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ON THE EXISTENCE OF A BOUNDED SOLUTION OF A NON-LINEAR DIFFERENTIAL EQUATION

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In this paper the existence of a solution of the differential equation

$$y'' - y = f(x, y, y') \tag{1}$$

which is bounded together with its first derivative on the whole real line, is proved under the condition that the function f is continuous and bounded.

Let us consider first the existence of a bounded solution of the linear differential equation

$$y'' - y = f(x), \tag{2}$$

where $f(x) \in C^\circ(I)$, $I = (-\infty, \infty)$ and $|f(x)| \leqq K$ on I. It is clear that this equation has at most one such solution. We are going to give the proof of the existence of a bounded solution of the equation (2) as follows. A hint of this proof was given in an exercise in the book [1], p. 297.

Homogeneous boundary-value problem

$$\begin{gathered} y'' - y = 0, \qquad x \in \langle -a, a\rangle \\ y(-a) = y(a) = 0, \end{gathered} \tag{3}$$

where $a > 0$, has only a trivial solution. Therefore the inhomogeneous boundary-value problem (2), (3) has one and only one solution

$$y_a(x) = \int_{-a}^{a} G_a(x, t) f(t)\, \mathrm{d}t, \qquad x \in \langle -a, a\rangle \tag{4}$$

where

$$G_a(x, t) = \begin{cases} \dfrac{(-\mathrm{e}^{-2a+t} + \mathrm{e}^{-t})(-\mathrm{e}^{2a+x} + \mathrm{e}^{-x})}{2(\mathrm{e}^{2a} - \mathrm{e}^{-2a})} & -a \leqq x \leqq t \\ \dfrac{(-\mathrm{e}^{2a+t} + \mathrm{e}^{-t})(-\mathrm{e}^{-2a+x} + \mathrm{e}^{-x})}{2(\mathrm{e}^{2a} - \mathrm{e}^{-2a})} & t \leqq x \leqq a \end{cases} \tag{5}$$

is the Green's function of the problem (2), (3). The solution $y_a(x)$ can be continued to the whole interval I. Further there exists

$$y(x) = \lim_{a\to\infty} y_a(x) = -\frac{1}{2} \int_{-\infty}^{\infty} \mathrm{e}^{-|x-t|} . f(t)\, \mathrm{d}t \tag{6}$$

where the convergence is uniform on every closed interval $\langle -b, b\rangle$, $b > 0$. For $x \in \langle -b, b\rangle$, $a \geqq b$ the inequality

$$\left| \int_{-a}^{a} G_a(x, t) f(t) \, dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-t|} . f(t) \, dt \right| \leqq \left| \int_{-a}^{a} G_a(x, t) f(t) \, dt + \right.$$

$$\left. + \frac{1}{2} \int_{-a}^{a} e^{-|x-t|} . f(t) \, dt \right| + \left| \frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-t|} . f(t) \, dt - \frac{1}{2} \int_{-a}^{a} e^{-|x-t|} . f(t) \, dt \right|$$

is true. Assuming that $a \geqq 2b > 0$ we estimate first the first term on the right side of the inequality.

$$\left| \int_{-a}^{a} G_a(x, t) f(t) \, dt + \frac{1}{2} \int_{-a}^{a} e^{-|x-t|} . f(t) \, dt \right| \leqq$$

$$\leqq \left| \int_{-a}^{x} \frac{(-e^{2a+t} + e^{-t})(-e^{-2a+x} + e^{-x})}{2(e^{2a} - e^{-2a})} f(t) \, dt + \frac{1}{2} \int_{-a}^{x} e^{t-x} . f(t) \, dt \right| +$$

$$+ \left| \int_{x}^{a} \frac{(-e^{-2a+t} + e^{-t})(-e^{2a+x} + e^{-x})}{2(e^{2a} - e^{-2a})} f(t) \, dt + \frac{1}{2} \int_{x}^{a} e^{x-t} f(t) \, dt \right| \leqq$$

$$\leqq \frac{K}{2(e^{2a} - e^{-2a})} \left\{ \int_{-a}^{x} | e^{t+x} + e^{-(t+x)} - e^{-2a}(e^{t-x} + e^{-(t-x)}) | \, dt + \right.$$

$$\left. + \int_{x}^{a} | e^{t+x} + e^{-(t+x)} - e^{-2a}(e^{t-x} + e^{-(t-x)}) | \, dt \right\} =$$

$$= \frac{K}{2(e^{2a} - e^{-2a})} (e^{a} + e^{-3a} - 2e^{-a})(e^{x} + e^{-x}) \leqq$$

$$\leqq \frac{2K e^{b}}{2(e^{2a} - e^{-2a})} (e^{a} + e^{-3a} - 2e^{-a}) =$$

$$= \frac{K . e^{b}}{(1 - e^{-4a})} (e^{-a} + e^{-5a} - 2e^{-3a}) < \frac{\varepsilon}{2}$$

for a sufficiently great a.

Similarly we get for a sufficiently great a, that for the second term the inequality

$$\frac{1}{2} \left| \int_{-\infty}^{\infty} e^{-|x-t|} . f(t) \, dt - \int_{-a}^{a} e^{-|x-t|} . f(t) \, dt \right| < \frac{\varepsilon}{2}$$

holds. Since (6) fulfils the conditions of the theorem on the differentiating of the parametric integral the following relation

$$(7) \qquad y'(x) = \frac{1}{2} \int_{-\infty}^{x} e^{t-x} . f(t) \, dt - \frac{1}{2} \int_{x}^{\infty} e^{-t+x} . f(t) \, dt = \int_{-\infty}^{x} e^{t-x} f(t) \, dt + y(x)$$

is true. Applying the same theorem to (7) we get

$$y''(x) = -\frac{1}{2} \int_{-\infty}^{x} e^{-x+t} . f(t) \, dt - \frac{1}{2} \int_{x}^{\infty} e^{-t+x} . f(t) \, dt + f(x) = y(x) + f(x).$$

We see that (6) is a solution of the differential equation (2). This ends the proof of

Lemma: *Let* $f(x) \in C^{\circ}(-\infty, \infty)$ *and* $|f(x)| \leqq K, K > 0, x \in (-\infty, \infty)$. *Then there exists one and only one solution* $y(x)$ *of the equation* (2) *which is bounded together with its first derivative in* $(-\infty, \infty)$. *This solution is given by the formula*

$$y(x) = -\frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-t|} . f(t) \, dt$$

and on each interval $\langle -b, b \rangle$ *is a uniform limit for* $a \to \infty$ *of the solutions* $y_a(x)$ *of the boundary-value problem* (2), (3).

Theorem: *Let* $f(x, y, y')$ *be a continuous bounded function of* $x, y, y' \in (-\infty, \infty)$. *Then there exists at least one solution of the equation* (1) *on the interval* $(-\infty, \infty)$ *which is bounded together with its first derivative in* $(-\infty, \infty)$.

Proof: On the basis of the lemma the solution of the integro-differential equation

$$(8) \qquad y(x) = -\frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-t|} . f(t, y(t), y'(t)) \, dt$$

will be a bounded solution of the equation (1) which has a bounded first derivative. The existence of the solution (8) we shall prove with the help of Tychonoff fixed point theorem.

Let $E = C^1(-\infty, \infty)$ be the space of functions on which a countable system of semi-norms

$$p_n[y(x)] = \max \left[\max_{x \in \langle -n, n \rangle} |y(x)|, \max_{x \in \langle -n, n \rangle} |y'(x)| \right], \qquad n = 1, 2, \ldots$$

is defined. By the family of these semi-norms E is complete locally convex space. Let $M = \{y(x) \in E : |y(x)| \leqq K, \ |y'(x)| \leqq 2K, \ x \in \in (-\infty, \infty)\}$, where the constant K is such that $|f(x, y, y')| \leqq K$ for all x, y, y'. The set M is closed, convex and bounded. Let

$$(9) \qquad Ty(x) = -\frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-t|} \, . f[t, y(t), y'(t)] \, dt$$

be the operator defined on the set M. We shall prove that it is continuous and compact on the set M and $TM \subset M$. Let $\varepsilon > 0$ and the natural n

$$10) \qquad n > \ln \frac{9K}{\varepsilon}$$

be arbitrary numbers. Then for the $y(x)$, $y_0(x) \in M$ and $x \in \langle -n, n \rangle$ the inequalities

$$(11) \quad |Ty(x) - Ty_0(x)| \leqq \frac{1}{2} \Big\{ \int_{-\infty}^{-2n} e^{-x+t} . |f[t, y(t), y'(t)] - f[t, y_0(t), y_0'(t)]| \, dt +$$

$$+ e^{-|x-t|} . \int_{-2n}^{2n} |f[t, y(t), y'(t)] - f[t, y_0(t), y_0'(t)]| \, dt +$$

$$+ \int_{2n}^{\infty} e^{x-t} . |f[t, y(t), y'(t)] - f[t, y_0(t), y_0'(t)]| \, dt \Big\} \leqq$$

$$\leqq \frac{1}{2} \Big\{ 2K \int_{-\infty}^{-2n} e^{-x+t} \, dt + \frac{\varepsilon}{9} \int_{-2n}^{2n} e^{-|x-t|} \, . \, dt + 2K \int_{2n}^{\infty} e^{x-t} \, dt \Big\} \leqq$$

$$\leqq K \, . \, e^{-n} + \frac{\varepsilon}{9} + K \, e^{-n} < \frac{\varepsilon}{3}$$

hold if $p_{2n}(y - y_0) < \delta$, where $\delta > 0$ is sufficiently small. Using the relation (7) we get

$$[Ty(x)]' = -\frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-t|} \, . f[t, y(t), y'(t)] \, dt + \int_{-\infty}^{x} e^{-x+t} \, . f[t, y(t), y'(t)] \, dt .$$

Then under the assumption (10) and considering (11)

$$(12) \qquad |[Ty(x)]' - [Ty_0(x)]'| \leqq \frac{\varepsilon}{3} +$$

$$+\int_{-\infty}^{x} \mathrm{e}^{-x+t}\,|\,f[t, y(t), y'(t)] - f[t, y_0(t), y_0'(t)]\,|\,\mathrm{d}t < \frac{\varepsilon}{3} +$$

$$+\int_{-\infty}^{\infty} \mathrm{e}^{-|x-t|}\,.\,|\,f[t, y(t), y'(t)] - f[t, y_0(t), y_0'(t)]\,|\,\mathrm{d}t < \frac{\varepsilon}{3} + \frac{2}{3}\varepsilon = \varepsilon.$$

It follows from the relations (11) and (12)

$$p_n[Ty(x) - Ty_0(x)] < \varepsilon, \quad \text{if} \quad p_{2n}[y(x) - y_0(x)] < \delta$$

and thus continuity of the operator (9) is proved.

The compactness of the operator (9) will be proved by the application of the Ascoli-Arzela theorem. It is therefore sufficient to show that $Ty(x)$ are equi-bounded and equi-continuous on each of the intervals $\langle -n, n\rangle$.

Let $y(x) \in M$. Then

$$|\,Ty(x)\,| = \left| -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-|x-t|}\,.\,f[t, y(t), y'(t)]\,\mathrm{d}t \right| \leqq \frac{K}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-|x-t|}\,\mathrm{d}t = K$$

$$|\,[Ty(x)]'\,| \leqq \left| \int_{-\infty}^{x} \mathrm{e}^{-x+t}\,.\,f[t, y(t), y'(t)]\,\mathrm{d}t \right| + K \leqq 2K.$$

Hence $TM \subset M$ and at the same time also the set TM is equibounded.

The equi-continuity of the functions from TM on the interval $\langle -n, n\rangle$ will be proved in the following way: Let $\varepsilon > 0$ be an arbitrary number and let

$$|\,x_1 - x_2\,| < \frac{\varepsilon}{4K}, \quad x_1 < x_2, \quad x_1, x_2 \in \langle -n, n\rangle.$$

Then

$$|\,Ty(x_1) - Ty(x_2)\,| = \frac{1}{2}\left| (\mathrm{e}^{-x_1} - \mathrm{e}^{-x_2}) \int_{-\infty}^{x_1} \mathrm{e}^{t}\,.\,f[t, y(t), y'(t)]\,\mathrm{d}t + \right.$$

$$+ \int_{x_1}^{x_2} (\mathrm{e}^{x_1-t} - \mathrm{e}^{-x_2+t})\,.\,f[t, y(t), y'(t)]\,\mathrm{d}t +$$

$$\left. + (\mathrm{e}^{x_1} - \mathrm{e}^{x_2}) \int_{x_2}^{\infty} \mathrm{e}^{-t}\,.\,f[t, y(t), y'(t)]\,\mathrm{d}t \right| \leqq$$

$$\leqq \frac{1}{2}\left[e^{-x_1} \mid x_1 - x_2 \mid K \int_{-\infty}^{x_1} e^t \, dt + K \int_{x_1}^{x_2} (e^{x_1-t} + e^{-x_2+t}) \, dt + \right.$$

$$\left. + e^{x_2} \mid x_1 - x_2 \mid \int_{x_2}^{\infty} e^{-t} \, dt \right] \leqq \frac{1}{2} (K \mid x_1 - x_2 \mid + 2K \mid x_1 - x_2 \mid +$$

$$+ K \mid x_1 - x_2 \mid) = 2K \mid x_1 - x_2 \mid < \frac{\varepsilon}{2},$$

$$\mid [Ty(x_1)]' - [Ty(x_2)]' \mid \leqq \frac{1}{2} \left| \int_{-\infty}^{\infty} (e^{-|x_1-t|} - e^{-|x_2-t|}) \, . f[t, y(t), y'(t)] \, dt \right| +$$

$$+ \left| \int_{-\infty}^{x_1} (e^{-x_1+t} - e^{-x_2+t}) \, . f[t, y(t), y'(t)] \, dt \right| +$$

$$+ \left| \int_{x_1}^{x_2} e^{-x_2+t} \, . f[t, y(t), y'(t)] \, dt \right| \leqq$$

$$\leqq 2K \mid x_1 - x_2 \mid + K \mid x_1 - x_2 \mid + K \mid x_1 - x_2 \mid = 4K \mid x_1 - x_2 \mid < \varepsilon,$$

what ends the proof of the equi-continuity of TM. It follows from what was proved that all assumptions of the Tychonoff fixed point theorem are fulfilled, so that there exists a fixed point of the operator (9) on M, which means that there exists at least one solution of the equation (1) which is bounded together with the first derivative on the whole real axis.

REFERENCES

[1] Birkhoff G., Rota G. C., *Ordinary differential equations*, Boston 1962.

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