Vladimír Ďurikovič

On the uniqueness of solutions and the convergence of successive approximations in the Darboux problem for certain differential equations of the type $u_{xy} = f(x, y, u, u_x, u_y)$

Archivum Mathematicum, Vol. 4 (1968), No. 4, 223--235

Persistent URL: `http://dml.cz/dmlcz/104670`

ON THE UNIQUENESS OF SOLUTIONS AND THE CONVERGENCE OF SUCCESSIVE APPROXIMATIONS IN THE DARBOUX PROBLEM FOR CERTAIN DIFFERENTIAL EQUATIONS OF THE TYPE $u_{xy} = f(x, y, u, u_x, u_y)$

VLADIMÍR ĎURIKOVIČ, BRATISLAVA

Received May 14, 1968

1. INTRODUCTION

B. Palczevski [4] has proved the convergence of Picard's successive approximations in the Darboux problem for the equations of the type $u_{xy} = f(x, y, u)$ under the conditions of the Krasnosielski and Krein type. These conditions for uniqueness had been generalized together with other conditions for uniqueness by J. S. W. WONG in the paper [7]. J. S. W. Wong has proved in his further paper [6] under these conditions not only the uniqueness of solutions but also the convergence of successive approximations in the Darboux problem for the equations of the type $u_{xy} = f(x, y, u)$. In the present paper we want to show that the conditions of the Krasnosielski-Krein and Nagumo-Perron-van Kampen [1] type generalized in a certain sense are making it possible to prove the uniqueness of solutions and convergence of successive approximations in the Darboux problem for the differential equations of the type $u_{xy} = f(x, y, u, u_x, u_y)$. For the proof of following theorems we shall use the results from the paper W. A. J. Luxemburg [3].

2. A THEOREM ON CONTRACTION

First of all we shall define the idea of the generalized metric space.

Let X be a non-void set; and let $d(x, y)$ be a non-negative real valued function $0 \leqq d(x, y) \leqq +\infty$ defined on the Cartezian product $X \times X$. Let the function $d(x, y)$ be satisfying for arbitrary elements $(x, y, z) \in X$ the following conditions:

a) $d(x, y) = 0$ if and only if $x = y$,
b) $d(x, y) = d(y, x)$,
c) $d(x, y) \leqq d(x, z) + d(z, y)$,

d) If the sequence $\{x_n\}_1^\infty$ of elements $x_n \in X$ is a d-Cauchy sequence, i.e. $\lim\limits_{m,n\to\infty} d(x_n, x_m) = 0$, then there exists an element $x \in X$ such, that $\lim\limits_{n\to\infty} d(x, x_n) = 0$.

An abstract set X on which a distance is defined in this way is called *generalized complete metric space*. It differs from the usual concept of complete metric space by the fact that not every pair of elements $(x, y) \in X$ necessarily has a finite distance $d(x, y)$.

Theorem 1 (Luxemburg [3]). *Let X be a generalized complete metric space and T a mapping of X into itself satisfying the following conditions:*

1°. *There exists a constant λ, $0 < \lambda < 1$, such, that*

$$d(Tx, Ty) \leqq \lambda d(x, y)$$

for all $(x, y) \in X$ with the distance $d(x, y) < +\infty$.

2°. *For every sequence of successive approximations $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$, where x_0 is an arbitrary element of X, there exists an index $N(x_0)$ such, that $d(x_N, x_{N+l}) < +\infty$ for all $l = 1, 2, \ldots$*

3°. *If x and y are two fixed points of mapping T, i.e. $Tx = x$ and $Ty = y$, then $d(x, y) < +\infty$.*

Then the equation $Tx = x$ has one and only one solution, and every sequence of successive approximations $\{x_n\}_1^\infty$ converges in the distance $d(x, y)$ to this unique solution.

3. THE FORMULATION OF THE DARBOUX PROBLEM

Let us introduce the following denotations and assumptions:

1. Let D denote the rectangle $0 \leqq x \leqq a$, $0 \leqq y \leqq b$, $a, b > 0$ and $D_1 = \{X\colon 0 < x \leqq a,\ 0 < y \leqq b\}$.

2. $E = D \times \{-\infty < u < +\infty\} \times \{-\infty < v < +\infty\} \times \{-\infty < w < < +\infty\}$ and $E_1 = D_1 \times \{-\infty < u < +\infty\} \times \{-\infty < v < +\infty\} \times \times \{-\infty < w < +\infty\}$.

3. Let the function $\varphi(x) \in C^1(\langle 0, a\rangle)$, $\psi(x) \in C^1(\langle 0, b\rangle)$ and $\varphi(0) = = \psi(0)$.

4. Let us denote the set of all functions $z(x, y) \in C^1(D)$ satisfying the conditions $z(x, 0) = \varphi(x)$ for $x \in \langle 0, a\rangle$, $z(0, y) = \psi(y)$ for $y \in \langle 0, b\rangle$ by $M(D)$.

5. Let $f(x, y, u, v, w)$ be a continuous function on the domain E. We shall understand by *the solution of the Darboux problem*

$$\frac{\partial^2 u}{\partial x\, \partial y} = f(x, y, u, u_x, u_y) \tag{1}$$

$$(2) \qquad u(x, 0) = \varphi(x), \quad u(0, y) = \psi(y), \quad \varphi(0) = \psi(0)$$

an arbitrary function $u(x, y) \in M(D)$ which has a continuous second partial derivative u_{xy} on D and identifies the equation (1).

Then the Darboux problem (1), (2) is equivalent to solving the integro-differential equation

$$(3) \qquad u(x, y) = g_0(x, y) + \int_0^x \int_0^y f[\sigma, \tau, u(\sigma, \tau), u_x(\sigma, \tau), u_y(\sigma, \tau)] \, d\sigma \, d\tau$$

where $g_0(x, y) = \varphi(x) + \psi(y) - \varphi(0)$. With respect to (3), *the sequence of Picard's successive approximations* $\{u_n\}_1^\infty$ is defined by the equation

$$(4) \qquad u_n(x, y) = g(x, y) + \int_0^x \int_0^y f[\sigma, \tau, u_{n-1}(\sigma, \tau), u_{n-1_x}(\sigma, \tau), u_{n-1_y}(\sigma, \tau)] \, d\sigma \, d\tau$$

and the sequences of the derivatives $\{u_{n_x}(x, y)\}_1^\infty$, $\{u_{n_y}(x, y)\}_1^\infty$ are determinated, by:

$$(4_1) \qquad \frac{\partial u_n(x, y)}{\partial x} = g_x(x, y) + \int_0^y f[x, \tau, u_{n-1}(x, \tau), u_{n-1_x}(x, \tau), u_{n-1_y}(x, \tau)] \, d\tau$$

$$(4_2) \qquad \frac{\partial u_n(x, y)}{\partial y} = g_y(x, y) + \int_0^x f[\sigma, y, u_{n-1}(\sigma, y), u_{n-1_x}(\sigma, y), u_{n-1_y}(\sigma, y)] \, d\sigma$$

for $n = 1, 2, \ldots$ and $(x, y) \in D$, where $g(x, y)$, $u_0(x, y)$ are arbitrary functions from the set $M(D)$ such, that $g_{xy} = 0$ on D.

4. THEOREMS ON THE EXISTENCE AND UNIQUENESS

In the following theorem we shall use the generalized conditions of Krasnosielski and Krein.

Theorem 2. *Let $f(x, y, u, v, w)$ be defined continuous and bounded on E and let it satisfy the conditions*

$$(5) \qquad | f(x, y, u_1, v_1, w_1) - f(x, y, u_2, v_2, w_2) | \leqq$$
$$\leqq \frac{k}{xy} \left(| u_1 - u_2 | + \frac{x}{\sqrt{k}} | v_1 - v_2 | + \frac{y}{\sqrt{k}} | w_1 - w_2 | \right), \quad k > 0$$

$$(6) \qquad | f(x, y, u_1, v_1, w_1) - f(x, y, u_2, v_2, w_2) | \leqq$$
$$\leqq \frac{C}{x^\beta y^\beta} (| u_1 - u_2 |^\alpha + x^\alpha | v_1 - v_2 |^\alpha + y^\alpha | w_1 - w_2 |^\alpha, \quad C > 0$$

with $0 < \alpha < 1$, $\beta < \alpha$ *and* $9k(1-\alpha)^2 < (1-\beta)^2$ *for all* $(x, y, u_j, v_j, w_j) \in \in E_1$, $j = 1, 2$. *Then there exists one and only one solution* $u(x, y)$ *of the Darboux problem* (1), (2) *and moreover Picard's sequence of successive approximations, which is defined by the equation* (4) *for any function* $u_0(x, y) \in M(D)$ *and* $g(x, y) \in M(D)$ *such, that* $g_{xy} = 0$ *on* D, *converges uniformly on* D *to this unique solution.*

Proof. For the proof of this theorem we shall apply Theorem 1. In this way we must choose a suitable generalized complete metric space X and an operator T mapping the space X into itself, and to show that this operator fulfils the conditions 1°, 2°, 3°.

On the set $M(D)$ let us define the distance

$$(7)\qquad \sup_{D_1}\left\{\frac{|z_1(x,y)-z_2(x,y)|+\dfrac{x}{\sqrt{k}}\left|\dfrac{\partial z_1(x,y)}{\partial x}-\dfrac{\partial z_2(x,y)}{\partial x}\right|}{(xy)^{p\sqrt{k}}}+\frac{\dfrac{y}{\sqrt{k}}\left|\dfrac{\partial z_1(x,y)}{\partial y}-\dfrac{\partial z_2(x,y)}{\partial y}\right|}{(xy)^{p\sqrt{k}}}\right\}$$

for an arbitrary pair of the elements $z_1, z_2 \in M(D)$, where p satisfies the inequalities $p^2k(1-\alpha)^2 < (1-\beta)^2$, $p^2k > 1$. Hence we immediately see, by the hypothesis $9k(1-\alpha)^2 < (1-\beta)^2$ that $p \in \big(3, 1/\sqrt{k}(1-\beta)/(1-\alpha)\big)$. The function $d(z_1, z_2)$ defined by relation (7) evidently fulfils the properties of the metric a), b), c), which are given in the part 2. From the inequality

$$(8)\qquad \max_{D}\left\{(ab)^{-p\sqrt{k}}\,|z_1-z_2|+\frac{a^{-p\sqrt{k}+1}\,b^{-p\sqrt{k}}}{\sqrt{k}}\left|\frac{\partial z_1}{\partial x}-\frac{\partial z_2}{\partial x}\right|+\frac{a^{-p\sqrt{k}}\,b^{-p\sqrt{k}+1}}{\sqrt{k}}\left|\frac{\partial z_1}{\partial y}-\frac{\partial z_2}{\partial y}\right|\right\}\leqq d(z_1, z_2)$$

follows that d-convergence of the sequence $\{z_n(x, y)\}_1^\infty$ of the functions $z_n(x, y) \in M(D)$ for $n = 1, 2, \ldots$ implies the convergence of the sequences

$$\{z_n(x, y)\}_1^\infty,\quad \left\{\frac{\partial z_n(x, y)}{\partial x}\right\}_1^\infty,\quad \left\{\frac{\partial z_n(x, y)}{\partial y}\right\}_1^\infty$$

in the sense of the distance

$$(8_1)\qquad d(z_1, z_2) = \max_{D} |z_1 - z_2|.$$

In addition we have the equalities

$$(8_2)\qquad \lim_{n\to\infty} z_n(x, y) = Z(x, y) \in M(D),\quad \lim_{n\to\infty} \frac{\partial z_n(x, y)}{\partial x} = \frac{\partial Z(x, y)}{\partial x},$$

$$\lim_{n\to\infty}\frac{\partial z_n(x,y)}{\partial y}=\frac{\partial Z(x,y)}{\partial y}.$$

Let the sequence $\{z_n(x,y)\}_1^\infty$ be d-Cauchy now, i.e. $\lim_{m,n\to\infty} d(z_m, z_n) = 0$. Then the inequality

$$d(z_m, z_n)=\sup_{D_1}\frac{|z_m-z_n|+\frac{x}{\sqrt{k}}|z_{m_x}-z_{n_x}|+\frac{y}{\sqrt{k}}|z_{m_y}-z_{n_y}|}{(xy)^{p\sqrt{k}}}<\varepsilon$$

holds for any $\varepsilon > 0$ and $m, n > N(\varepsilon)$, where $N(x) > 0$ is real valued function. This implies that the sequences

$$\left\{\frac{1}{(xy)^{p\sqrt{k}}}z_n\right\}_1^\infty,\quad\left\{\frac{x}{\sqrt{k}\,(xy)^{p\sqrt{k}}}z_{n_x}\right\}_1^\infty,\quad\left\{\frac{y}{\sqrt{k}\,(xy)^{p\sqrt{k}}}z_{n_y}\right\}_1^\infty$$

converge uniformly on the domain D_1 and from (8_2) it follows that the inequalities

$$\frac{1}{(xy)^{p\sqrt{k}}}|z_n-Z|<\frac{\varepsilon}{3}\quad\text{for}\quad n>N_1(\varepsilon),$$

$$\frac{x}{\sqrt{k}\,(xy)^{p\sqrt{k}}}|z_{n_x}-Z_x|<\frac{\varepsilon}{3}\quad\text{for}\quad n>N_2(\varepsilon),$$

$$\frac{y}{\sqrt{k}\,(xy)^{p\sqrt{k}}}|z_{n_y}-Z_y|<\frac{\varepsilon}{3}\quad\text{for}\quad n>N_3(\varepsilon)$$

are satisfied on D_1. If we denote $N_0(\varepsilon) = \max(N_1, N_2, N_3)$ then we have $d(z_n, Z) \leqq \varepsilon$ for $n > N_0(\varepsilon)$, i.e. $\lim_{n\to\infty} d(z_n, Z) = 0$. This ends the proof of property d). Consequently the set $M(D)$, on which the distance is defined by the equality (7), is the required generalized complete metric space X.

The operator T defined by the relation

$$(9)\qquad Tu(x,y)=g(x,y)+\int_0^x\int_0^y f\left(\sigma,\tau,u(\sigma,\tau),\frac{\partial u(\sigma,\tau)}{\partial x},\frac{\partial u(\sigma,\tau)}{\partial y}\right)d\sigma\,d\tau$$

for $(x, y) \in D$ maps the space X into itself. Moreover the following relations

$$(9_1)\qquad (Tu)_x=\frac{\partial}{\partial x}Tu(x,y)=g_x(x,y)+$$

$$+\int_0^y f\left(x,\tau,u(x,\tau),\frac{\partial u(x,\tau)}{\partial x},\frac{\partial u(x,\tau)}{\partial y}\right)d\tau$$

$$(9_2) \qquad (Tu)_y = \frac{\partial}{\partial y} Tu(x, y) =$$

$$= g_y(x, y) + \int_0^x f\left(\sigma, y, u(\sigma, y), \frac{\partial u(\sigma, y)}{\partial x}, \frac{\partial u(\sigma, y)}{\partial y}\right) d\sigma,$$

hold on D. Therefore, the problem to find the solution of the Darboux problem (1), (2) or of the integro-differential equation (3) is transformed to the problem of finding the fixed point of the mapping T on the set $M(D)$. The sequence of Picard's approximations (4) is equivalent to the sequence $\{u_n = Tu_{n-1}\}_1^\infty$ and sequences of the derivatives (4_1), (4_2) are equivalent to the sequences $\{u_{n_x} = (Tu_{n-1})_x\}_1^\infty$, $\{u_{n_y} = (Tu_{n-1})_y\}_1^\infty$ for any function $u_0(x, y) \in M(D)$.

Proof of the property 1°. Let u_1, u_2 be two arbitrary functions from the space X with $d(u_1, u_2) < +\infty$. Then by the hypothesis (5) we have

$$|Tu_1 - Tu_2| \leqq \int_0^x \int_0^y |f(\sigma, \tau, u_1, u_{1_x}, u_{1_y}) - f(\sigma, \tau, u_2, u_{2_x}, u_{2_y})| \, d\sigma \, d\tau \leqq$$

$$\leqq k \int_0^x \int_0^y \frac{|u_1 - u_2| + \frac{\sigma}{\sqrt{k}} |u_{1_x} - u_{2_x}| + \frac{\tau}{\sqrt{k}} |u_{1_y} - u_{2_y}|}{(\sigma\tau)^{p\sqrt{k}}} (\sigma\tau)^{p\sqrt{k}-1} \, d\sigma \, d\tau \leqq$$

$$\leqq d(u_1, u_2) \frac{(xy)^{p\sqrt{k}}}{p}$$

for $(x, y) \in D_1$. Similarly we obtain in D_1 the following inequalities

$$\frac{x}{\sqrt{k}} \left| \frac{\partial}{\partial x} Tu_1 - \frac{\partial}{\partial x} Tu_2 \right| \leqq$$

$$\leqq x\sqrt{k} \int_0^y \frac{|u_1 - u_2| + \frac{x}{\sqrt{k}} |u_{1_x} - u_{2_x}| + \frac{\tau}{\sqrt{k}} |u_{1_y} - u_{2_y}|}{(x\tau)^{p\sqrt{k}}} (x\tau)^{p\sqrt{k}-1} \, d\tau \leqq$$

$$\leqq d(u_1, u_2) \frac{(xy)^{p\sqrt{k}}}{p}$$

$$\frac{y}{\sqrt{k}} \left| \frac{\partial}{\partial y} Tu_1 - \frac{\partial}{\partial y} Tu_2 \right| \leqq$$

$$\leqq y\sqrt{k}\int_0^x \frac{|u_1-u_2|+\frac{\sigma}{\sqrt{k}}|u_{1_x}-u_{2_x}|+\frac{y}{\sqrt{k}}|u_{1_y}-u_{2_y}|}{(\sigma y)^{p\sqrt{k}}}(\sigma y)^{p\sqrt{k}-1}\,d\sigma \leqq$$

$$\leqq d(u_1, u_2)\,\frac{(xy)^{p\sqrt{k}}}{p}.$$

From the given inequalities it follows

$$d(Tu_1, Tu_2) \leqq \lambda d(u_1, u_2),$$

where $\lambda = 3/p$. From there, the first condition of Theorem 1 is proved.

Now we shall use the boundedness of the function $f(x, y, u, v, w)$ on E and the assumption (6) to prove condition 2°. We denote $M = \sup_E |f(x, y, u, v, w)|$. Then by (4), (4_1), (4_2) for any function $u_0(x, y) \in X$:

$$|u_2(x, y) - u_1(x, y)| \leqq 2Mxy,$$

$$(10)\qquad \left|\frac{\partial u_2(x, y)}{\partial x} - \frac{\partial u_1(x, y)}{\partial x}\right| \leqq 2My,\quad \left|\frac{\partial u_2(x, y)}{\partial y} - \frac{\partial u_1(x, y)}{\partial y}\right| \leqq 2Mx,$$

holds in the domain D. By relations (10) and (6) the estimations

$$|u_3(x, y) - u_2(x, y)| \leqq$$

$$\int_0^x\int_0^y |f(\sigma, \tau, u_2, u_{2_x}, u_{2_y}) - f(\sigma, \tau, u_1, u_{1_x}, u_{1_y})|\,d\sigma\,d\tau \leqq$$

$$\leqq C\int_0^x\int_0^y \frac{|u_1-u_2|^\alpha + \sigma^\alpha|u_{1_x}-u_{2_x}|^\alpha + \tau^\alpha|u_{1_y}-u_{2_y}|^\alpha}{\sigma^\beta\tau^\beta}\,d\sigma\,d\tau \leqq$$

$$\leqq 3C\int_0^x\int_0^y \frac{(2M)^\alpha\sigma^\alpha\tau^\alpha}{\sigma^\beta\tau^\beta}\,d\sigma\,d\tau \leqq 3C(2M)^\alpha(xy)^{\alpha-\beta}(xy),$$

for $(x, y) \in D_1$.

Similarly, it is possible to show that

$$\left|\frac{\partial u_3(x, y)}{\partial x} - \frac{\partial u_2(x, y)}{\partial x}\right| \leqq 3C(2M)^\alpha(xy)^{\alpha-\beta}\,y,$$

$$\left|\frac{\partial u_3(x, y)}{\partial y} - \frac{\partial u_2(x, y)}{\partial y}\right| \leqq 3C(2M)^\alpha(xy)^{\alpha-\beta}\,x,$$

in the domain D_1. The following inequalities can be proved by the mathematical induction for an arbitrary $n = 0, 1, 2, \dots$ and $(x, y) \in D_1$

(11)
$$\begin{aligned}
&| u_{n+3}(x, y) - u_{n+2}(x, y) | \leqq \\
&\leqq (3C)^{1+\alpha+\dots+\alpha^n} (2M)^{\alpha^{n+1}} (xy)^{(\alpha-\beta)(1+\alpha+\dots+\alpha^n)} (xy), \\
&\left| \frac{\partial u_{n+3}(x, y)}{\partial x} - \frac{\partial u_{n+2}(x, y)}{\partial x} \right| \leqq \\
&\leqq (3C)^{1+\alpha+\dots+\alpha^n} (2M)^{\alpha^{n+1}} (xy)^{(\alpha-\beta)(1+\alpha+\dots+\alpha^n)} y, \\
&\left| \frac{\partial u_{n+3}(x, y)}{\partial y} - \frac{\partial u_{n+2}(x, y)}{\partial y} \right| \leqq \\
&\leqq (3C)^{1+\alpha+\dots+\alpha^n} (2M)^{\alpha^{n+1}} (xy)^{(\alpha-\beta)(1+\alpha+\dots+\alpha^n)} x.
\end{aligned}$$

From the above mentioned relations (11) it follows that

(12)
$$\begin{aligned}
&| u_{n+3}(x, y) - u_{n+2}(x, y) | + \frac{x}{\sqrt{k}} | u_{n+3_x}(x, y) - u_{n+2_x}(x, y) | + \\
&+ \frac{y}{\sqrt{k}} | u_{n+3_y}(x, y) - u_{n+2_y}(x, y) | \leqq \\
&\leqq (3C)^{1+\alpha+\dots+\alpha^n} (2M)^{\alpha^{n+1}} \left(1 + \frac{2}{\sqrt{k}}\right) (xy)^{(\alpha-\beta)(1+\alpha+\dots+\alpha^n)+1}.
\end{aligned}$$

The hypothesis $p^2k(1-\alpha)^2 < (1-\beta)^2$ guarantees the existence of the number $N(p)$ such that for $n \geqq N(p)$ we have

$$\begin{aligned}
(\alpha-\beta)(1+\alpha+\dots+\alpha^n) + 1 &= (1-\beta)(1+\alpha+\dots+\alpha^n) + \alpha^{n+1} = \\
&= \frac{1-\beta}{1-\alpha}(1-\alpha^{n+1}) + \alpha^{n+1} > p\sqrt{k}.
\end{aligned}$$

Consequently $d(u_{n+1}, u_n) < +\infty$ for $n \geqq N(p) + 2$. Then on the basis of property c) and of distance (7) we conclude that condition 2° is proved.

Proof of 3°. Let us suppose that $u, v \in X$ are two fixed points of the mapping T, i.e. $Tu = u$, $Tv = v$. Using the method from the proof of condition 2° we obtain for the difference of the functions u, v and their derivatives u_x, u_y, v_x, v_y estimates (11) and inequality (12) too. Hence it follows $d(u, v) < +\infty$. Thereby, we have proved the existence, the uniqueness of the solution of the integro-differential equation (3) and the uniform convergence of successive approximations (4) to this solution for any function $u_0(x, y) \in M(D)$. The proof of Theorem 2 is given.

In the following two theorems we shall generalize the Nagumo-Perron van Kampen's assumption of paper [6] and use it to consider the Darboux problem (1), (2).

Let us assume that T is the operator defined by relation (9) and $TM(D)$ is the set of all the images of the set $M(D)$ under the mapping T. If we denote the complete metric space which was obtained by the completion of the metric space $[TM(D), d_2]$ in the sense of the distance

$$(13)\quad d_2(z_1, z_2) = \max_D \left(| z_1 - z_2 | + \left| \frac{\partial z_1}{\partial x} - \frac{\partial z_2}{\partial x} \right| + \left| \frac{\partial z_1}{\partial y} - \frac{\partial z_2}{\partial y} \right| \right)$$

by $[M^*(D), d_2]$, then the following theorem holds:

Theorem 3. *Let $f(x, y, u, v, w)$ be a function defined and continuous on E and let it fulfil the following conditions:*

$$(14)\qquad | f(x, y, u, v, w) | \leqq A(xy)^p,\ p \geqq 0,\ A > 0$$

for $(x, y, u, v, w) \in E$.

$$(15)\qquad | f(x, y, u_1, v_1, w_1) - f(x, u_2, v_2, w_2) | \leqq$$
$$\leqq \frac{C}{(xy)^r} (| u_1 - u_2 |^q + x^q | v_1 - v_2 |^q + y^q | w_1 - w_2 |^q)$$

for $(x, y, u_j, v_j, w_j) \in E_1$, $j = 1, 2$, where $q \geqq 1$, $c > 0$, $q(1 + p) - r = p$, $3C(2A)^{q-1}/(p + 1)^q < 1$. Then there exists one and only one solution $u(x, y) \in M^*(D)$ of the Darboux problem (1), (2), and moreover the sequence of Picard's approximations defined by (4) for any functions $g(x, y)$, $u_0(x, y) \in M(D)$with $g_{xy} = 0$ in D, converges uniformly on D to this unique solution.

Proof: The proof will be given similarly as that of Theorem 1. The set $M^*(D)$ is a subset of the set $M(D)$, by (13). On the set we can define the distance

$$(16)\quad d(z_1, z_2) = \sup_{D_1} \left\{ \frac{| z_1(xy) - z_2(x, y) | + x \left| \frac{\partial z_1(x, y)}{\partial x} - \frac{\partial z_2(x, y)}{\partial x} \right|}{(xy)^{p+1}} + \right.$$
$$\left. + \frac{y \left| \frac{\partial z_1(x, y)}{\partial y} - \frac{\partial z_2(x, y)}{\partial y} \right|}{(xy)^{p+1}} \right\}.$$

The operator T by (9) mapps the set $M^*(D)$ into itself. From the inequalities

$$(17)\qquad \max_D \left\{ (ab)^{-p-1} | z_1 - z_2 | + a^{-p} b^{-p-1} \left| \frac{\partial z_1}{\partial x} - \frac{\partial z_2}{\partial x} \right| + \right.$$
$$\left. + a^{-p-1} b^{-p} \left| \frac{\partial z_1}{\partial y} - \frac{\partial z_2}{\partial y} \right| \right\} \leqq d(z_1, z_2)$$

property d) of the metric space $X = [M^*(D), d]$ follows:

The d-Cauchy convergence of the sequence $\{z_n(x, y)\}_1^\infty$ of the function $z_n(x, y) \in M^*(D)$ implies the convergence of the sequence $\{z_n(x, y)\}_1^\infty$ and sequences $\left\{\frac{\partial z_n(x, y)}{\partial x}\right\}_1^\infty$, $\left\{\frac{\partial z_n(x, y)}{\partial y}\right\}_1^\infty$ in the sence of the distance (8_1) and

$$\lim_{n\to\infty} z_n(x, y) = Z(x, y)$$

$$\lim_{n\to\infty} \frac{\partial z_n(x, y)}{\partial x} = \frac{\partial Z(x, y)}{\partial x}, \quad \lim_{n\to\infty} \frac{\partial z_n(x, y)}{\partial y} = \frac{\partial Z(x, y)}{\partial y}$$

for $(x, y) \in D$. Hence we see that the sequence $\{z_n(x, y)\}_1^\infty$ converges in distance (13) to the function $Z(x, y)$ and $Z(x, y) \in X$. Then we shall show analogically to Theorem 2 that the $\lim\limits_{n\to\infty} d(z_n, Z) = 0$.

The proof of the condition 1°. Let $z_1(x, y)$, $z_2(x, y)$ be two arbitrary elements of X with the distance $d(z_1, z_2) < +\infty$. Then from (13) and (14) we obtain:

$$(18) \qquad |z_1(x, y) - z_2(x, y)| \leqq \frac{2A}{p+1} (xy)^{p+1}$$

$$x\,|z_{1_x}(x, y) - z_{2_x}(x, y)| \leqq \frac{2A}{p+1} (xy)^{p+1},$$

$$y\,|z_{1_y}(x, y) - z_{2_y}(x, y)| \leqq \frac{2A}{p+1} (xy)^{p+1}$$

for $(x, y) \in D$. Moreover, by (15) and (18) we have the following estimates in the domain D_1:

$$|Tz_1(x, y) - Tz_2(x, y)| \leqq$$

$$\leqq C \int_0^x \int_0^y \frac{|z_1 - z_2|^q + \sigma^q |z_{1_x} - z_{2_x}|^q + \tau^q |z_{1_y} - z_{2_y}|^q}{(\sigma\tau)^r} d\sigma\, d\tau \leqq$$

$$\leqq C \left(\frac{2A}{p+1}\right)^{q-1} \int_0^x \int_0^y (\sigma\tau)^{(p+1)(q-1)-r+p+1} \times$$

$$\times \frac{|z_1 - z_2| + \sigma |z_{1_x} - z_{2_x}| + \tau |z_{1_y} - z_{2_y}|}{(\sigma\tau)^{p+1}} d\sigma\, d\tau \leqq$$

$$\leqq C \frac{(2A)^{q-1}}{(p+1)^q} d(z_1, z_2)\, (xy)^{p+1},$$

$$x \left| \frac{\partial}{\partial x} Tz_1(x, y) - \frac{\partial}{\partial x} Tz_2(x, y) \right| \leqq$$

$$\leqq Cx \int_0^y \frac{|z_1 - z_2|^q + x^q |z_{1_x} - z_{2_x}|^q + \tau^q |z_{1_y} - z_{2_y}|^q}{(x\tau)^r} d\tau \leqq$$

$$\leqq C \frac{(2A)^{q-1}}{(p+1)^q} d(z_1 z_2)\,(xy)^{p+1},$$

$$y \left| \frac{\partial}{\partial y} Tz_1(x, y) - \frac{\partial}{\partial y} Tz_2(x, y) \right| \leqq$$

$$\leqq Cy \int_0^x \frac{|z_1 - z_2|^q + \sigma^q |z_{1_x} - z_{2_x}|^q + y^q |z_{1_y} - z_{2_y}|^q}{(\sigma y)^r} d\sigma \leqq$$

$$\leqq C \frac{(2A)^{q-1}}{(p+1)^q} d(z_1, z_2)\,(xy)^{p+1}.$$

From the last inequalities it follows that $d(Tz_1, Tz_2) \leqq \dfrac{3C(2A)^{q-1}}{(p+1)^q} d(z_1, z_2)$ Thus condition 1° is proved.

Condition 2° follows directly from (18) because hence we have $d(u_n, u_{n+1}) \leqq \dfrac{6A}{p+1} < +\infty$ for a $n = 1, 2, \ldots$ From (18) we obtain condition 3° too.

Remark 1. The assuption (14) of Theorem 3 guarantees the boundedness of the function $f(x, y, u, v, w)$ in E. In the following theorem we shall show that this assumption is not necessary.

Theorem 4. *Let the function $f(x, y, u, v, w)$ be continuous in E and let it satisfy the following conditions:*

(19) $$|f(x, y, u, v, w)| \leqq A(x, y)(xy)^p, \quad -1 < p < 0$$

for all $(x, y, u, v, w) \in E_1$. The function $A(x, y)$ is integrable in the domain D, in the interval $\langle 0, a \rangle$ with respect to the variable x, in $\langle 0, b \rangle$ with respect to y and it satisfies the inequalities $0 \leqq A(x, y) \leqq A_0$, $A(x, y) \leqq A_0 x^{-p}$, $A(x, y) \leqq A_0 y^{-p}$ for $A_0 \geqq 0$ in D. Let further, the inequality

(20) $$|f(x, y, u_1, v_1, w_1) - f(x, y, u_2, v_2, w_2)| \leqq$$
$$\leqq \frac{C(x, y)}{(xy)^r} (|u_1 - u_2|^q - x^{q(p+1)} |v_1 - v_2|^q + y^{q(p+1)} |w_1 - w_2|^q),$$

for all $(x, y, u_j, v_j, w_j) \in E_1$, $j = 1, 2$, where $q \geqq 1$ and $q(p+1) - r = p$, $\dfrac{C_0}{(p+1)^2}\left[\dfrac{2A_0}{(p+1)^2}\right]^{q-1} < 1$ hold. The function $C(x, y)$ is integrable in the domain D, in the interval $\langle 0, a \rangle$ and $\langle 0, b \rangle$ with respect to the variable x and y, and furthermore the inequalities $0 \leqq C(x, y) \leqq C_1$, $C(x, y) \leqq$

$\leqq C_2x^{-p}$, $C(x, y) \leqq C_3y^{-p}$, $C_0 = C_1 + C_2 + C_3$ hold in D, where C_1, C_2, C_3 are suitable positive constants. Then there exists one and only one solution of the Darboux problem (1), (2), and moreover the sequence of Picard's approximations defined by (4) for arbitrary functions $g(x, y)$, $u_0(x, y) \in M(D)$ with $g_{xy} = 0$ in D, converges uniformly on D to this unique solution.

Proof. The operator T is defined by the relation (9) as in the preceding theorems. Analogically to Theorem 3 it can also be shown that the set $M^*(D)$ with distance

$$(21)\quad d(z_1, z_2) = \sup_{D_1} \left\{ \frac{|z_1(x,y) - z_2(x,y)| + x^{p+1}\left|\dfrac{\partial z_1(x,y)}{\partial x} - \dfrac{\partial z_2(x,y)}{\partial x}\right|}{(xy)^{p+1}} + \right.$$
$$\left. + \frac{y^{p+1}\left|\dfrac{\partial z_1(xy)}{\partial y} - \dfrac{\partial z_2(x,y)}{\partial y}\right|}{(xy)^{p+1}} \right\}$$

is a complete generalized metric space. Let us denote it by X. From (13) and (19) it follows that

$$(22)\quad |z_1 - z_2| \leqq 2 \int_0^x\int_0^y A(\sigma,\tau)(\sigma\tau)^p \,d\sigma\, d\tau \leqq 2A_0 \frac{(xy)^{p+1}}{(p+1)^2},$$

$$|z_{1_x} - z_{2_x}| \leqq 2 \int_0^y A(x,\tau)(x\tau)^p \,d\tau \leqq 2A_0 \frac{y^{p+1}}{(p+1)^2},$$

$$|z_{1_y} - z_{2_y}| \leqq 2 \int_0^x A(\sigma,y)(\sigma y)^p \,d\sigma \leqq 2A_0 \frac{x^{p+1}}{(p+1)^2}$$

in D_1 for any pair of functions $z_1, z_2 \in M^*(D)$. From inequalities (22) and assumption (20) we obtain the estimates

$$|Tz_1 - Tz_2| \leqq \int_0^x\int_0^y \left\{ \frac{C(\sigma,\tau)(|z_1 - z_2|^q + \sigma^{q(p+1)}|z_{1_x} - z_{2_x}|^q)}{(\sigma\tau)^r} + \right.$$
$$\left. + \frac{C(\sigma,\tau)\,\tau^{q(p+1)}\,|z_{1_y} - z_{2_y}|^q}{(\sigma\tau)^r} \right\} d\sigma\, d\tau \leqq$$

$$\leqq \left[\frac{2A_0}{(p+1)^2}\right]^{q-1} d(z_1, z_2) \int_0^x\int_0^y C(\sigma,\tau)(\sigma\tau)^{(p+1)(q-1)-r+p+1} \,d\sigma\, d\tau \leqq$$

$$\leqq \left[\frac{2A_0}{(p+1)^2}\right]^{q-1} \mathrm{d}(z_1, z_2)\,(xy)^{p+1} \frac{C_1}{(p+1)^2},$$

$$\left|\frac{\partial}{\partial x} Tz_1 - \frac{\partial}{\partial x} Tz_2\right| \leqq \left[\frac{2A_0}{(p+1)^2}\right]^{q-1} d(z_1, z_2)\, y^{p+1} \frac{C_2}{(p+1)^2},$$

$$\left|\frac{\partial}{\partial y} Tz_1 - \frac{\partial}{\partial y} Tz_2\right| \leqq \left[\frac{2A_0}{(p+1)^2}\right]^{q-1} d(z_1, z_2)\, x^{p+1} \frac{C_3}{(p+1)^3}$$

for $(x, y) \in D_1$. Hence we have the inequality

$$d(Tz_1, Tz_2) \leqq \left[\frac{2A_0}{(p+1)^2}\right]^{q-1} \frac{C_0}{(p+1)^2} d(z_1, z_2)$$

Condition 1° of Theorem 1 is proved. The proof of the properties 2°, 3° we get from the inequalities (22).

REFERENCES

[1] Kampen van E. R., *Notes on systems of ordinary differential equations*, American Journal of Math. 63 (1941), pp. 371—376.

[2] Luxemburg W. A. J., *On the convergence of successive approximations in the theory of ordinary differential equations II*, Indag. Math. 20 (1958), pp. 540—546.

[3] Luxemburg W. A. J., *On the convergence of successive approximations in the theory of ordinary differential equations III*, Nieuw Archief voor Wiskunde (3), VI (1958), pp. 93—98.

[4] Palczewski B., *On the uniqueness of solutions and the convergence of successive approximations in the Darboux problem under the conditions of the Krasnosilski and Krein type*, Ann. polon. Math. 14 (1964), pp. 183—190.

[5] Ф. Трикоми: *Лекции по уравнениям в частных производных*, Москва 1957, pp 205—216.

[6] Wong J. S. W., *On the convergence of successive approximations in the Darboux problem*, Ann. Polon. Math. XVII (1966), pp. 329—336.

[7] Wong J. S. W., *Remarks on uniqueness theorem of solutions of the Darboux problem*, Canad. Math. Bull. 8 (1965), pp. 791—796.

Department of Mathematics
J. A. Komenský University,
Bratislava