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A NOTE ON TOPOLOGY COMPATIBLE WITH THE ORDERING

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This paper is dealing with two problems formulated in [1]. The definitions and concepts are the same as in [1]. For completeness we shall state some of them. If (P, u) is a topological space than by $O(u)$ we shall denote the system of all open sets in (P, u). If (P, u), (P, v) are two topological spaces we say that the topology u is finer than v, if $O(u) \supseteq O(v)$. We note $u \leqq v$.

We say that an ordered set P is finitely separable if there exists points $x_1, \ldots, x_n \in P$ such that $P \subseteq (x_1] \cup \ldots \cup (x_n] \cup [x_1) \cup \ldots \cup [x_n)$.

The principal concepts in [1] are the concepts of topology compatible and strongly compatible with the ordering.

Definition 1: Let P be an ordered set and u a topology on P. We say that u is compatible with the ordering if u is a T_1-topology and if for every pair $a, b \in P$, $a < b$, there exist a neighbourhood O_1 of the point a and a neighbourhood O_2 of the point b so that

$$x \in O_1 \Rightarrow x < b \text{ or } x \parallel b$$
$$y \in O_2 \Rightarrow y > a \text{ or } y \parallel a$$

hold.

Definition 2: Let P be an ordered set, u a topology on P. We say that u is strongly compatible with the ordering, if u is a T_1-topology and if for every pair $a, b \in P$, $a < b$, there exist a neighbourhood O_1 of the point a and a neighbourhood O_2 of the point b such that

$$x \in O_1,\, y \in O_2 \Rightarrow x < y \text{ or } x \parallel y.$$

Some types of topologies compatible with the ordering, for example an interval topology, has the character that if is T_2-topology it is strongly compatible with the ordering. Problem 4.21 in [1] asks whether this character has also the ideal topology, i.e. the topology which has as subbasis of open sets totally irreducible ideals and totally irreducible dual ideals (see [2]). Its solution gives the following theorem.

Theorem 1: *Let P be an ordered set. If the ideal topology on P is T_2-topology it is strongly compatible with the ordering.*

Proof: Let us denote the ideal topology on P by ι. Let $a, b \in P$, $a < b$ and ι be a T_2-topology. Then there exist open sets $O_1, O_2 \in O(\iota)$, so that $a \in O_1, b \in O_2, O_1 \cap O_2 = \varnothing$ holds. Furthermore $O_1 = \bigcup_i \bigcap_{k=1}^{n_i} O_{i,k}$, $O_2 = \bigcup_r \bigcap_{s=1}^{m_r} O_{r,s}$, where $O_{i,k}$, $O_{r,s}$ are totally irreducible ideals or totally irreducible dual ideals resp. There exists i_0, r_0 so that $a \in \bigcap_{k=1}^{n_{i_0}} O_{i_0,k}$, $b \in \bigcap_{s=1}^{m_{r_0}} O_{r_0,s}$ and evidently $\bigcap_{k=1}^{n_{i_0}} O_{i_0,k} = \bigcap_{1 \leqq k \leqq n'_{i_0}} O_{i_0,k} \cap \bigcap_{n'_{i_0} < k \leqq n_{i_0}} O_{i_0,k}$ and $\bigcap_{s=1}^{m_{r_0}} O_{r_0,s} = \bigcap_{1 \leqq s \leqq m'_{r_0}} O_{r_0,s} \cap \bigcap_{m'_{r_0} < s \leqq m_{r_0}} O_{r_0,s}$, where $O_{i_0,k}$ for $1 \leqq k \leqq n'_{i_0}$, $O_{r_0,s}$ for $1 \leqq s \leqq m'_{r_0}$ are totally irreducible ideals and $O_{i_0,k}$ for $n'_{i_0} < k \leqq$ $\leqq n_{i_0}$, $O_{r_0,s}$ for $m'_{r_0} < s \leqq m_{r_0}$ are totally irreducible dual ideals.

Let us denote $A = \bigcap_{1 \leqq k \leqq n'_{i_0}} O_{i_0,k}$, $B = \bigcap_{n'_{i_0} < k \leqq n_{i_0}} O_{i_0,k}$, $A' = \bigcap_{1 \leqq s \leqq m'_{r_0}} O_{r_0,s}$, $B' = \bigcap_{m'_{r_0} < s \leqq m_{r_0}} O_{r_0,s}$. Due to the fact that each ideal is a semiideal*) and each dual ideal a dual semiideal, A, A' are semiideals and B, B' dual semiideals. Because $a < b$, $a \in B$, it holds $b \in B$. Further $b \bar{\in} A$ as on the contrary case we get a contradiction with the assumption $O_1 \cap O_2 = \varnothing$. Analogously we prove that $a \in A'$, $a \bar{\in} B'$ holds.

Put $O_a = A \cap A'$, $O_b = B \cap B'$. It is O_a, $O_b \in O(\iota)$, $a \in O_a$, $b \in O_b$, $O_a \cap O_b = \varnothing$. Let us assume that there exist $x, y \in P$ so that $x \in O_a$, $y \in O_b$ and $x \geqq y$. O_a being the semiideal, $y \in O_a$ holds and we get a contradiction.

We have constructed to the elements $a, b \in P$, $a < b$, the neighbourhoods O_a, O_b with the demanded properties.

The second problem in [1], problem 5.7., is a question when there exists the greatest element in $\mathscr{S}(P)$, where $\mathscr{S}(P)$ is the system of all topologies on P which are compatible with the ordering. We give a partial solution of this problem;**) a necessary condition for the existence of the greatest element.

First of all we give one needful construction.

Definition 3:: Let P be an ordered set, $a, b, c \in P$, $a < b \leqq c$.

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*) A semiideal is a subset A of an ordered set P with the property: $x \in A$, $y \leqq x$ implies $y \in A$.

**) The autor has solved this problem completely. The solution will appear in a forthcoming paper.

Put
$$\mathfrak{M}_1 = \{(x] \mid x < c\}$$
$$\mathfrak{M}_2 = \{P - [x) \mid c \bar{\in} [x)\}$$
$$\mathfrak{N}_1 = \{[x) \mid x > b\}$$
$$\mathfrak{N}_2 = \{P - (x] \mid b \bar{\in} (x]\}$$
$$\mathfrak{B} = \{P - \{x\} \mid b < x \leqq c\}$$
$$B = P - \{a, b\}$$

Let $u(a, b, c)$ be the topology on P, which has as subbasis of open ses the system $S(u) = \mathfrak{M}_1 \cup \mathfrak{M}_2 \cup \mathfrak{N}_1 \cup \mathfrak{N}_2 \cup \mathfrak{B} \cup \{B\}$.

Lemma 1: *It holds* (*i*) $X \in \mathfrak{M}_2 \cup \mathfrak{N}_2 \cup \mathfrak{B} \Rightarrow b \in X$

(*ii*) $X \in \mathfrak{N}_1 \Rightarrow X \cap (P - [b)) = \emptyset$

(*iii*) $X \in \mathfrak{M}_2 \cup \mathfrak{N}_2 \Rightarrow (b, c] \subseteqq X$

Proof is evident.

Lemma 2: *Let P be an ordered set, $a, b, c \in P$, $a < b \leqq c$. Then* $u(a,b,c) \in \mathscr{S}(P)$.

Proof: Let us denote $u \equiv u(a, b, c)$. We shall prove that u is a T_1-topology. Let $x, y \in P$, $x \neq y$.

(1) $$x < y$$

If $b \bar{\in} (x]$ then $O = P - (x] \in \mathfrak{N}_2 \subseteq O(u)$, $x \bar{\in} O$, $y \in O$.

If $\bar{b} \in (x]$ so $b \leqq x < y$ i.e. for $O = [y)$ it is $O \in \mathfrak{N}_1 \subseteq O(u)$, $x \bar{\in} O$, $y \in O$.

(2) $$x > y$$

Let $c \bar{\in} [x)$. Then for $O = P - [x)$ there holds $O \in \mathfrak{M}_2 \subseteq O(u)$, $x \bar{\in} O$, $y \in O$.

If $c \in [x)$ then $c \geqq x > y$ and for $O = (y]$ there holds $O \in \mathfrak{M}_1 \subseteq O(u)$, $x \bar{\in} O$, $y \in O$.

(3) $$x \parallel y$$

If $x = b$ then it is sufficient to put $O = B$. Let $x \neq b$. If $b \bar{\in} (x]$ then for $O = P - (x]$ it is $O \in \mathfrak{N}_2 \subseteq O(u)$, $x \bar{\in} O$, $y \in O$. If $c \bar{\in} [x)$ then for $O = P - [x)$ is $O \in \mathfrak{M}_2 \subseteq O(u)$, $x \bar{\in} O$, $y \in O$. It remains the case $b \in (x]$, $c \in [x)$, i.e. $b < x \leqq c$ as simultaneously we assume $x \neq b$. Then for $O = P - \{x\}$ it is $O \in \mathfrak{B} \subseteq O(u)$, $x \bar{\in} O$, $y \in O$.

We have proved that u is a T_1-topology. Let $x, y \in P$, $x < y$. According to (1) there exists a dual semiideal $O_2 \in O(u)$ such that $x \bar{\in} O_2$, $y \in O_2$. According to (2) there exists a semiideal $O_1 \in O(u)$ such that $x \in O_1$, $y \bar{\in} O_1$. Let us assume that there exists $z \in O_1$ so that $z \geqq y$. Due to the fact that O_1 is a semiideal it is $y \in O_1$ what si a contradiction. Analogously for O_2. We have proved that $u \in \mathscr{S}(P)$.

Now we can prove a theorem partially solving problem 5.7. in [1].

Theorem: 2 *Let P be an ordered set which contains an infinite bounded interval $[x, y]$, where if x is a minimal and y a maximal element in P, either x is the least or y the greatest element in P. If $\mathcal{S}(P)$ has the greatest element then the set P is finitely separable.*

Proof: If x is a minimal and y a maximal element in P than according to the assumption the set P contains either the least or the greatest element, i.e. it is finitely separable.

Do not let x be a minimal element in P. We put $x \equiv b$, $y \equiv c$. There exists $a \in P$ such that $a < b$. Let us study the topology $u \equiv u(a, b, c)$. Let O be an open set of the topology u such one that $a \in O \subseteq P - [b)$. It holds $O = \bigcup_i \bigcap_{j=1}^{n_i} O_{i,j}$ where $O_{i,j} \in S(u)$. Let us denote $O_i = \bigcap_{j=1}^{n_i} O_{i,j}$. We can suppose that $O_i \neq \emptyset$ for each i holds. Because $a \in O$, $b \bar{\in} O$ there are these possibilities for chosen $i = i_0$.

(α) $$a \in O_{i_0},\ b \bar{\in} O_{i_0}$$

Then there exists j_0 such that $a \in O_{i_0, j_0}$, $b \bar{\in} O_{i_0, j_0}$. Because $O_{i_0, j_0} \in S(u)$ then according to the assertion of (i) lemma 1 it is $O_{i_0, j_0} \in \mathfrak{M}_1 \cup \mathfrak{N}_1 \cup \{B\}$. But $a \bar{\in} B$ so that $O_{i_0 j_0} \neq B$. From (ii) lemma 1 it follows that $O_{i_0, j_0} \bar{\in} \mathfrak{N}_1$. Then $O_{i_0, j_0} \in \mathfrak{M}_1$ i.e. $O_{i_0} \subseteq O_{i_0, j_0} \subseteq (c]$.

(β) $$a, b \bar{\in} O_{i_0}.$$

From (ii) lemma 1 it follows that $O_{i_0, j} \bar{\in} \mathfrak{N}_1$ for each j. Let $N = \{1, 2, \dots, n_{i_0}\}$ be the set of indices j, $N_1 = \{j \in N \mid O_{i_0, j} \neq B\}$, $N_2 = \{j \in N \mid O_{i_0, j} \in \mathfrak{B}\}$.

Let us assume that there does not exist $j \in N$ such that $O_{i_0, j} \in \mathfrak{M}_1$. Because $b \bar{\in} O_{i_0}$, $j_0 \in N$ exists such that $b \bar{\in} O_{i_0, j_0}$. From (i) lemma 1 and from the previous result one gets that $O_{i_0, j_0} = B$ i.e. $N - N_1 \neq \emptyset$. It holds $P - [b) \supseteq O_{i_0} = \bigcap_{j \in N} O_{i_0, j} = B \cap \bigcap_{j \in N_1} O_{i_0, j} \supseteq (b, c] \cap \bigcap_{j \in N_1} O_{i_0, j}$. Because $(b, c] \cap (P - [b)) = \emptyset$ so $(b, c] \cap \bigcap_{j \in N_1} O_{i_0, j} = \emptyset$. From the previous result and from (iii) lemma 1 follows that $(b, c] \cap \bigcap_{j \in N_2} O_{i_0, j} = \emptyset$. But here is the contradiction with the fact that the set $(b, c]$ is infinite and the set N_2 finite.

Therefore it exists $j \in N$ such that $O_{i_0, j} \in \mathfrak{M}_1$ i.e. $O_{i_0} \subseteq O_{i_0, j} \subseteq (c]$. (*)We have proved that if $O \in O(u)$, $a \in O \subseteq P - [b)$ so it holds $O \subseteq (c]$.

Let ν be the greatest element of $\mathcal{S}(P)$ and ι the interval topology on P.

It is $O(\nu) \subseteq O(u) \cap O(\iota)$. Because $a < b$ there exists $O \in O(\nu)$, $a \in O \subseteq$ $\subseteq P - [b)$. According to (*) there is $O \subseteq (c]$ and $O \in O(\iota)$. It holds $O =$ $= \bigcup_{\iota} \bigcap_{j=1}^{n_\iota} O_{\iota,j}$ where $O_{\iota,j} = P - (x_j^i]$ or $P - [x_j^i)$ resp. Then $\bigcap_{j=1}^{n_\iota} O_{\iota,j} \subseteq$ $\subseteq (c]$ i.e. $\bigcap_{j=1}^{n'_\iota} (P - [x_j^i)) \cap \bigcap_{j=n'_\iota+1}^{n_\iota} (P - (x_j^i]) \subseteq (c]$ so that $P - (\bigcup_{j=1}^{n'_\iota} [x_j^i) \cup$ $\cup \bigcup_{j=n'_\iota+1}^{n_\iota} (x_j^i]) \subseteq (c]$. It means that $P = [x_1^i) \cup \dots \cup [x^i_{n'_\iota}) \cup (x^i_{n'_\iota+1}] \cup$ $\cup \dots \cup (x^i_{n_\iota}] \cup (c]$ i.e. the set P is finitely separable.

If x is a minimal element in P and y is not maximal we shall make the proof dually.

The theorem can not be reversed. We shall show that there exists an ordered set P fulfilling the assumption of the previous theorem, which is finitely separable and $\mathscr{S}(P)$ has not the greatest element.

Example 1: Let P be an ordered set constructed according to the diagram where the interval $(b, c]$ is a chain of the type w^* and $\{b_1, \dots,$ $\dots, b_n, \dots\}$ is an antichain.

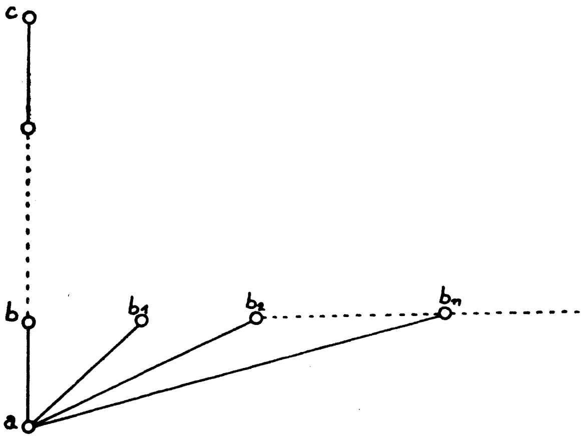

Fig. 1.

A set P is finitely separable because $P = [a)$. Let $O \in O(u)$, $a \in O \subseteq P - [b)$ where $u \equiv u(a, b, c)$. According to the assertion (*) mentioned in the proof of previous theorem it is $O \subseteq (c]$ so that $O = \{a\}$.

We shall show that $\{a\} \overline{\in} O(\iota)$. This will prove that $\mathscr{S}(P)$ has not the greatest element.

Let us assume $\{a\} \in O(\iota)$ i.e. $\{a\} = \bigcap_i \bigcup_{j=1}^{n_i} O_{i,j}$, where $O_{i,j} = P - (x_j^i]$ or $P - [x_j^i)$ resp. There exists i_0 such that $\{a\} = \bigcap_{j=1}^{n_{i_0}} O_{i_0,j} = \bigcap_{j=1}^{n_{i_0}} (P - [x_j^{i_0}))$. But this is impossible as the set $\{b_1, b_2, \ldots, b_n, \ldots\}$ is infinite.

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