic correspondences.* $\sim$-*strategic collections*

**1.** We have called elements of exp $\mathbf{P}$ *aims*. We say that an aim $\mathbf{A}$ *can be enforced by a (free) strategy* $\boldsymbol{\sigma}$ *from* $x(\in P)$ iff $\mathrm{s}(x, \boldsymbol{\sigma}) \subset \mathbf{A}$. Thus $[\{\mathrm{s}(x, \boldsymbol{\sigma})\}]_{\mathbf{P}}$ is the collection of all aims which can be enforced by $\boldsymbol{\sigma}$ from $x$. Hence $\mathbf{A}$ can be enforced by $\boldsymbol{\sigma}$ from all positions iff $\mathrm{s}(\boldsymbol{\sigma}) \subset \mathbf{A}$ (since $\bigcup_{x \in P} \mathrm{s}(x, \boldsymbol{\sigma}) =$ $= \bigcup_{x \in P} (\mathrm{s}(\boldsymbol{\sigma}) \cap \mathbf{L}x) = \mathrm{s}(\boldsymbol{\sigma})$). (Cf. § 1.4.)

We say that $\mathbf{A}$ *can be* $(\sim, u)$ *-enforced from* $x$ iff there exists $\boldsymbol{\sigma} \in \sim\!\boldsymbol{S}(u)$ such that $\mathbf{A}$ can be enforced by $\boldsymbol{\sigma}$ from $x$.

**2.** For a memory relation $\sim$ and for regular correspondences of Corr $(P, \exp P)$ we define the corresponding $\sim$-*strategic correspondences* (we shall denote them by the symbol $\sim$ of the memory relation and by the corresponding thick small Latin letter, e.g. $\sim\!\boldsymbol{v}$ for $\sim, v$) in the following way:

$$\sim\!\boldsymbol{u} \in \text{Corr}\,(P, \exp \mathbf{P})$$

$$x(\sim\!\boldsymbol{u})\,\mathbf{A} \Leftrightarrow \mathrm{s}(x, \boldsymbol{\sigma}) \subset \mathbf{A} \text{ for some } \boldsymbol{\sigma} \in \sim\!\boldsymbol{S}(u)$$

for all $x \in P$, $\mathbf{A} \subset \mathbf{P}$ (i.e. $x(\sim\!\boldsymbol{u})\mathbf{A}$ iff $\mathbf{A}$ can be $(\sim, u)$-enforced from $x$). Clearly

$$u \subset v \Rightarrow \sim\!\boldsymbol{u} \subset \sim\!\boldsymbol{v} \tag{1}$$

(since $\sim\!\boldsymbol{S}(u) \subset \sim\!\boldsymbol{S}(v)$ if $u \subset v$),

(2) $$x(\sim \boldsymbol{u}) = [\{s(x, \boldsymbol{\sigma}) | \boldsymbol{\sigma} \in \sim S(u)\}]_{\mathbf{P}}$$

for all $x \in P$, hence

(3) $$\sim \boldsymbol{u} \text{ is an RM-correspondence,}$$

(4) $$\sim \boldsymbol{u}\mathbf{P} = P,$$

(5) $$(P, \emptyset) \text{ is the type of } \sim \boldsymbol{u}$$

(§ 2 (15)). Because $s(x, \boldsymbol{\sigma}) \subset \mathbf{X} \cap \mathbf{L}x = \boldsymbol{\Gamma}x$ for $\boldsymbol{\sigma} \in S(u)$, $\Gamma = \Gamma_u$,

(6) $$u \text{ has the graph } \Gamma \Rightarrow \sim \boldsymbol{u} \text{ is } \boldsymbol{\Gamma}\text{-generable}$$

(see (2), and § 2 (27)); we can also say

(7) $u$ is a $\Gamma$-generable game correspondence $\Rightarrow \sim \boldsymbol{u}$ is $\boldsymbol{\Gamma}$-generable

(see (6), (9)). Clearly

(8) $$x(\sim \boldsymbol{u})\mathbf{A} \Leftrightarrow \boldsymbol{\Sigma}_u(\sim, \ \mathbf{A} \cup (\mathbf{P} - \mathbf{L}x).$$

For all $x \subset P$, $A \subset P$ there holds

$$\mathbf{A}(x, A) = \mathbf{A}(x, A) \cup (\mathbf{P} - \mathbf{L}x).$$

From this and from (8) the assertion

(9) $$u \approx v \Leftrightarrow \sim \boldsymbol{u} = \sim \boldsymbol{v}$$

follows by means of theorem 26.1 of § 2. Therefore, a $\sim$-strategic correspondence may be considered as an object defined for a regular (i.e. only regular correspondences containing, compare § 2.14) class of the M-decomposition on Corr ($P$, exp $P$), and for distinct regular classes of this decomposition these objects are distinct.

**3.** For a memory relation $\sim$ and for regular correspondences of Corr ($P$, exp $P$) we define the coresponding $\sim$*-strategic collections* (we shall denote them by the symbol $\sim$ of the memory relation and by the corresponding thick great Gothic letter, e.g. $\sim \mathfrak{V}$ for $\sim$, $v$) in the following way:

$$\sim \mathfrak{U} := \{\mathbf{A} \mid \mathbf{A} \subset \mathbf{P}, \text{ there exists } \boldsymbol{\sigma} \in \sim S(u) \text{ such that } s(\boldsymbol{\sigma}) \subset \mathbf{A}\},$$

i.e. $\mathbf{A} \in \sim \mathfrak{U}$ iff $\boldsymbol{\Sigma}_u(\sim, \mathbf{A})$. Analogous results (to those in 2) hold for $\sim$-strategic collections:

(10) $$u \subset v \Rightarrow \sim \mathfrak{U} \subset \sim \mathfrak{V}$$

(11) $$\sim \mathfrak{U} = [\{(s\boldsymbol{\sigma}) \mid \boldsymbol{\sigma} \in \sim S(u)\}]_{\mathbf{P}},$$

hence

(12) $$\sim \mathfrak{U} \text{ is a non-empty RM-collection}$$

(13) $\qquad u$ has the graph $\Gamma \Rightarrow \sim \mathfrak{U}$ is $\boldsymbol{X}$-generable,

where $\boldsymbol{X} := \boldsymbol{X}_{\Gamma}$ ($\mathrm{s}(\boldsymbol{\sigma}) \subset \boldsymbol{X}$ for $\boldsymbol{\sigma} \in S(u)$, $\Gamma := \Gamma_u$; see (11), § 2 (24)),

(14) $\quad u$ is a $\Gamma$-generable game correspondence $\Rightarrow \sim \mathfrak{U}$ is $\boldsymbol{X}$-generable

(see (13), (17)). It is possible to express easily $\sim \boldsymbol{u}$ by means of $\sim \mathfrak{U}$:

(15) $$x(\sim \boldsymbol{u}) = [\{\mathbf{L}x\}]_{\boldsymbol{P}} \sqcap \sim \mathfrak{U}$$

(for all $x \in P$); namely $x(\sim \boldsymbol{u}) = [\{\mathrm{s}(x, \boldsymbol{\sigma}) \mid \boldsymbol{\sigma} \in \sim S(u)\}]_{\boldsymbol{P}} = [\{\mathbf{L}x\} \sqcap \sqcap \{\mathrm{s}(\boldsymbol{\sigma}) \mid \boldsymbol{\sigma} \in \sim S(u)\}]_{\boldsymbol{P}} = [\{\mathbf{L}x\}]_{\boldsymbol{P}} \sqcap [\{\mathrm{s}(\boldsymbol{\sigma}) \mid \boldsymbol{\sigma} \in \sim S(u)\}]_{\boldsymbol{P}} = [\{\mathbf{L}x\}]_{\boldsymbol{P}} \sqcap \sqcap \sim \mathfrak{U}$ (see (2), (11), § 2 (18)). Similary can be derived

(16) $\qquad u$ has the graph $\Gamma \Rightarrow x(\sim \boldsymbol{u}) = [\{\boldsymbol{\Gamma}x\}]_{\boldsymbol{P}} \sqcap \sim \mathfrak{U}$.

From § 2.26.1 we obtain

(17) $$u \approx v \Leftrightarrow \sim \mathfrak{U} = \sim \mathfrak{V}.$$

### b) "*Absolute*" *strategies*

4. Under an *aim-collection* an element of exp exp $\boldsymbol{P}$ is meant. (E.g. $\sim$-strategic collections are aim-collections.) Aim-collections will be denoted by letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$. The phrase "$\mathfrak{A}$ is (*well*) *ordered*" means that $\mathfrak{A}$ is (well) ordered under the set-theoretical inclusion.

5. We say that $\boldsymbol{\sigma}$ is a $(\sim, \mathfrak{A})$-*absolute u-strategy* iff $\boldsymbol{\sigma}$ is a $\sim$-acceptable $u$-strategy such that if $x(\sim \boldsymbol{u})\boldsymbol{A}$, $\boldsymbol{A} \in \mathfrak{A}$, then $\boldsymbol{A}$ can be enforced by $\boldsymbol{\sigma}$ from $x$. Clearly, $\boldsymbol{\sigma} \in \sim S(u)$ is a $(\sim, \mathfrak{A})$-absolute $u$-strategy iff $\mathrm{s}(x, \boldsymbol{\sigma}) \subset \subset \bigcap_{\boldsymbol{A} \in \mathfrak{A} \cap x(\sim \boldsymbol{u})} \boldsymbol{A}$ ($\bigcap_{\boldsymbol{A} \in \emptyset} \boldsymbol{A} := \boldsymbol{P}$) for each $x \in P$. Therefore, the assertion "a $(\sim, \mathfrak{A})$-absolute u-strategy exists" can be expressed by means of the strategic predicate $\boldsymbol{\Sigma}_u$.

6. Let $=$ mean the equality on $\boldsymbol{Z}$ here.

We shall write also "$\boldsymbol{A}$ can be $u$-enforced", "$\boldsymbol{u}$", "$\mathfrak{U}$" (and similarly), "$\mathfrak{A}$-absolute" instead of "$\boldsymbol{A}$ can be $(=, u)$-enforced", "$= \boldsymbol{u}$", "$= \mathfrak{U}$" (and similarly), "$(=, \mathfrak{A})$-absolute", respectively.

We shall write also "$\boldsymbol{A}$ can be plainly $u$-enforced", "$\mathring{\boldsymbol{u}}$", "$\mathring{\mathfrak{U}}$" (and similarly), "plainly $\mathfrak{A}$-absolute" instead of "$\boldsymbol{A}$ can be $(\overset{\circ}{\sim}$, u)-enforced" "$\overset{\circ}{\sim} \boldsymbol{u}$", "$\overset{\circ}{\sim} \mathfrak{U}$" (and similarly), "$(\overset{\circ}{\sim}, \mathfrak{A})$-absolute", respectively. $\overset{\circ}{\sim}$( is the greatest memory relation, compare § 3,0, § 2.20, § 1.2.).

7. **Lemma.** *Let $\mathfrak{A}$ be an aim-collection, $\sim$ be a memory relation. If an $\mathfrak{A}$-absolute u-strategy exists, then exactly one of the following cases occurs:*

(i) $\sim S(u)$ *and the set of all* $\mathfrak{A}$-*absolute u-strategies are disjoint.*

(ii) *The set of all* $(\sim, \mathfrak{A})$-*absolute u-strategies is the intersection of* $\sim S(u)$ *and the set of all* $\mathfrak{A}$-*absolute u-strategies.*

(The *proof* is simple.)

**8.0.** In the following (in § 3) we shall investigate those two "extreme" cases: $\mathfrak{A}$-*absolute*, and *plainly* $\mathfrak{A}$-*absolute u-strategies*. There is a trivial sufficient condition for the existence of an $\mathfrak{A}$-absolute $u$-strategy:

**8.1. Lemma.** *Let* $\mathfrak{A}$ *be a well ordered aim-collection. Then an* $\mathfrak{A}$-*absolute u-strategy exists.*

*Proof.* Without loss of generality we shall assume $\boldsymbol{P} \in \mathfrak{A}$. For each $x \in P$ we put $\boldsymbol{A}(x) := \min \{\boldsymbol{A} \mid \boldsymbol{A} \in \mathfrak{A}, x\boldsymbol{u}\boldsymbol{A}\}$ (where the minimum is taken under the set-theoretical inclusion), and we choose some $\boldsymbol{\sigma}_x \in S(u)$ such that $\mathrm{s}(x, \boldsymbol{\sigma}_x) \subset \boldsymbol{A}(x)$. Let $\boldsymbol{\sigma}^0$ be the $u$-strategy which satisfies the condition: $\boldsymbol{\sigma}^0\mathbf{z} = \boldsymbol{\sigma}_{z_0}\mathbf{z}$ for all $\mathbf{z} \in \boldsymbol{Z}$. If $x \in P$, $\boldsymbol{\sigma} \in S(u)$, $\boldsymbol{A} \in \mathfrak{A}$, $\mathrm{s}(x, \boldsymbol{\sigma}) \subset \boldsymbol{A}$, then evidently $\mathrm{s}(x, \boldsymbol{\sigma}^0) = \mathrm{s}(x, \boldsymbol{\sigma}_x) \subset \boldsymbol{A}(x) \subset \boldsymbol{A}$, i.e. $\boldsymbol{\sigma}^0$ is an $\mathfrak{A}$-absolute $u$-strategy. Q.E.D.

**8.2.** Simple examples show that for the existence of an $\mathfrak{A}$-absolute $u$-strategy it is not sufficient to suppose only that $\mathfrak{A}$ is ordered, and further, if $\mathfrak{A}$ is finite, then the supposition that $\mathfrak{A}$ is (well) ordered must not be omitted; namely in both the cases $\bigcap_{\boldsymbol{A} \in \mathfrak{A} \cap x\boldsymbol{u}} \boldsymbol{A} = \emptyset$ can happen for some $x$.

From 8.1 there follows

**8.3. Corollary.**

$$\mathfrak{U} = \bigcap_{x \in P} x\boldsymbol{u}. \tag{18}$$

*Proof.* From (15) we conclude $\bigcap_{x \in P} x\boldsymbol{u} \supset \mathfrak{U}$, since $x\boldsymbol{u} = \mathfrak{U} \sqcap \sqcap [\{\mathbf{L}x\}]_{\boldsymbol{P}} \supset \mathfrak{U} \sqcap \{\boldsymbol{P}\} = \mathfrak{U}$. Let $\boldsymbol{A} \in \bigcap_{x \in P} x\boldsymbol{u}$; from 8.1 there follows the existence of an $\{\boldsymbol{A}\}$-absolute $u$-strategy $\boldsymbol{\sigma}^0$. Because $x\boldsymbol{u}\boldsymbol{A}$ holds for all $x \in P$, $\mathrm{s}(x, \boldsymbol{\sigma}^0) \subset \boldsymbol{A}$ holds for each $x$, i.e. $\boldsymbol{A} \in \mathfrak{U}$. Hence (18) is valid. Q.E.D.

**8.4.** The direct analogue of Lemma 8.1 for plainly $\mathfrak{A}$-absolute $u$-strategies does not hold, even for a one-element aim-collection $\mathfrak{A}$ a plainly $\mathfrak{A}$-absolute $u$-strategy need not exist. (E.g. $P = \{1, 2\}$, $P_0 = \emptyset$, $\mathrm{pr}_1 u = \{(1, \{1\}), (1, \{2\}), (2, \{1\})\}$, $\boldsymbol{A} = \{(1, 1, 1, 1, \ldots), (2, 1, 2, 1, \ldots)\}$, $\mathfrak{A} = \{\boldsymbol{A}\}$. Evidently $\boldsymbol{A} \in (\bigcap_{x \in P} x\mathring{\boldsymbol{u}}) - \mathfrak{U}$.)

**8.5.** It is important to give some non-trivial sufficient conditions for the existence of a plainly $\mathfrak{A}$-absolute $u$-strategy. This problem is solved (for our purposes) by the main result of § 3 — Theorem 11. But first of all we introduce three definitions; let us note that the property (I, Γ) and several other ones are investigated in § 6.

**9.1.** For $\mathbf{x} \in \mathbf{P}$, $0 \leqq m < 1 + l(\mathbf{x})$, we define the sequence $\mathbf{x}^{[m]} = (x_k^{[m]})\ 0 \leqq k < 1 + l(\mathbf{x}^{[m]})$ (the *mth remainder* of $\mathbf{x}$): $l(\mathbf{x}^{[m]}) := l(\mathbf{x}) - m$ $(\omega_0 - m := \omega_0$ for $0 \leqq m < \omega_0)$, $x_k^{[m]} := x_{k+m}$ for $0 \leqq k < 1 + l(\mathbf{x}^{[m]})$. Clearly $\mathbf{x}^{[m]} \in \mathbf{P}$, $\mathbf{x}^{[0]} = \mathbf{x}$.

**9.2.** We say that an aim $\mathbf{A}$ has the *property* (I, Γ) iff for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}_\Gamma$, and for each $k$, $0 \leqq k < 1 + \min(l(\mathbf{x}), l(\mathbf{y}))$, there holds

$$[\mathbf{x} \in \mathbf{A} \wedge (x_0, \ldots, x_k) = (y_0, \ldots, y_k) \wedge (\mathbf{x}^{[k]} \notin \mathbf{A} \vee \mathbf{y}^{[k]} \in \mathbf{A})] \Rightarrow \mathbf{y} \in \mathbf{A}.$$

**9.3.** We say that an aim-collection has the property (I, Γ) iff each of its elements has the property (I, Γ).

Let us present a useful result:

**10. Theorem.** *Let an aim* $\mathbf{A}$ *have the property* (I, Γ), *where* Γ *is the graph of u. Let* $\sigma_0, \sigma$ *be plain u-strategies such that* $\sigma_0 z = \sigma z$ *for all* $z \in Z \cap A$, *where* $A = \{x \mid x \in P, \mathrm{s}(x, \sigma_0) \subset \mathbf{A}\}$. *Then* $\mathrm{s}(x, \sigma) \subset \mathbf{A}$ *for all* $x \in A$.

*Proof.* Let $x \in A$, $\mathbf{x} \in \mathrm{s}(x, \sigma) - \mathbf{A}$. There exists $m$, $0 \leqq m < 1 + l(\mathbf{x})$, such that $x_k \in A$ for each $k$, $0 \leqq k < m$, and $x_m \notin A$ (if such $m$ does not exist, then $\mathbf{x} \in \mathrm{s}(x, \sigma_0) \subset \mathbf{A}$, which would be a contradiction), and there exists $\mathbf{y} \in \mathrm{s}(x_m, \sigma_0) - \mathbf{A}$. Consequently $(x_0, \ldots, x_m, y_1, y_2, \ldots) \in \mathrm{s}(x, \sigma_0) \subset \mathbf{A}$, and hence $\mathbf{x} \in \mathbf{A}$ (since $\mathbf{A}$ has the property (I, Γ)), which is a contradiction. Therefore $\mathrm{s}(x, \sigma) \subset \mathbf{A}$ for $x \in A$. Q.E.D.

**11. Theorem.** *Let* $\mathfrak{A}$ *be a well ordered aim-collection with the property* (I, Γ), *where* Γ *is the graph of u. Then a plainly* $\mathfrak{A}$*-absolute u-strategy exists.*

*Proof.* Without loss of generality we shall assume $\mathbf{P} \in \mathfrak{A}$. Let us "mark" all elements of the set $\mathring{S}(u)$ by ordinal numbers: $\mathring{S}(u) = \{\sigma_\eta \mid 0 \leqq \eta < \xi\}$ ($\eta, \xi$ will always be ordinal numbers). For each $x \in P$ we denote $\mathbf{A}(x) := \min\{\mathbf{A} \mid \mathbf{A} \in \mathfrak{A}, x\mathring{\boldsymbol{u}}\mathbf{A}\}$ (where the minimum is taken under the set-theoretical inclusion), and we put $\eta(x) := \min\{\eta \mid 0 \leqq \eta < \xi, \mathrm{s}(x, \sigma_\eta) \subset \mathbf{A}(x)\}$. Let $\sigma^0$ be the plain $u$-strategy such that $\sigma^0 z = \sigma_{\eta(z)} z$ for all $z \in Z$.

Let $x \in P$, $\mathbf{x} \in \mathrm{s}(x, \sigma^0)$ be chosen fixedly for the remaining part of this proof. For each $\mathbf{A} \in \mathfrak{A}$ the following assertions $(i)$ — $(iii)$ (in which $0 \leqq m < 1 + l(\mathbf{x})$) hold:

$(i)$ If $m > 0$, $x_{m-1}\mathring{\boldsymbol{u}}\mathbf{A}$, $\mathbf{x}^{[m]} \in \mathbf{A}$, then $\mathbf{x}^{[m-1]} \in \mathbf{A}$.

(Namely, under the suppositions there exists $\mathbf{y} \in \mathrm{s}(x_{m-1}, \sigma_{\eta(x_{m-1})}) \subset$

$\subset \boldsymbol{A}(x_{m-1}) \subset \boldsymbol{A}$ such that $y_1 = x_m$; thus $\mathbf{x}^{[m-1]} \in \boldsymbol{A}$, since $\boldsymbol{A}$ has the property (I, $\Gamma$). — If we besides suppose $x_k \overset{\circ}{\boldsymbol{u}} \boldsymbol{A}$ for all $k$, $0 \leqq k < m$, then from $(i)$ it follows that $\mathbf{x}^{[m-r]} \in \boldsymbol{A}$ for $r = 0, \ldots, m$; hence there holds):

$(ii)$ If $\mathbf{x}^{[m]} \in \boldsymbol{A}$, $x_k \overset{\circ}{\boldsymbol{u}} \boldsymbol{A}$ for each $k$, $0 \leqq k < m$, then $\mathbf{x} \in \boldsymbol{A}$.

Further

$(iii)$ If $m > 0$, $x_{m-1} \overset{\circ}{\boldsymbol{u}} \boldsymbol{A}$, $\mathrm{s}(x_m, \sigma_{\eta(x_{m-1})}) \not\subset \boldsymbol{A}$, then $\mathbf{x}^{[m-1]} \in \boldsymbol{A}$.

(Namely, under the suppositions there exists $\mathbf{y} \in \mathrm{s}(x_{m-1}, \sigma_{\eta(x_{m-1})}) \subset \boldsymbol{A}$ such that $y_1 = x_m$, $\mathbf{y}^{[1]} \notin \boldsymbol{A}$; thus $\mathbf{x}^{[m-1]} \in \boldsymbol{A}$, since $\boldsymbol{A}$ has the property (I, $\Gamma$).)

Now, there occurs exactly one of the following cases $(\alpha)$, $(\beta)$:

$(\alpha)$ For each $k$, $0 < k < 1 + l(\mathbf{x})$, there holds $\mathrm{s}(x_k, \sigma_{\eta(x_{k-1})}) \subset \boldsymbol{A}(x_{k-1})$. Then $\boldsymbol{A}(x_k) \subset \boldsymbol{A}(x_{k-1})$ for these $k$, and —because $\mathfrak{A}$ is well ordered— there exists $n$, $0 \leqq n < 1 + l(\mathbf{x})$, such that $\boldsymbol{A}(x_j) = \boldsymbol{A}(x_n)$ for all $j$, $n \leqq j < 1 + l(\mathbf{x})$. Hence, the sequence $(\eta(x_j))_{n \leqq j < 1+l(\mathbf{x})}$ is non-increasing, consequently there exists $m$, $j \leqq m < 1 + l(\mathbf{x})$, such that $\eta(x_j) = \eta(x_m)$ for all $j$, $m \leqq j < 1 + l(\mathbf{x})$. But then $\mathbf{x}^{[m]} \in \mathrm{s}(x_m, \boldsymbol{\sigma}_{\eta(x_m)}) \subset \boldsymbol{A}(x_m)$.

$(\beta)$ There exists $n$, $0 < n < 1 + l(\mathbf{x})$, such that $\mathrm{s}(x_j, \sigma_{\eta(x_{j-1})}) \subset \boldsymbol{A}(x_{j-1})$ for all $j$, $0 < j < n$, but $\mathrm{s}(x_n, \sigma_{\eta(x_{n-1})}) \not\subset \boldsymbol{A}(x_{n-1})$. Then $\boldsymbol{A}(x_j) \subset \boldsymbol{A}(x_{j-1})$ for all $j$, $0 < j < n$. If we put $\boldsymbol{A} := \boldsymbol{A}(x_{n-1})$ in $(iii)$, then we get $\mathbf{x}^{[m]} \in \boldsymbol{A}(x_m)$, where $m := n - 1$.

Thus in both the cases $(\alpha)$, $(\beta)$ there exists $m$, $0 \leqq m < 1 + l(\mathbf{x})$, such that $\mathbf{x}^{[m]} \in \boldsymbol{A}(x_m)$, and $\boldsymbol{A}(x_j) \subset \boldsymbol{A}(x_{j-1})$ for all $j$, $0 < j \leqq m$. Let us choose $\boldsymbol{A} := \boldsymbol{A}(x_0)$ $(= \boldsymbol{A}(x))$; then $\mathbf{x}^{[m]} \in \boldsymbol{A}$, $x_k \overset{\circ}{\boldsymbol{u}} \boldsymbol{A}$ for all $k$, $0 \leqq k < m$. From $(ii)$ it follows that $x \in \boldsymbol{A} = \boldsymbol{A}(x)$.

Therefore we conclude: $\mathrm{s}(x, \sigma^0) \subset \boldsymbol{A}(x)$ for all $x \in P$, but $\boldsymbol{A}(x) \subset \boldsymbol{A}$ for each $\boldsymbol{A} \in \mathfrak{A}$ such that $x \overset{\circ}{\boldsymbol{u}} \boldsymbol{A}$. Consequently, $\sigma^0$ is a plainly $\mathfrak{A}$-absolute $u$-strategy.

Q.E.D.

**12.** Let us give a simple example of well ordered (even one-element) aim-collection $\mathfrak{A}$ such that a plainly $\mathfrak{A}$-absolute $u$-strategy exists, but $\mathfrak{A}$ has not the property (I, $\Gamma$), where $\Gamma$ is the graph of $u$:

$P = \{0, 1, 2\}$, $P_0 = \emptyset$, $\mathrm{pr}_1\, u = \{(0, \{0\}), (1, \{0\}), (1, \{1\}), (2, \{1\})\}$, $\boldsymbol{A} = \{\mathbf{x} \mid \mathbf{x} \in \boldsymbol{P}, x_2 = 0\}$, $\mathfrak{A} = \{\boldsymbol{A}\}$.

From Theorem 11 there immediately follows

**13. Corollary.** *Let $u$ be a $\Gamma$-generable game correspondence, let $\mathfrak{A}$ be a well ordered aim-collection with the property* (I, $\Gamma$). *Then a plainly $\mathfrak{A}$-absolute $u$-strategy exists.*

*(To be continued.)*

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