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SOME GEOMETRIC REMARKS ABOUT DISPERSIONS

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1. O. Borůvka has developed a very interesting theory of second order linear differential equations based on the distribution of the zeros of the solutions. We refer to the monograph [1] for all definitions. Professor Borůvka's book contains a number of geometric interpretations and applications of the theory, based on centroaffine differential geometry. In this Note, we interprete a number of results on oscillatory equations within the framework of *unimodular* centroaffine differential geometry.

We consider an arc $x(\tau) = [u(\tau), v(\tau)]$ in the plane. The variable τ belongs to some interval $I \subset \boldsymbol{R}$. The coordinate functions $u(\tau)$, $v(\tau)$ are supposed to be of class C^2. Furthermore, we suppose that there exists a continuous determination of the angle

$$\alpha(\tau) = \arctan u(\tau)/v(\tau)$$

which is monotone increasing and that the Wronskian

$$\begin{vmatrix} u(\tau) & v(\tau) \\ u'(\tau) & v'(\tau) \end{vmatrix} > 0.$$

Then the radius vector of the point on the curve turns about the origin in a monotone and continuous way and the curve is concave towards the origin in the neighborhood of any of its points.

The unimodular centroaffine parameter is chosen so that the Wronskian becomes a constant. The usual choice is a parameter t defined (up to an additive constant) by

$$\begin{vmatrix} u(t) & v(t) \\ \mathrm{d}u/\mathrm{d}t & \mathrm{d}v/\mathrm{d}t \end{vmatrix} = 1.$$

By definition, $t_1 - t_0$ is twice the area covered by the radius vector in its motion from $x(t_0)$ to $x(t_1)$. The two vectors $x(t)$, $x'(t)$ are linearly independent and can serve as a frame for the curve. The Frenet equations of this frame are

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q(t) & 0 \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}$$

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$q(t)$ is the unimodular centroaffine curvature. The curve $x(t)$ is solution of the differential equation

$$x'' + q(t)\, x = 0, \qquad q(t) > 0, \qquad x(0) = x_0.$$

The unimodular centroaffine curvature is easily expressed in terms of euclidean quantities. Let Θ be the oriented angle of the tangent line to the curve and the x-axis, $h(\Theta)$ the distance of the tangent line from the origin (the support function) and ϱ the radius of curvature. Then $\Theta = \Theta(t)$ and $q(t) = \{\varrho(\Theta)\, h^3(\Theta)\}^{-1}$. Also, the curve $x'(t)$ is the image of the polar reciprocal of the tangents to $x(t)$ in a rotation of angle $+\pi/2$. The unimodular centroaffine parameter t^* of x' (twice the area) is given by $dt^* = q(t)\, dt$.

2. Let us consider now an equation

$$x'' + \lambda q(t)\, x = 0 \tag{1}$$

where $q(t) > 0$ is periodic of period π and $\lambda > 0$ is a parameter. By $\varphi_k(\lambda, t)$ we denote the k-th dispersion of the first kind [1] for the parameter value λ. *The k-th interval of instability of* (1) [7] *collapses to a point if and only if there exists a value $\lambda = \lambda_k$ for which*

$$\varphi_k(\lambda_k, t) = t + \pi. \tag{2}$$

We have to show that (2) implies that the coordinate functions $u(t)$, $v(t)$ of $x(t)$ both are k-th eigenfunctions of the Liapounoff boundary value problem

$$\begin{aligned} x(\pi) &= -x(0) \qquad && k \text{ odd} \\ x(\pi) &= x(0) && k \text{ even.} \end{aligned}$$

Since the order of an eigenfunction of a Sturm-Liouville equation is given by the number of its zeros in the interval in question, we only have to show that the functions satisfy the boundary value condition. Choose $x(0) = (1, 0)$, $x'(0) = (0, 1)$. For k even, the monotonicity of $\alpha(t)$ and the condition (2) together imply $u(\pi) > 0$, $v(\pi) = 0$. Now, if $u(\pi) \neq 1$, the fact that t is twice the area implies that the curve $x(t)$, $t > \pi$, cuts the y-axis for the first time at a point that is nearer to (farther from) the origin than the point at which $x(t)$, $t > 0$, cuts for the first time if $u(\pi) > 1\,[u(\pi) < 1]$. Hence, there is a first point of intersection of the two arcs in the first quadrant. By (2), the points 0, $x(t)$, $x(t + \pi)$ are collinear for all t. Hence, $x(t_0) = x(t_0 + \pi)$ at the point of intersection. For the areas covered by the radius vector we have

$$2A_1 = \int_0^{t_0} \begin{vmatrix} u & v \\ du & dv \end{vmatrix} = t_0$$

and

$$2A_2 = \int_{\pi}^{t_0+\pi} \begin{vmatrix} u & v \\ du & dv \end{vmatrix} = t_0.$$

This is a contradiction since $A_2 - A_1 \neq 0$ (fig. 1). Hence, $u(\pi) = 1$. The proof for k odd proceeds along the same lines.

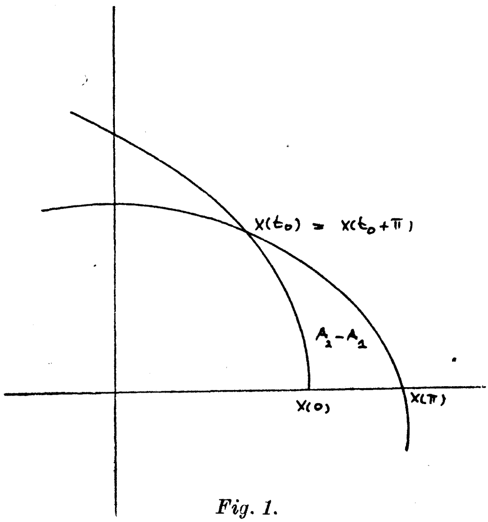

Fig. 1.

3. Since $\varrho(\Theta) = h(\Theta) + h''(\Theta)$ (see, e.g. [3]), the equations (1) which satisfy (2) can be constructed in the following way.

Let $h(\Theta)$ be a C^2 periodic function of period $k\pi$, subject to

(a) $$h(\Theta) > 0$$

(b) $$h(\Theta) + h''(\Theta) > 0$$

(c) $$\int_0^{k\pi} \{h^2(\Theta) - h'^2(\Theta)\} \, d\Theta = \pi.$$

The integral to the left hand side of (c) represents twice the surface area covered by the radius vector of a curve of support function $h(\Theta)$. The area element in terms of the support angle is $1/2h \, ds(\Theta) = 1/2h(\Theta) \, \varrho(\Theta) \, d\Theta$. Hence, the equation in Θ

$$(h + h'') hx'' - [(h + h'') h]' x' + (h + h'')^2 x = 0$$

can be transformed into the form (1) *with* $\lambda_k = 1$ *by choosing as independent variable*

$$t = \int_0^{\Theta} (h + h'')\, h \,\mathrm{d}\Theta$$

and all equations (1) *which satisfy* (2) *for* $\lambda_k = 1$ *can be obtained in that way*; at least for continuous $q(t)$. For the transformed equation,

$$\frac{1}{q(t)} = \left\{ h[\Theta(t)] + \frac{d^2 h[\Theta(t)]}{\mathrm{d}\Theta^2} \right\} h^3[\Theta(t)].$$

It is an interesting open problem to construct the functions $q(t)$ for which there exists an increasing sequence $\lambda_{j_1} < \lambda_{j_2} < \ldots$ such that $\varphi_{j_s}(\lambda_{j_s}, t)$ satisfies (2). The only known results in this direction are due to Hochstadt; see, e.g., [7].

4. In Liapounoff's theory of stability, the integral

$$I(\lambda) = \lambda\pi \int_0^{\pi} q(t)\,\mathrm{d}t$$

plays an important role. If $\varphi_1(\lambda, t) = t + \pi$, it is known [4] that

$$8 \leq I(\lambda) \leq \pi^2 \tag{3}$$

In particular, for $\lambda = 1$ is follows from [1], p. 139, that

$$f(t) = f(t + \pi), \quad f \in C^2, \quad f(c) = f'(c) = 0,$$

$$\int_0^{\pi} \frac{\exp\,[-2f(\sigma)] - 1}{\sin^2 (\sigma - c)}\,\mathrm{d}\sigma = 0$$

implies

$$0 \leq \int_0^{\pi} \{f'^2(t) + 2f'(t) \cot (t - c)\}\,\mathrm{d}t < \pi - \frac{8}{\pi} \approx 0.595. \tag{4}$$

The left hand side of (4) follows from the analytic conditions without much trouble. Similarly, the right hand side of (3) can be established analytically by known inequalities concerning the eigenvalues of (1). Geometrically, the upper bound for $I(\lambda)$ is a classical inequality of Santaló [8]. On the other hand, the left hand side of (3) says that the areas A of a symmetric oval and A^* of its polar reciprocal with respect to the center of symmetry satisfy

$$AA^* \geq 8$$

with equality only for the parallelogram (for which $q(t)$ is not a function but a point-mass distribution). No analytic proof is known for this inequality.

The inequality (3) shows the extraordinary strength of the result ([1], p. 136, 138) that for $\varphi(1, t) = t + \pi$ we have at least four solutions to $q(t) = 1$ in $(0, \pi)$. In fact, it is possible to approximate the parallelogram by convex, analytic curves and in that way to find C_ω periodic functions $q(t)$ for which $\varphi(1, t) = t + \pi$ and

$$0.81 \sim \frac{8}{\pi^2} < \mathbf{q} = \frac{1}{\pi} \int_0^\pi q(t)\, dt < \frac{8}{\pi^2} + \varepsilon .$$

For such a function, the minimal deviation from the mean is over 20 %. Usually, one expects theorems that a certain function takes on its mean value at least four times [3].

5. The unimodular centroaffine curvature presents some interesting problems connected with the so-called "vertex theorems". A vertex of a curve in a geometry is a point where the corresponding curvature has a relative extremum.

For a smooth oval (a closed, convex curve), the existence of two vertices follows from the Weierstrass theorem. This number two can occur as the number of unimodular centroaffine vertices, e.g., for the circle referred to any interior point other than its center. If the origin is at the centroid of a curve or at the centroid of the polar reciprocal, the number of vertices is ≥ 6. For a symmetric oval referred to its center, the number of vertices is ≥ 8 [6]. The characterization of all interior points for which the number of unimodular centroaffine vertices is > 2 is an open question. The result in [1], p. 136, can be interpreted as a theorem on relative vertices of a type encountered also in Minkowski geometry (cf. [2], p. 324, 3). Consider two arcs $x(t)$, $y(t)$ of the type described in sec. 1. Let $x(t^*)$ be the point of smallest parameter value $t^* > t$ collinear with 0 and $x(t)$ and denote by $\varphi_x(t) = 1/2(t^* - t)$ the area of the convex domain bounded by the arc $\widehat{x(t)\, x(t^*)}$ and the segment $\overline{x(t)\, x(t^*)}$. The result of [1] can be reformulated as:

If $\varphi_x(t) = \varphi_y(t + c)$ *for* $t_0 \leqq t < t_0^*$, *then the unimodular centroaffine curvature of* $x(t)$ *is equal to that of* $y(t + c)$ *at least four times in* $t_0 \leq t < t_0^*$.

As a special case, we consider a smooth, oriented oval C and a point P in the interior of C. The Grassmann space of the oriented lines l through P is homeomorphic to a circle Γ of center P. We denote by $(l \cap C)_i$, $i = 1, 2$, the i-th point of intersection of l and C in the direction of l. The surface area of the part of the interior of C bounded by by the segment $(l \cap C)_1 (l \cap C)_2$ and the arc of C pointing from $(l \cap C)_1$ towards $(l \cap C)_2$ is denoted by $\varphi_C(l)$. The surface area of the sector cut out from C by the rays from P to points $(l_1 \cap C)_1$ and $(l_2 \cap C)_1$ is denoted by $A(l_1, l_2)$. The unimodular centro-affine curvature, for the origin at P, of C at $Q \in C$ is denoted by $\varkappa_{C,P}(Q)$. Then we have:

Given two C^2, oriented ovals C_1, C_2 and points P_i $(i = 1, 2)$ in the interior of C_i. If there exists a homeomorphism $f: \Gamma_1 \to \Gamma_2$ of the oriented lines through P_1 to the oriented lines through P_2 such that

$$A(l, l_0) = A[f(l), f(l_0)]$$
$$\varphi_{C_1}(l) = \varphi_{C_2}[f(l)]$$

for all $l \in \Gamma_1$ and a fixed $l_0 \in \Gamma_1$, then, for at least eight different lines $l \in \Gamma_1$,

$$\frac{\varkappa_{C_1, P_1}[(l \cap C_1)_1]}{\varkappa_{C_2, P_2}\{[f(l) \cap C_2]_1\}} = 1$$

and the ratio of the two unimodular centroaffine curvatures has at least eight relative extrema.

In particular, such a map f exists (by parallelism) between the lines through the center of a centrally symmetric oval of surface area A and those through the center of a circle of radius $(A/\pi)^{1/2}$:

The unimodular centroaffine curvature of a centrally symmetric oval of surface area A computed for the center of symmetry is equal to π^2/A^2 at at least eight different points.

By (3), the average $\bar{\varkappa} = (1/A) \int_0^A \varkappa \, dt$ of the unimodular affine curvature of a symmetric oval satisfies

$$\frac{8}{A^2} \leq \bar{\varkappa} \leq \frac{\pi^2}{A^2}.$$

In all cases known today, the centroaffine unimodular curvature of an oval has either 2 or ≥ 6 extrema. This seems to be connected with the fact that the curvature appears as coefficient in a Hill equation [5].

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