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CATEGORIES OF ORDERED SETS

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INTRODUCTION

In this paper the categories are studied, in which objects are ordered sets and morphisms various kinds of homomorphic (i.e. isotone) maps. The basic properties of A-, B-, C- homomorphisms have been described in [6]. Some aspects of Čulík's results are generalized in section 2. In sections 5 and 6 the completeness in the sense of [9] is dealt with. Section 7 is devoted to the problem of so called "*stable ordering of full-relation subobjects*", section 8 to a study of automorphisms class group. In section 9 categories with ordered sets of morphisms are defined. Section 10 deals with the operations in the category of ordered sets with distinguished elements.

Let us recall that in several questions dealing with ordered sets there are also important several other kinds of mappings, e.g. convergence mappings (let us mention [10], [13] as ones of the most recent papers on this subject) or strong homomorphisms (see [4]) important in constructing of universal categories (see [11]). Many results on categories of ordered sets can be found in [1].

The studied categories are examples of the structured categories in the sense of [8] (see also related considerations in [4]). Further, it is possible to assign to every ordered set A a topology on A (e.g. so called left-topology, where the system $\{\{x : x \leqq a\} : a \in A\}$ forms a subbase of the system of open sets) so that isotone mappings are continuous. Categories of topological spaces have been studied in many papers (e.g. [12], [14], [15], [23]). A generalisation of the results of this paper to more general types of structured categories, especially a comparison with results on categories of topological spaces are intended to be content of further research.

1. BASIC NOTIONS

1.1. The definitions of basic notions of the theory of categories are taken from [16]. The class of all objects or morphisms of a category $\mathscr{K}$ is denoted as $O(\mathscr{K})$ or $M(\mathscr{K})$ respectively. In diagrams $\rightarrowtail$ denotes a monomorphism, $\twoheadrightarrow$ an epimorphism. If it is necessary to emphazise that a set of morphisms of a into b in the category $\mathscr{K}$ is considered we write $H_{\mathscr{K}}(a, b)$. In the most cases morphisms are denoted by small Greek letters. Recall that for $\alpha: a \to b$, $\beta: b \to c$ it is $\alpha\beta: a \to c$. If the objects of a category $\mathscr{K}$ are sets, then they are denoted by capital letters. If it is said nothing else (e.g. in $\mathscr{K}_6$—see below—the objects are non empty sets) the empty set is always taken in considerations, too. If A, B are two sets, then a mapping φ of the set A in the set B is a triple $[A, B, F]$, where $F \subset A \times B$ and for each $a \in A$ there exists exactly one element $b \in B$ such at $\langle a, b \rangle \in F$. If A, B are sets with some structure, then we suppose that the symbols A, B represent also this structure. Mostly, in the sequel, F represents the whole triple. If, especially, $A = \emptyset$, then $[\emptyset, B, \emptyset]$ is the only mapping of $\emptyset$ in B. If $B \neq \emptyset$, there exists no mapping of B in $\emptyset$. Let us add that Gödel—Bernays system is taken as a foundation of the set theory. If $\alpha: A \to B$, $X \subset A$ and $y \in B$, then $(y)\,\alpha^{-1} = \{x: x \in A, (x)\,\alpha = y\}$, $(X)\,\alpha = \{y: y \in B, y = (x)\,\alpha$ for a certain $x \in X\}$.

If F is a functor mapping a category $\mathscr{K}$ in a category $\mathscr{L}$, the values of F in an object a and a morphisms α are denoted as $(a)\,F$, $(\alpha)\,F$.

1.2. Binary relation (further only relation) ϱ on a set A is a subset of $A \times A$ (i.e. $\varrho \subset A \times A$). The set A provided with ϱ is denoted also as (A, ϱ). Let (A_1, ϱ_1), (A_2, ϱ_2) be two sets with relations, φ a mapping of A_1 in A_2 such that $x\varrho_1 y \Rightarrow (x)\,\varphi\varrho_2(y)\,\varphi$ for all $x, y \in A_1$. Then φ is called a homomorphism of the set (A_1, ϱ_1) in the set (A_2, ϱ_2). Let ϱ be a relation on a set A, $X \subset A$. Let $\varrho' = \varrho \cap X \times X$. Let ι be the identity mapping of X in A, i.e. $(x)\,\iota = x$ for all $x \in X$. Then ι is called the inclusion mapping of (X, ϱ') in (A, ϱ). So, if inclusion mapping of a subset X of a set (A, ϱ) is talked about, we have in mind X provided with the restriction of the relation ϱ (X is then called full-relation subobject). Moreover, if a distinguished point a or x is defined in A or X respectively, then the described mapping ι will be called an inclusion mapping, if $x = a$.

1.3. If ϱ is a reflexive, antisymetric and transitive relation on a set A, then (A, ϱ) is an ordered set. As a rule, instead of ϱ the symbol $\leqq$ (and its modifications) is written. If every two elements of an ordered set A are comparable, A is called a chain, if every two distinct elements of A are incomparable, the A is an antichain.

Let $(A, \leqq)$, $(B, \leqq)$ be two ordered sets and φ a homomorphism of A in B. Consider the following properties of φ.

(1) $$x < y \Rightarrow (x)\,\varphi < (y)\,\varphi.$$

(2) $x \parallel y \Rightarrow (x)\,\varphi \parallel (y)\,\varphi$ ($x \parallel y$ means x and y are incomparable).

(3) $$x \parallel y \Rightarrow (x)\,\varphi \parallel (y)\,\varphi \quad \text{or} \quad (x)\,\varphi = (y)\,\varphi.$$

φ is called an A-homomoprhism (B-homomorphism, C-homomorphism, respectively) if (2)[(1) and (3), (3) respectively] is valid (see [6]).

Let X be a subset of an ordered set A and for $y \in A - X$ following property be satisfied: if $x_1, x_2 \in X$ then

$$y < x_1 \Rightarrow y < x_2,$$

$$y > x_1 \Rightarrow y > x_2.$$

Such a subset X will be called an embedded subset of A. In [6] 2.1., 3.1., 4.1. there is proved:

1.3.a. *If φ is A- (B- or C-) homomorphism of A in B, then $\{(b)\,\varphi^{-1}: b \in (A)\,\varphi\}$ is a decomposition of A in embedded chains (antichains or ordered sets respectively).*

As for the decompositions of the sets the terminology of the book [3] will be used. If R is a decomposition on a set A, then $(a)\,\varkappa$ means that element of R, for which $a \in (a)\,\varkappa$. So, $\varkappa$ is a mapping of A onto R and it is called the canonical mapping for the decomposition R. If φ is a mapping of a set X in a set Y then $\{(y)\,\varphi^{-1}; y \in (X)\,\varphi\}$ is a decomposition of X and the mapping $\overline{\varphi}$ of this decomposition in Y induced by φ is defined by $[(y)\,\varphi^{-1}]\,\overline{\varphi} = y$.

We shall deal with the following categories.

Notation	Objects	Morphisms
$\mathscr{K}_1$	all sets with relation	homomorphisms
$\mathscr{K}_2$	all ordered sets	homomorphisms
$\mathscr{K}_3$	all ordered sets	A-homomorphisms
$\mathscr{K}_4$	all ordered sets	B-homomorphisms
$\mathscr{K}_5$	all ordered sets	C-homomorphisms
$\mathscr{K}_6$	all ordered sets with distinguished element	homomorphisms mapping the distinguished element in distinguished element.

It is evident that all properties of the category are satisfied in all these cases. The objects of the category $\mathscr{K}_6$ are denoted as $(A, a, \leqq)$, where A is the corresponding set and a the distinguished element of A.

1.4. Evidently following assertions are valid.

a) *$\mathscr{K}_2$ is a full subcategory in $\mathscr{K}_1$.*

b) *$\mathscr{K}_i$ is a subcategory in $\mathscr{K}_2$ for $i = 3, 4, 5$.*

c) *$\mathscr{K}_3$ and $\mathscr{K}_4$ are subcategories of $\mathscr{K}_5$.*

2. INVERSIBLE MAPPING, MONOMORPHISM, EPIMORPHISM

2.1. Let (A, ϱ), $(A_1, \varrho_1) \in O(\mathscr{K}_1)$, $\varphi \in H_{\mathscr{K}_1}[(A, \varrho), (A_1, \varrho_1)]$ a one-to-one mapping of A onto (in) A_1 and

(a) $$x\varrho y \equiv [(x)\ \varphi]\ \varrho_1[(y)\ \varphi]$$

for all $x, y \in A$. Then φ is a relation-isomorphism (a relationisomorphic mapping) of (A, ϱ) onto (in) (A_1, ϱ_1). If, instead of (a) $x\varrho y \equiv [(y)\ \varphi]\ \varrho_1[(x)\ \varphi]$ is valid, φ is called a relation-antiisomorphism (a relation-antiisomorphic mapping).

2.2. *If* $i = 1, \ldots 6$, $A, B \in O(\mathscr{K}_i)$, $\varphi \in H_{\mathscr{K}_i}(A, B)$,
a) φ *is inversible in* $\mathscr{K}_i$, *if and only if* φ *is a relation-isomorphism.*
b) φ *is a monomorphism in* $\mathscr{K}_i$, *if and only if* φ *is one-to-one.*
c) φ *is an epimorphism in* $\mathscr{K}_i$, *if and only if* φ *is onto.*

Proof. b) and c) is proved for $\mathscr{K}_2$ in [1], 1.2.4 p. 27 and 1.2.8., p. 29). Other cases are quite similar. Also a) is evident.

2.3. *Let* $\mathscr{K}$ *be a full subcategory in* $\mathscr{K}_i$ $(i = 2, 3, 4, 5)$, $\alpha \in H_{\mathscr{K}}(A, B)$ *a monomorphism in* $\mathscr{K}_i$. *Then* α *is one-to-one.*

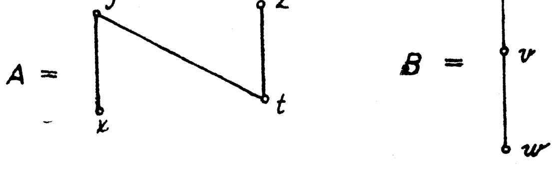
Proof. Let $i = 3, 4, 5$. Let $x \in (A)\ \alpha$. Then $(x)\ \alpha^{-1}$ is an embedded chain, antichain or ordered set, respectively, in the set A. Let $y \in (x)\ \alpha^{-1}$. Define $\beta: A \to A$ so: $(z)\ \beta = z$ for z non $\in (x)\ \alpha^{-1}$, $(z)\ \beta = y$ for $z \in (x)\ \alpha^{-1}$. β is an element of $M\ (\mathscr{K}_i)$ and $\beta\alpha = \alpha$. As α is a monomorphism, β is uniquely determined, so card $(x)\ \alpha^{-1} = 1$.

Let $i = 2$. Notation will be as above. Admit card $(x)\ \alpha^{-1} > 1$. Let $x_1, x_2 \in (x)\ \alpha^{-1}$, $x_1 \neq x_2$. Let $(z)\ \beta_j = x_j$ for $z \in A$ and $j = 1, 2$. Then $\beta_j \in M(\mathscr{K}_2)$, $\beta_1\alpha = \beta_2\alpha$ and $\beta_1 \neq \beta_2$, a contradiction.

2.4. 2.3 *is not valid for* $i = 1,6$.

Proof. a) $i = 1$. Let $A = \{x, y\}$ with the relation $\{\langle x, y\rangle\}$, $B = \{z\}$ with the relation $\{\langle z, z\rangle\}$. Then the full subcategory in $\mathscr{K}_1$ with A and B as objects has the mapping $\varphi: A \to B$, $(x)\ \varphi = (y)\ \varphi = z$ as a monomorphism.

b) $i = 6$. Let A and B have the following Hasse-diagrams with x and v as distinguished elements. Then $\varphi: A \to B$ $(x)\ \varphi = (z)\ \varphi = v$, $(y)\ \varphi = u$,



$(t)\,\varphi = w$ is a monomorphism in the full subcategory of category $\mathscr{K}_6$ with A and B as objects.

2.5. *Let $\mathscr{K}$ be a full subcategory of $\mathscr{K}_2$ or $\mathscr{K}_6$, α an epimorphism in $\mathscr{K}$, $\alpha\colon A \to B$, $A \neq \emptyset$. Then α is onto.*

Proof. a) Let $\mathscr{K}$ be a subcategory of $\mathscr{K}_2$. Admit $x \in B - (A)\,\alpha$.

Let $x_1, y_1 \in B$, $x_1 < y_1$. Define $\beta_1, \beta_2\colon B \to B$ in the following way: $(z)\,\beta_1 = y_1$ for all $z \geqq x$, $(z)\,\beta_1 = x_1$ otherwise; $(z)\,\beta_2 = y_1$ for all $z > x$, $(z)\,\beta_2 = x_1$ otherwise. Clearly $\alpha\beta_1 = \alpha\beta_2$ and $\beta_1, \beta_2 \in M(\mathscr{K})$. So α is not an epimorphism.

If B is an antichain, take in the place of x_1 and y_1 two distinct elements. The constructions of β_1 and β_2 run as above.

Note. If $\mathscr{K}$ contains B with card $B \geqq 2$, then the assertion is true also for $A = \emptyset$. If $\mathscr{K}$ does not contain such an object, then every morphism is epimorphism.

b) Let $\mathscr{K}$ be a full subcategory of $\mathscr{K}_6$. As above, let $x \in B - (A)\,\alpha$. Let b be the distinguished element of B. Clearly $x \neq b$.

First, suppose x comparable with b, for instance $x > b$. Construct β_1, β_2 as in a) with $x_1 = b$, $y_1 = x$. If $x < b$, put $x_1 = x$, $y_1 = b$.

At second, let $x \parallel b$. If there is an element $x' \in B - \{b\}$ comparable with b, the procedure is as above taking x' instead of x, if $x' > b$. If $x' < b$, then β_1, β_2 can be constructed in the dual way, i.e.

$(z)\,\beta_1 = x'$ for all $z \leqq x$, $(z)\,\beta_1 = b$ otherwise.

$(z)\,\beta_2 = x'$ for all $z < x$, $(z)\,\beta_1 = b$ otherwise.

So, let b be incomparable with all the elements of the set $B - \{b\}$. Let card $B - \{b\} \geqq 2$. Then on $B - \{b\}$ the constructions of β_1 and β_2 run as in a) and put $(b)\,\beta_1 = (b)\,\beta_2 = b$. If $B - \{b\} = \{x\}$, put $(b)\,\beta_1 = (x)\,\beta_1 = b$, β_2 the identity.

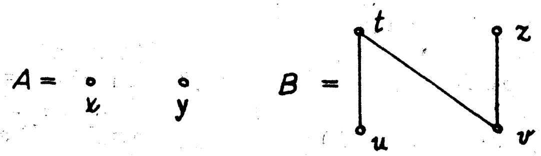
In all cases $\beta_1, \beta_2 \in M(\mathscr{K})$, $\alpha\beta_1 = \alpha\beta_2$, so α is not an epimorphism in $\mathscr{K}$.

2.6. *Proposition* 2.5. *is not valid for $\mathscr{K}_1$, $\mathscr{K}_3$, $\mathscr{K}_4$, $\mathscr{K}_5$.*

Proof. Define the following mappings α_1, α_5: $A \to B$, which are not onto, nevertheless can be easily proven to be epimorphisms in the corresponding two object full subcategories of $\mathscr{K}_1$ or $\mathscr{K}_5$, respectively.

ad $i = 1$. $A = \{x\}$ with the relation $\emptyset$, $B = \{y, z\}$ with the relation $\{\langle y, y\rangle\}$ and $(x)\,\alpha_1 = z$.

ad $i = 5$. Let Hasse diagrams of A and B be as follows and $(x)\,\alpha_5 = u$, $(y)\,\alpha_5 = v$.



Further, $\alpha_5 \in M(\mathcal{K}_3) \cap M(\mathcal{K}_4)$. So, according to 1.4.c. α_5 is an epimorphism in the full subcategories spanned by the objects A and B in $\mathcal{K}_3$ and $\mathcal{K}_4$.

3. V-SUBCATEGORIES

3.1. A non empty class V of ordered sets will be called a variety, if it contains with an element A all elements relation-isomorphic to A. So
$A \in V$, $B \in O(\mathcal{K}_2)$, B relation-isomorphic to $A \Rightarrow B \in V$.

Let V be a variety and $\mathcal{K}$ a subcategory of $\mathcal{K}_2$ such that
1. There exists at least one non empty object in $O(\mathcal{K})$.
2. If $\varphi \in M(\mathcal{K})$, φ: $A \to B$, then $(x)\, \varphi^{-1} \in V$ for all $x \in B$.
Then $\mathcal{K}$ is called a V-subcategory in $\mathcal{K}_2$.

3.2. *V being a variety a V-subcategory exists if and only if V contains a one-point set.*
Clear.

3.3. *Let V be a variety containing a one-point set. Following assertions are equivalent.*
1. *The greatest V-subcategory exists.*
2. *V possesses the following properties:*
a) *V is closed under the lexicografical summation, i.e. for A_i, $B \in V$, $i \in B$ it is $\sum_{i \in B} A_i \in V$.*
b) *If $(B, \varrho) \in V$ and ϱ_1 is an ordering of B with $\varrho_1 \subset \varrho$ (so ϱ is an extension of ϱ_1), then $(B, \varrho_1) \in V$.*

Proof. Let $\mathcal{K}$ be the greatest of all V-subcategories. Let $B \in V$, $C = \{c\}$, $\varphi: B \to \{c\}$ (so $b \in B \Rightarrow (b)\, \varphi = c$). $\varphi \in M(\mathcal{K})$ since the category with B and C as objects and with the identidy mappings together with φ as morphisms is a V-subcategory, so contained in $\mathcal{K}$. Let $P = \Sigma_{i \in B} A_i$, $A_i \in V$, be the lexicografical sum (i.e. the set of all $\langle i, a \rangle$, where $i \in B$, $a \in A_i$ and $\langle i, a \rangle \leqq \langle i', a' \rangle$ if and only if $i < i'$ or $i = i'$ and $a \leqq a'$). Let $\langle i, a \rangle\, \psi = i$ for all $i \in B$, $a \in A_i$. By similar arguments as for φ, $\psi \in M(\mathcal{K})$. So $\psi\varphi \in M(\mathcal{K})$. Then $(c)\, (\psi\varphi)^{-1} = P$, so $P \in V$. Hence a) for V is satisfied.

Further, (B, ϱ_1) being the set of b), χ the mapping of (B, ϱ_1) in (B, ϱ) defined by identity is a morphism of $\mathcal{K}$. So $\chi\varphi \in M(\mathcal{K})$ (φ as above). But $(c)\, (\chi\varphi)^{-1} = (B, \varrho_1)$. We get $(B, \varrho_1) \in V$.
On the contrary, let a) and b) satisfied for V.
Define the category $\mathcal{K}$ as follows. $O(\mathcal{K}) = O(\mathcal{K}_2)$ and
$\varphi \in M(\mathcal{K}) \equiv \varphi \in M(\mathcal{K}_2)$ and, if φ: $A \to B$, then $(x)\, \varphi^{-1} \in V$ for $x \in B$.
$\mathcal{K}$ is realy a subcategory of $\mathcal{K}_2$. Namely, if φ: $A \to B$, ψ: $B \to C$, $\varphi, \psi \in M(\mathcal{K})$, $c \in C$, then $(c)\, \psi^{-1} \in V$ and $b \in B \Rightarrow (b)\, \varphi^{-1} \in V$. Put

$S = (c)\,(\varphi\psi)^{-1}$, $S' = \sum_{b \in (c)\psi^{-1}} (b)\,\varphi^{-1}$. By a) $S' \in V$. Let $x \in S$, $(x)\,\varphi \in (c)\,\psi^{-1}$. Put $(x)\,\chi = \langle (x)\,\varphi, x\rangle$. χ is clearly a homomorphic mapping of S onto S' and is one-to-one. So, by b) and the definition of the variety, $S \in V$.

As one-point sets are elements of G, identity maps are elements of $M(\mathcal{K})$.

3.4. Let V be a variety. V-subcategory $\mathcal{K}$ of $\mathcal{K}_2$ is called a regular V-subcategory of K_2 if
$\varphi\colon A \to B$, $\varphi \in M(\mathcal{K})$, $x \in B \Rightarrow (x)\,\varphi^{-1}$ is embedded in A (see 1.3).

3.5. *Let V be a variety. Let there exist the greatest regular V-subcategory $\mathcal{K}$ of $\mathcal{K}_2$. Then $V = \{X\colon X$ one-point set$\}$ or $V = \{X\colon X$ one-point set or empty set$\}$.*

Proof. Admit $(A, \varrho) \in V$, card $A \geqq 2$. Let (B, ϱ_1) be isomorphic to (A, ϱ), $A \cap B = \emptyset$ and c non $\in A \cup B$. Put $D = A \cup B \cup \{c\}$ and $\varrho'' = \varrho \cup \varrho_1 \cup \{\langle c, a\rangle : a \in A\} \cup \{\langle c, c\rangle\}$. ϱ'' is clearly an ordering of D. $(D, \varrho'') \in O(\mathcal{K})$, as $\mathcal{K}'$, where $O(\mathcal{K}') = \{D\}$ and $M(\mathcal{K}')$ contains only the identity on D is a regular V-subcategory. By similar arguments one proves $O(\mathcal{K}_2) - \{\emptyset\} \subset O(\mathcal{K})$. Let $E = \{x, y, z\}$, $x \neq y \neq z \neq x$ and $x \geqq y$, $y \leqq z$, Let $\varphi\colon (D, \varrho'') \to E$, $(A)\,\varphi = \{x\}$, $(B)\,\varphi = \{z\}$, $(c)\,\varphi = y$. $\mathcal{K}$ being the greatest regular V-subcategory, $\varphi \in M(\mathcal{K})$.

Now, we shall prove that two-point antichain is an element of V. Let $F = \{u, v\}$ be an antichain, $u_1, v_1 \in A$ two distinct elements of A. Put $(u)\,\chi = u_1$, $(v)\,\chi = v_1$. Clearly $\chi \in M(\mathcal{K})$ and $(x)\,(\chi\varphi)^{-1} = F$. So $F \in V$. Now, the subset $\{x, z\}$ of E is embedded and $\{x, z\} \in V$. If $G = \{s, t\}$, $s < t$, $(x)\,\psi = (z)\,\psi = t$, $(y)\,\psi = s$, then $\psi \in M(\mathcal{K})$ and hence $\varphi\psi \in M(\mathcal{K})$. Nevertheless, $(t)\,(\varphi\psi)^{-1} = A \cup B$ and $A \cup B$ is not embedded in D.

Notes. a) Let $\mathcal{K}$ be the following category: $O(\mathcal{K}) = \mathcal{K}_2$, $M(\mathcal{K}) = \{\varphi\colon \varphi \in M(\mathcal{K}_2)$, φ one-to-one onto $\}$, then $\mathcal{K}$ is the greatest regular V-subcategory for $V = \{X\colon X$ one-point set$\}$.

b) If "onto" in the definition of $M(\mathcal{K})$ is omitted, one gets the greatest regular V-subcategory for $V = \{X\colon X$ one pointed or empty set$\}$.

3.6. Let V be a variety closed under the lexicografical summation containing one-point sets and $\emptyset$. Define the category $\mathcal{K}$ in the following way.

1. $O(\mathcal{K}) = O(\mathcal{K}_2)$.
2. $\varphi \in M(\mathcal{K}) \equiv \varphi \in M(\mathcal{K}_2)$ and, if $\varphi\colon A \to B$, following property is satisfied

(*) If X is embedded in B, $X \in V$, then $(X)\,\varphi^{-1} \in V$ and $(X)\,\varphi^{-1}$ is embedded in A.

Theorem. *$\mathcal{K}$ is a maximal regular V-subcategory of $\mathcal{K}_2$.*

Proof.

1. $\mathcal{K}$ is a subcategory of $\mathcal{K}_2$. Namely

1.1. Identity mappings are clearly elements of $M(\mathcal{K})$.

1.2. Let $\varphi: A \to B$, $\psi: B \to C$, φ and ψ satisfying (*) and $\varphi, \psi \in M(\mathcal{K}_2)$ Let $X \subset C$, $X \in V$, X embedded in C. Then $(X)\,\psi^{-1}$ is embedded in B and $(X)\,\psi^{-1} \in V$, so $(X)\,\psi^{-1}\varphi^{-1}$ is embedded in A and $(X)\,\psi^{-1}\varphi^{-1} \in V$. So $\varphi\psi$ satisfies (*).

2. $\mathcal{K}$ is a regular V-subcategory.

In proving that it suffices to put in 1.2. X equal to one-point set.

3. Admit the existence of a regular V-subcategory $\mathcal{K}'$, for which $M(\mathcal{K}') \underset{\supset}{\neq} M(\mathcal{K})$. Let $\chi \in M(\mathcal{K}') - M(\mathcal{K})$, $\chi: A \to B$. Let X be embedded in B, $X \in V$, $(X)\,\chi^{-1}$ not embedded in A. Let B' be the decomposition on B, the elements of which are the set X and one-point subsets $\{y\}$, where $y \in B - X$. Define the order on B' in such a way:

$$\{x\} \leqq \{y\} \equiv x \leqq y \qquad \text{for} \qquad x, y \in B - X.$$

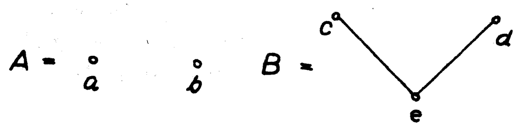
$$\{x\} \leqq X \equiv (z \in X \Rightarrow x \leqq z) \qquad \text{for} \qquad x \in B - X.$$

$$X \leqq \{x\} \equiv (z \in X \Rightarrow z \leqq x) \qquad \text{for} \qquad x \in B - X.$$

Let φ be the canonical mapping of B onto B'. Clearly $\varphi \in M(\mathcal{K})$, so $\varphi \in M(\mathcal{K}')$. But $(X)\,(\chi\varphi)^{-1}$ is not embedded in A, which contradicts $\mathcal{K}'$ to be a regular V-subcategory in $\mathcal{K}_2$.

3.7. *Let V consist of all chains (all antichains, $V = O(\mathcal{K}_2)$, respectively). Then $\mathcal{K}_3$ ($\mathcal{K}_4$ or $\mathcal{K}_5$, respectively) is a regular V-subcategory and $M(\mathcal{K}_i) \subset \subset M(\mathcal{K})$ ($\mathcal{K}$ constructed in 3.6). It is $\mathcal{K}_i \neq \mathcal{K}$.*

Proof. Clearly in all cases V contains one point sets and is closed under lexicografical summation. Assertion on $\mathcal{K}_i$ to be a regular V-subcategory follows from 1.3. $M(\mathcal{K}_i) \subset M(\mathcal{K})$ follows by [6], 2.2., 4.2. and remark on p. 507. For proving $\mathcal{K}_i \neq \mathcal{K}$ let us take A and B with Hasse diagrams as follows and (a) $\varphi = c$, (b) $\varphi = e$. Then $\varphi \in M(\mathcal{K})$ (for $i = 3, 4, 5$) and φ is not a C-homomorphism.



3.8. Notes. 1. If $\emptyset$ non $\in V$, all other assumption of 3.6. being satisfied, take only epimorphisms of $\mathcal{K}_2$ in the construction corresponding to that of $\mathcal{K}$. Resulting category is a maximal regular V-subcategory, too. Proof runs as for $\mathcal{K}$.

2. One can prove by means of the axiom of choice existence of maximal regular V-subcategory for every variety in $\mathscr{K}_2$. In 3.6., i.e. in the case that V is closed under the lexicografical summation, no use of axiom of choice has been made. It is an open question, if in general case axiom of choice is needed.

4. WEAKLY INITIAL OBJECT, WEAKLY TERMINAL OBJECT, GENERATOR AND COGENERATOR

4.1. Let $\mathscr{K}$ be a category, $a \in O(\mathscr{K})$.

1. a is called weakly initial (terminal) if for every $y \in O(\mathscr{K})$ with eventual exception for one object of $\mathscr{K}$, $H_{\mathscr{K}}(a, y) \neq \emptyset$ ($H_{\mathscr{K}}(y, a) \neq \emptyset$).

2. a is called a generator (cogenerator) of $\mathscr{K}$, if a is weakly initial (terminal) and for $\alpha, \beta \in H_{\mathscr{K}}(b, c)$, $\alpha \neq \beta$ there exists $\xi \in H_{\mathscr{K}}(a, b)$ ($\xi \in H_{\mathscr{K}}(c, a)$) so that $\xi\alpha \neq \xi\beta (\alpha\xi \neq \beta\xi)$.

4.2. Notes. Ad 1. In [17] p. 42 initial and terminal objects are defined. There uniqueness of a mapping $a \to y$ ($y \to a$) is demanded and no exception allowed.

Ad 2. In [16] p. 22 one speaks about entire (coentire) objects, definition of which is as for generator (cogenerator) again without any exception. In definition of generator (cogenerator) in [18], p. 72 one does not require for a to be weakly initial (terminal).

4.3. a) *Let* $(A, \varrho) \in O(\mathscr{K}_1)$. (A, ϱ) *is weakly initial in* $\mathscr{K}_1$, *if and only if* $\varrho = \emptyset$.

b) *Each object of* $\mathscr{K}_2$ ($\mathscr{K}_5$ *or* $\mathscr{K}_6$) *is weakly initial.*

c) $(A, \leqq) \in O(\mathscr{K}_3)$ *is weakly initial in* $\mathscr{K}_3$ *exactly when* $(A, \leqq)$ *is a chain.*

d) $(A, \leqq) \in O(\mathscr{K}_4)$ *is weakly initial in* $\mathscr{K}_4$, *when it is an antichain.*

Proof. Ad a) Let $(S, \emptyset) \in O(\mathscr{K}_1)$. Then $H_{\mathscr{K}}((A, \varrho), (S, \emptyset)) \neq \emptyset \Rightarrow \varrho = \emptyset$. The converse is clear.

Ad b) Let $(S, \leqq) \in O(\mathscr{K}_i)$ ($i = 2, 5, 6$), $S \neq \emptyset$. Let $x \in S$(if $i = 6$ let x be the distinguished element of S). Put $\varphi: A \to S$, $(a)\varphi = x$ for all $a \in A$. Clearly $\varphi \in H_{\mathscr{K}_i}(A, S)$.

Ad c) Let $(S, \leqq)$ be chain. Then $H_{\mathscr{K}_3}(A, S) \neq \emptyset \Rightarrow (A, \leqq)$ is a chain.

Ad d) Similarly as in ad c).

4.4. We evidently get

In $\mathscr{K}_i$ ($i = 1, \dots, 5$) *initial object is empty.*

In $\mathscr{K}_6$ *initial object is each one-point set.*

4.5. a) $(A, \varrho) \in O(\mathscr{K}_1)$ *is weakly terminal in* $\mathscr{K}_1$ *exactly when there exists* $x \in A$ *such that* $\langle x, x\rangle \in \varrho$.

b) *In* $\mathscr{K}_i$ ($i = 2, 5, 6$) *every non empty set is weakly terminal.*

c) *In* $\mathscr{K}_3$ *and* $\mathscr{K}_4$ *no weakly terminal objects exist.*

Proof. Ad a) Clear.

Ad b) As in 4.3. b).

Ad c) Let $(A, \leqq) \in O(\mathscr{K}_3)$ $(O(\mathscr{K}_4))$ be an antichain (chain) with a cardinality $\mathfrak{m}$. Let $B \in O(\mathscr{K}_3)$ $(O(\mathscr{K}_4))$, $\varphi \in H(A, B)$. Then card $[(A)\varphi] \geqq \mathfrak{m}$. So no weakly terminal object exists.

4.6. a) $(A, \varrho) \in O(\mathscr{K}_1)$ *is terminal, if and only if* $A = \{x\}$, $\varrho = \{\langle x, x\rangle\}$.

b) $(A, \varrho) \in O(\mathscr{K}_i)$ $(i = 2, 5, 6)$ *is terminal, if and only if A is a one point set.*

Proof. Clear.

4.7. a) *In* $\mathscr{K}_i$ $(i = 1, 2, 3, 4, 5)$ *every non empty weakly initial object is a generator. Empty object is not a generator.*

b) *In* $\mathscr{K}_6$ (A, a, ϱ) *is a generator, if it contains at least two connected components.*

Proof. a) Clear

Ad b) Let (A, a, ϱ) have at least two connected components. Let (N, n, ν), $(P, p, \pi) \in O(\mathscr{K}_6)$, $\varphi, \gamma\colon N \to P$, $\varphi \neq \gamma$. Let c be such an element of N that $(c)\varphi \neq (c)\gamma$. Clearly $c \neq n$. Let us define $\chi \in H(A, N)$ as follows: If K is the component of A containing a and if $y \in K$, then $(y)\chi = n$, $(y)\chi = c$ otherwise. It is $\chi \in H(A, N)$ and $\chi\varphi \neq \chi\gamma$.

On the contrary, let (A, a, ϱ) be a generator in $\mathscr{K}_6$, N a three-point antichain with elements 1, 2, 3 and 1 being the distinguished element of N. Let φ be the identity mapping in $H(N, N)$ and define $\gamma \in H(N, N)$ in the following way: $(1)\gamma = 1$, $(2)\gamma = 3$, $(3)\gamma = 2$. Let $\chi\varphi \neq \chi\gamma$ for a certain $\chi \in H(A, N)$. The $(1)\chi^{-1}$ and $(2)\chi^{-1}$ [$(3)\chi^{-1}$ respectively] are set theoretical sums of disjunctive systems of connected components of A.

4.8. *Let* $i = 1, \ldots, 6$. $\mathscr{K}_i$ *is a concrete category.*

Proof follows from 4.7 and by 5 in § 4 in [16].

Note. For $i = 1, \ldots, 5$ our definition of generator differs from that of [16] but the exceptial object in $\mathscr{K}_1, \ldots, \mathscr{K}_5$ is the empty set. The mapping constructed in § 4 in [16] is an embedding of $\mathscr{K}_i$ in the category of all sets. The image of $\emptyset$ is $\emptyset$. Let us mention that 4.8 is also an immediate consequence of representation of the objects of K_i $(i = 1, \ldots, 6)$ as algebraic structures ([4], chapter IV).

4.9. a) $(A, \varrho) \in O(\mathscr{K}_1)$ *is a cogenerator in* $\mathscr{K}_1$ *exactly when two dinstinct elements* $x, y \in A$ *exist such that* $\{x, y\} \times \{x, y\} \subset \varrho$.

b) $(A, \varrho) \in O(\mathscr{K}_2)$ *is a cogenerator exactly when* (A, ϱ) *is not an antichain.*

c) $(A, a, \varrho) \in O(\mathscr{K}_6)$ *is a cogenerator of* $\mathscr{K}_6$ *exactly when there exist elements* $x, y \in A$ *such that* $x < a < y$.

d) $\mathscr{K}_3, \mathscr{K}_4, \mathscr{K}_5$ *have no cogenerators.*

Proof. Ad a) Let (a, ϱ) possess the described property. Let (N, ν), $(P, \pi) \in O(\mathscr{K}_1)$, $\varphi, \gamma \in H(N, P)$, $\varphi \neq \gamma$. Choose $z \in N$ such that $(z)\varphi \neq$

$\neq (z)\,\gamma$. Define χ: $P \to A$ in the following way: For $v \in P$, $v\pi[(z)\,\varphi]$, $v \neq (z)\,\gamma \Rightarrow (v)\,\chi = x$, $(v)\,\chi = y$ otherwise. In consequence of the assumption on x and y $\chi \in H(P, A)$ and clearly $\varphi\chi \neq \gamma\chi$.

On the contrary let (A, ϱ) be a cogenerator in $\mathscr{K}_1$. Put $N = \{1, 2\}$, $\nu = N \times N$. Let φ be the identity mapping $N \to N$ and define γ as follows: $(1)\,\gamma = 2$, $(2)\,\gamma = 1$. Let $\chi \in H(N, A)$ with $\varphi\chi \neq \gamma\chi$. Then $(1)\,\chi \neq (2)\,\chi$ and $[(1)\,\chi]\,\varrho\,[(1)\,\chi]$, $[(2)\,\chi]\,\varrho[(2)\,\chi]$, $[(1)\,\chi]\,\varrho[(2)\,\chi]$, $[(2)\,\chi]\,\varrho[(1)\,\chi]$.

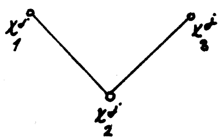
Ad b) Let $a_1, a_2 \in A$, $a_1 < a_2$. Let $(N, \leqq)$, $(P, \leqq) \in O(\mathscr{K}_2)$, $\varphi, \gamma \in H(N, P)$, $\varphi \neq \gamma$ and $(x)\,\varphi \neq (x)\,\gamma$ for certain $x \in N$. Notation for φ and γ will be chosen so that $(x)\,\varphi > (x)\,\gamma$ or $(x)\,\varphi \parallel (x)\,\gamma$. Define χ: $P \to A$ as follows. If $z \in P$, then $z \geqq (x)\,\varphi \Rightarrow (z)\,\chi = a_2$, $(z)\,\chi = a_1$ otherwise. Then $\chi \in H(P, A)$ and $\varphi\chi \neq \gamma\chi$.

On the contrary, let (A, ϱ) be a cogenerator in $\mathscr{K}_2$. Put $N = \{1\}$, $P = \{1, 2\}$ (with the ordering $1 < 2$). Let φ, γ: $N \to P$, $(1)\,\varphi = 1$, $(1)\,\gamma = 2$. Let $\chi \in H(P, A)$, $\varphi\chi \neq \gamma\chi$. Then $(1)\,\chi \neq (2)\,\chi$ and $(1)\,\chi < (2)\,\chi$. So A is not an antichain.

Ad c) Let A possess the required properties, $(N, n, \leqq)$, $(P, p, \leqq) \in O(\mathscr{K}_6)$, φ, $\gamma \in H(N, P)$, $(z)\,\varphi \neq (z)\,\gamma$ for a certain $z \in N$. Suppose $(z)\,\varphi > (z)\,\gamma$ or $(z)\,\varphi \parallel (z)\,\gamma$. Let $p \geqq (z)\,\varphi$. Define χ: $P \to A$ as follows: $v \geqq (z)\,\varphi \Rightarrow (v)\,\chi = a$, $(v)\,\chi = x$ otherwise. If p non $\geqq (z)\,\varphi$ then the definition of χ runs as follows: $v \geqq (z)\,\varphi \Rightarrow (v)\,\chi = y$, $(v)\,\chi = a$ otherwise. It is $\chi \in H(P, A)$, $\varphi\chi \neq \gamma\chi$.

To prove the converse, it suffices to take in considerations the object $N = \{1, 2, 3\}$ with the ordering $1 < 2 < 3$, 2 being the distinguished element. Let φ: $N \to N$, $(1)\,\varphi = 2$, $(2)\,\varphi = 2$, $(3)\,\varphi = 3$, γ be the identity on N. Let $(A, a, \leqq)$ be a cogenerator in $\mathscr{K}_6$ and $\varphi\chi \neq \gamma\chi$ for a suitable $\chi \in H(N, A)$. Then $(2)\,\chi = a$, $(1)\,\chi \neq a$, so $(1)\,\chi < a$. Similarly, the existence of y can be proved.

Ad d) One gets the assertion for $\mathscr{K}_3$ and $\mathscr{K}_4$ from 4.5.c). Let us prove d) for $\mathscr{K}_5$ as follows. Take a cardinal number $\mathfrak{m}$ and the sets A_j with Hasse diagram where j runs through a chain J with the cardinality $\mathfrak{m}$



and $A_j \cap A_i = \emptyset$ for $i \neq j$. Put $A = \bigcup_{j \in J} A_j$, the orderings of A_j be kept and let $x^i_1, x^i_2 < x^j_1$ for $i < j$. There exist no embedded subsets in A, but one-point subsets and A alone. So if $\varphi \in H_{\mathscr{K}_5}(A, B)$, $B \in O(\mathscr{K}_5)$, then φ is a mapping on one point of B or a relation-isomorphic mapping. Admit that B is a cogenerator. Then card $B \geqq \mathfrak{m}$ for all cardinals $\mathfrak{m}$.

4.10. Note. In fact, following proposition has been proved in the proof of 4.9.c.

Let V be a variety, $\mathcal{K}$ a regular V-subcategory in $\mathcal{K}_2$, for which $O(\mathcal{K}) = O(\mathcal{K}_2)$ and all mappings of one point sets be morphisms of $\mathcal{K}$. Then $\mathcal{K}$ possesses no cogenerators.

4.11. Now, let us add some results on the category $\mathcal{K}_2$.

Let $\varphi, \psi \in M(\mathcal{K}_2)$. According to [15] p. 251 we shall write $\varphi \downarrow \psi$, if and only if $\varphi\gamma_1 = \varphi\gamma_2 \Rightarrow \psi\gamma_1 = \psi\gamma_2$ for all $\gamma_1, \gamma_2 \in M(\mathcal{K}_2)$. $\varphi \uparrow \psi$ is defined in dual way. A monomorphism μ is said to be an i-mapping, if $\mu \downarrow \beta \Rightarrow \beta = \beta_1\mu$ for a suitable β_1. Similarly, an epimorphism ν is a p-mapping if $\nu \uparrow \beta \Rightarrow \beta = \nu\beta_1$ (Kowalsky calls i-mappings injections, p-mappings projections. These terms are reserved for concepts related to direct and free joins in this paper).

One can easily prove

4.11. a) $\varphi\colon N \to M$, $\psi\colon P \to M$, $\varphi \downarrow \psi \Leftrightarrow (N)\,\varphi \supset (P)\,\psi$.

b) $\varphi\colon M \to N$, $\psi\colon M \to P$, $\varphi \uparrow \psi \Leftrightarrow (x)\,\varphi = (x')\,\varphi \Rightarrow (x)\,\psi = (x')\,\psi$ *for all* $x, x' \in M$.

(Compare with the resuls in [15] p. 251.)

c) *A monomorphism $\mu \in M(\mathcal{K}_2)$ is an i-mapping, if and only if it is a relation-isomorphic mapping.*

Proof. Let μ be a relation-isomorphic mapping $A \to B$ and $\mu \downarrow \beta$, $\beta\colon C \to B$. According to 4.11. a) $(A)\,\mu \supset (C)\,\beta$. Let μ_1 denote the isomorphism $(A)\,\mu \to A$ inverse to the isomorphism $A \to (A)\,\mu$ induced by μ. Let β_1 be the mapping $C \to (A)\,\mu$ induced by β. Then $\beta = \beta_1\mu_1\mu$.

Let a monomorphism $\mu\colon A \to B$ be an i-mapping. Admit the existence of $x, y \in A$, $x \parallel y$ and $(x)\,\mu < (y)\,\mu$. Let ι be the inclusion mapping of $(A)\,\mu$ in B. Then $\mu \downarrow \iota$ and ι clearly has no factorisation by means of μ.

d) *Epimorphism $\nu \in M(\mathcal{K}_2)$, $\nu\colon M \to N$ is a p-mapping, if and only if following equivalence is valid for all $x, y \in M$. (*) $(x)\,\nu < (y)\,\nu \equiv$ there exist $x_i, y_i \in M$, $i = 1, \ldots, n$, $x_i < y_{i+1}$, $x < y_1$, $x_n < y$, $(x_i)\,\nu = (y_i)\,\nu$.*

Proof. Let (*) be satisfied, $\nu \uparrow \beta$, $\beta\colon M \to P$. By 4.11.b) for all $x, x' \in M$ $(x)\,\nu = (x')\,\nu \Rightarrow (x)\,\beta = (x')\,\beta$. Define $\beta_1\colon N \to P$ as follows: If $y \in N$, let $y_1 \in (y)\,\nu^{-1}$ and put $(y)\,\beta_1 = (y_1)\beta$. If $y_1' \in (y)\,\nu^{-1}$, too, then $(y_1')\,\nu = (y_1)\,\nu$, so $(y_1')\,\beta = (y_1)\,\beta$. Hence the definition of β_1 does not depend on the choice of y_1. Let $y, y' \in N$, $y < y'$. Let $y_1 \in (y)\,\nu^{-1}$, $y_1' \in (y')\,\nu^{-1}$. (*) implies $(y)\,\beta_1 \leqq (y')\,\beta_1$. So $\beta_1 \in M(\mathcal{K}_2)$ Clearly $\beta = \nu\beta_1$.

Let ν be a p-mapping.

Define on $P = \{(z)\,\nu^{-1}\colon z \in N\}$ the relation by the equivalence

$(z)\,\nu^{-1}\varrho \leqq (z_1)\,\nu^{-1} \equiv$ there exist x and x_1, $(x)\,\nu = z$,

$$(x_1)\,\nu = z_1 \quad \text{and} \quad x \leqq x_1.$$

Transitive hull $\leqq$ of ϱ is an ordering of P and the canonical mapping $\varkappa: M \to P$ is an homomorphism, so morphism in $\mathscr{K}_2$. By 4.11. b$\nu \uparrow \varkappa$. Let $\varkappa = \nu\varkappa_1$. If $(x)\,\nu < (y)\,\nu$, then $(\varkappa)\,\varkappa \leqq (y)\,\varkappa$ and (*) follows.

4.12. By standart consideration following proposition can be proved. $A \in O(\mathscr{K}_2)$ is an injective object, if and only if A is a chain, A is a projective object, if and only if A is an antichain, (definitions of injective and projective objects see e.g. [18] pp. 69, 71).

5. DIFFERENCE KERNEL AND COKERNEL

5.1. Let $\mathscr{K}$ be a category. Let $\varphi, \gamma: a \to b$, $\varphi, \gamma \in M(\mathscr{K})$. Let $\psi \in M(\mathscr{K})$, ψ: $c \to a$ with the following properties

a) $\psi\varphi = \psi\gamma$.

b) If $\mu\varphi = \mu\gamma$, then there exists a unique $\nu \in M(\mathscr{K})$ so that $\mu = \nu\psi$.

Then ψ is called a difference kernel of φ and γ (see [9] p. 21).

Difference cokernel is defined in the dual way.

5.2. *Let φ, $\gamma \in M(\mathscr{K}_i)$ $(i = 1, \ldots, 6)$, φ, $\gamma \in H(A, B)$. Then difference kernel of φ and γ exists.*

Proof. Let $K = \{x: (x)\,\varphi = (x)\,\gamma\}$ and K be provided with the reduction of the relation defined on A. If $i = 6$ and a is the distinguished element of A, then $a \in K$ and will be supposed to be the distinguished element of K. Let ι be the inclusion mapping of K in A. Clearly $\iota \in H_{\mathscr{K}_i}(K, A)$. Let $\chi\varphi = \chi\gamma$ for a certain $X \in O(\mathscr{K}_i)$ and a certain $\chi \in H(X, A)$. Hence $z \in X \Rightarrow (z)\,\chi \in K$. So χ induces in a natural way χ': $X \to K$. It is $\chi = \chi'\iota$. As ι is a monomorphism in $\mathscr{K}_i$, χ' is determined uniquely.

5.3. The investigations on cokernels are little more complicated. Let $A, B \in O(\mathscr{K}_i)$, $i = 1, \ldots, 6$, $\varphi, \gamma \in H(A, B)$. Let $i = 1$. The decomposition R on B is defined as follows. If $X_a = \{(a)\,\varphi, (a)\,\gamma\}$ for $a \in A$ ($(a)\,\varphi = (a)\,\gamma$ is admitted), then R is the finest decomposition on B, for which every X_a is contained in some element of R. If ϱ is the relation of B then the relation ϱ' on R is defined as follows. $X_1, X_2 \in R$, $X_1\varrho'X_2 \equiv$ $\equiv$ there exist $x_1 \in X_1$, $x_2 \in X_2$ so that $x_1\varrho x_2$. We shall prove that the canonical mapping $\varkappa$ of B onto R is a difference cokernel of φ and γ. First, $\varkappa \in H(B, R)$ by construction of ϱ'. Further, $\varrho\varkappa = \gamma\varkappa$ by definition of R. Let $\chi \in O(\mathscr{K}_1)$, $\chi \in H(B, X)$, $\varphi\chi = \gamma\chi$. We shall prove that $R^* =$ $= \{(x)\,\chi^{-1} : x \in (B)\,\chi\}$ is a covering of R. As for all $a \in A$ $(a)\,\varphi\chi = (a)\,\gamma\chi$, $(a)\,\varphi$ and $(a)\,\gamma$ are elements of the same class of R^*. By minimality of R, R^* is a covering of R. Define on R^* the relation ϱ^* in a similar way as ϱ' has been defined. χ induces $\bar{\chi} : R^* \to X$, $\bar{\chi} \in M(\mathscr{K}_1^*)$. Let $\varkappa'$ be the canonical mapping of R onto R^*, clearly $\varkappa' \in M(\mathscr{K}_1)$ and $\chi = \varkappa\varkappa'\bar{\chi}$. As $\varkappa$ is an epimorphism, the factorisation of χ in $\varkappa\chi''$ is unique.

For the cases $\mathscr{K}_2$ and $\mathscr{K}_6$ the consideration is quite similar. As for R, the following properties are demanded:

1. If $a \in A$, then $X' \in R$ exists so that $X_a \subset X'$.

2. If ϱ'' is the transitive hull of the relation ϱ', then ϱ'' is an ordering of R (transitive hull of the relation ϱ is the least transitive relation containing ϱ).

3. R is the finest decomposition with the properties 1. an 2.

The existence of R can be proved as follows. Let R be the infimum of all decompositions R_j on B satisfying 1. (this system is not empty, as it contains the coarsest decomposition). So $R = \bigwedge_{j \in J} R_j$ Clearly 1. holds for R. Define ϱ'' as in 2. Let $X_1, X_2 \in R$, $X_2 \varrho'' X_1$, $X_1 \varrho'' X_2$. Then there exist sets $Y_1, \ldots, Y_n$, $Y_1 = X_1$, $Y_k = X_2$, $Y_n = X_1$, $Y_i \in R$, $i = 1, \ldots, n$, k a suitable number among them, so that there exist elements y_i, y_i', $y_i \in Y_i$ for every i, for which $y_1 \leqq y_2'$, $y_2 \leqq y_3'$, $\ldots$, $y_n \leqq y_1'$ hold. Let $j \in J$. Then by 2. for R_j there exists an element $X_j \in R_j$ containing all y_i, y_i'. Let $X = \bigcap_{j \in J} X_j$. Then $X = X_1 = X_2$. So ϱ'' is an antisymetrical relation. Reflexivity is clear.

As for R^*, it is immediately seen that R^* satisfies 1. and 2., so R^* a covering of R and the rest of the consideration made for $i = 1$ is valid also in this case (for $i = 6$ the distinguished elements of R and R^* are that containing the distinguished element of B). So we get

5.4. *In $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_6$, every two morphisms φ, $\gamma \in H(A, B)$ possess a difference cokernel.*

5.5. *Let γ, $\varphi \in H_{\mathscr{K}_i}(A, B)$ $i = 3, 4, 5$. Morphisms γ and φ have a difference cokernel if and only if there exists a decomposition R of B in embedded chains (for $i = 3$), antichains ($i = 4$) or ordered sets ($i = 5$) satisfying 1. from 5.3.*

Proof is analogical to that of 5.4 and follows from the description of the decomposition R^* taken from 1.3.9.

5.6. *Let φ, $\gamma \in H_{\mathscr{K}_5}(A, B)$. Then φ and γ possess a difference cokernel.*

Proof. The coarsest decomposition on B fullfils the conditions from 5.5.

5.7, *For $i = 3$ or $i = 4$, φ, $\gamma \in H_{\mathscr{K}_i}$ (A, B) exist such that they do not possess any difference cokernel.*

Proof. Let $A = \{a_1, a_2\}$, $a_1 \parallel a_2$, $B = \{b_1, b_2, b_3, b_4\}$, $b_1 < b_2$, $b_3 < b_4$. Let $(a_1)\,\varphi = b_4$, $(a_2)\,\varphi = b_2$, $(a_1)\,\gamma = b_1$, $(a_2)\,\gamma = b_3$. It is $X_{a_1} = \{b_1, b_4\}$, $X_{a_2} = \{b_2, b_3\}$, so the coarsest decomposition of B is the only decomposition of B consisting of embedded subsets of B and satisfying 1. from 5.3., nevertheless conditions of 5.5 are not fullfilled neither for $i = 3$ nor $i = 4$.

6. DIRECT AND FREE JOINS

6.1. Let $\mathscr{K}$ be a category, $S = \{a_j\}_{j\in J}$ an indexed system of its objects. Say that an object $c \in O(\mathscr{K})$ together with morphisms $\alpha_j: c \to a_j$ is a fastdirect join of the system S, if for every system of morphisms $\beta_j: d \to a_j$ there exists $\gamma: d \to c$ such that $\gamma\alpha_j = \beta_j$ for all $j \in J$. α_j is called a projection (similarly as for direct join).

Note. If $J = \emptyset$, then a is a fastdirect (direct) join of S if and only if a is a weakly terminal (terminal) object of $\mathscr{K}$.

6.2. *Let $J \neq \emptyset$, $\{A_j\}_{j\in J}$ be a system of objects from $\mathscr{K}_i$ $(i = 1, 2, 3, 5)$. Let at least one of these objects be empty. Then the direct join of $\{A_j\}_{j\in J}$ exists and it is the empty est.*

Clear.

6.3. *Let C be a fastdirect join of a system $\{A_j\}_{j\in J}$ $(A_j \neq \emptyset$ from $\mathscr{K}_j$ $i = 1, \ldots, 6)$. Let α_j be corresponding projection. Then α_j is an epimorphism and for every two distinct indices $j', j'' \in J$ and every $x \in A_{j'}$, $y \in A_{j''}$*

$$(x)\,\alpha_{j'}^{-1} \cap (y)\,\alpha_{j''}^{-1} \neq \emptyset.$$

Proof. $i = 2, 3, 4, 5$. Admit $j_1 \in J$ exists such that α_{j_1} is not an epimorphism. Let a non $\in (C)\,\alpha_{j_1}$ and $D_1 = \{d_1\}$ be one point set. Define $\beta_{j_1}: (d)\,\beta_{j_1} = a$, β_j arbitrary for $j \neq j_1$. Evidently $\gamma: D \to C$ with demanded properties does not exist.

Let $x \in A_{j'}$, $y \in A_{j''}$, $D_2 = \{d_2\}$ and define $(d_2)\,\beta_{j'} = x$, $(d_2)\,\beta_{j''} = y$, $\beta_j, j \neq j', j''$ arbitrary. Then $(d_2)\,\gamma \in (x)\,\alpha_{j'}^{-1} \cap (y)\,\alpha_{j''}^{-1}$.

$i = 1$. The proof runs as above, only relation of D_i $(i = 1, 2)$ is to be considered empty.

$i = 6$. Take in above consideration A_{j_1} instead of D_1, identity mapping in the place of β_{j_1}, an antichain $D_3 = \{d, d'\}$, d' the distinguished element instead of D_2 and define $\beta_{j'}$, $\beta_{j''}$ so that $(d)\,\beta_{j'} = x$, $(d)\,\beta_{j''} = y$. This is possible as $d \parallel d'$.

6.4. *Let $(A_j, \varrho_j) \in O(\mathscr{K}_i)$, $j \in J$, $i = 1, 2, 6$. Then $\{(A_j, \varrho_j)\}_{j\in J}$ has a direct join.*

Proof. For $J = \emptyset$ see **4.6** and note in 6.1. Let $J \neq \emptyset$. Let P be the cartesian product of the sets A_j, π a relation on P defined as follows: $(\ldots, x_j, \ldots)\,\pi(\ldots, y_j, \ldots) \equiv x_j\varrho_j y_j$ for all $j \in J$. Let α_j denote the projection of P onto A_j. If $i = 6$, let the element of P, all coordinates of which are distinguished elements, be the distinguished element of P. The proof of the fact that P is a direct join of $\{(A_j, \varrho_j)\}_{j\in J}$ with the projections α_j is straightforward.

6.5. *Let A_1, A_2 be two objects of $\mathscr{K}_3$. Let* card *$A_j > 1$, $j = 1, 2$. Let $\{A_j\}_{j\in J}$ be a system of objects in $\mathscr{K}_3$, $1, 2 \in J$. Then a fastdirect join of this system does not exist.*

Proof. Admit that C is a fastdirect join of the system $\{A_j\}_{j\in J}$, α_j denotes the corresponding projection. Let R_j be the decomposition of C which corresponds to α_j. According to 6.3. card $R_j > 1$ for $j = 1, 2$. According to 1.3.a the elements of R_j are chains embedded in C. Further, $X \in R_1$ $X' \in R_2 \Rightarrow X \cap X' \neq \emptyset$. Let us consider two cases.

a) A_1, A_2 are chains. Then C, as $(A_j)\,\alpha_j^{-1}$ since α_j is A-homomorphism, must be a chain, too. Let $X_1, X_2 \in R_1$, $X_3, X_4 \in R_2$, $X_1 \neq X_2$, $X_3 \neq X_4$. Choose the notation in such a way that $x_1 \in X_1$, $x_2 \in X_2 \Rightarrow x_1 < x_2$; $x_3 \in X_3, x_4 \in X_4 \Rightarrow x_3 < x_4$. Let $x' \in X_1 \cap X_3$, $x'' \in X_1 \cap X_4$, $x''' \in X_2 \cap X_3$ $x'''' \in X_2 \cap X_4$. So $x' < x'' < x''''$, $x' < x''' < x''''$. It is $x'' \neq x'''$ and C is a chain. Admit $x''' < x''$. Then X_1 is not embedded in C. If $x'' < x'''$, X_4 is not embedded in C, so we get a contradiction.

b) Choose the notation so that A_1 be not a chain. Then X_1 and X_2 in R_1 exist such that $x_1 \in X_1$, $x_2 \in X_2 \Rightarrow x_1 \parallel x_2$. Let $X \in R_2$, $x' \in X_1 \cap X$, $x'' \in X_2 \cap X$. As X is a chain, x', x'' are comparable, a contradiction.

6.6. *Let $A_j \in O(\mathscr{K}_3)$, $A_j \neq \emptyset$, $j \in J$,* card $J \geqq 2$. *Let at most one of the set A_j possess more than one element. Then $\{A_j\}_{j \in J}$ possesses a directed join exactly when A_j is a chain for all $j \in J$. If there exists $j_1 \in J$ such that A_{j_1} is not a chain, then no fastdirect join of $\{A_j\}_{j \in J}$ exists.*

Proof. The assertion is clear if all the sets A_j are one-point sets. Do not let A_{j_1} be a one-point set and let it be a chain. Then put $C = A_{j_1}$, α_{j_1} being the identity mapping on A, α_j, $j \neq j_1$ the mapping of A_j on (one-point) set A_j. C with these morphisms is clearly a direct join. Now, let card $A_{j_1} > 1$ and A_{j_1} be not a chain. Admit C with $\{\alpha_j\}_{j \in J}$ to be a fastdirect join of $\{A_j\}_{j \in J}$. As α_{j_1} is an epimorphism by 6.3, C is not a chain and so $(C)\,\alpha_j$ for $j \in J$, $j \neq j_1$ cannot be a one-point set, which contradicts card $A_j = 1$ for $j \neq j_1$.

6.7. *Let $A_j \in O(\mathscr{K}_4)$ for $j \in J$,* card $J \geqq 2$, *$A_j \neq \emptyset$ and at least one of these objects, say A_1, be not an antichain. Then a fastdirect join of $\{A_j\}_{j \in J}$ does not exist.*

Proof. Let $x_1, x_2 \in A_1$, $x_1 < x_2$. Let $A_2 \in \{A_j\}_{j \in J}$ (so we suppose $2 \in J$). Admit there exists a fastdirect join of $\{A_j\}_{j \in J}$. Denote it by C. $(x_1)\,\alpha_1^{-1}$ and $(x_2)\,\alpha_1^{-1}$ are embedded antichains in C and for $y_1 \in (x_1)\,\alpha_1^{-1}$ $y_2 \in (x_2)\,\alpha_1^{-1}$ it is always $y_1 < y_2$. If R_2 is the decomposition of C corresponding to α_2 and $X \in R_2$, take $y_1 \in X \cap (x_1)\,\alpha_1^{-1}$, $y_2 \in X \cap (x_2)\,\alpha_1^{-1}$. As X is an antichain, it is impossible to have $y_1 < y_2$.

6.8. *Let $A_j \in O(\mathscr{K}_4)$ be an antichain for $j \in J \neq \emptyset$. Then $\{A_j\}_{j \in J}$ has a direct join.*

Proof. Let P be the general cartesian product of the sets A_j. Every two distinct elements of P are considered to be incomparable, i.e. P is an antichain. α_j denotes the usual projection. Let $D \in O(\mathscr{K}_4)$ and $\beta_j : D \to A_j$. Let $(d)\,\gamma = (\ldots, (d)\,\beta_j, \ldots)$ for $d \in D$. Then $\gamma \in H(D, P)$ and $\beta_j = \gamma\alpha_j$. γ is clearly unique.

6.9. *Let $A_1, A_2 \in O(\mathscr{K}_5)$,* card $A_1 > 1$, card $A_2 > 1$ *and at least one* *of these objects be not an antichain. Let J be a set, $1, 2 \in J$, $A_j \in O(\mathscr{K}_5)$ for $j \in J$. Then the fastdirect join of the system $\{A_j\}_{j\in J}$ does not exist.*

Proof. Let e.g. A_1 be not an antichain. Admit that C is a fastdirect join of $\{A_j\}_{j\in J}$, α_j denotes the corresponding projection. Elements x_1 and x_2 of A_1 exist such that $x_1 < x_2$. Then $y_1 \in (x_1)\,\alpha_1^{-1}$, $y_2 \in (x_2)\,\alpha_1^{-1}$ implies $y_1 < y_2$. R_2 denoting the decomposition induced by α_2, let $X' = (x_3)\,\alpha_1^{-1}$, $X'' = (x_4)\,\alpha_1^{-1}$ be two distinct elements of R_2. We can suppose $x_3 < x_4$ or $x_3 \parallel x_4$. In the first case, $y' \in X'$, $y'' \in X'' \Rightarrow y' < y''$. Choose $y' \in X' \cap (x_1)\,\alpha_1^{-1}$, $y'' \in X'' \cap (x_1)\,\alpha_1^{-1}$, $y''' \in X' \cap (x_2)\,\alpha_1^{-1}$, $y'''' \in X'' \cap (x_2)\,\alpha_1^{-1}$. We have $y' < y'''$, $y''' < y''$. Simultaneously y', $y'' \in (x_1)\,\alpha_1^{-1}$, y''' non $\in (x_1)\,\alpha_1^{-1}$. So $(y_1)\,\alpha_1^{-1}$ is not embedded in C.

Let $x_3 \parallel x_4$. Then $y' \in X', y'' \in X'' \Rightarrow y' \parallel y''$. Nevertheless by preceeding considerations $y' \in X' \cap (x_1)\,\alpha_1^{-1}$, $y'' \in X'' \cap (x_2)\,\alpha_1^{-1} \Rightarrow y' < y''$, a contradiction.

6.10. *Let $A_j \in O(\mathscr{K}_5)$, $A_j \neq \emptyset$, $j \in J \neq \emptyset$. Let at most one of these object contain more than one element. Then a direct join of $\{A_j\}_{j\in J}$ exists.*

Proof. Let all A_j' s be one point sets. Then every one point set is a direct join of $\{A_j\}_{j\in J}$. Suppose A_{j_1} to be not a one point set. Then A_{j_1} is a direct join of $\{A_j\}_{j\in J}$. The projection α_{j_1} is the identity map.

6.11. *Let $A_j \in O(\mathscr{K}_5)$, $j \in J \neq \emptyset$, $A_j \neq \emptyset$. Let all A_j be antichains. Then a direct join of $\{A_j\}_{j\in J}$ exists.*

Proof. Let P be a cartesian product of A_j, considered to be an antichain, α_j the usual projection. Let $\beta_j : D \to A_j$. Put, as in 6.8. $(d)\,\gamma = (\ldots, (d)\,\beta_j, \ldots)$ for all $d \in D$. As all elements of every connected component of D are mapped on the same element from A_j, $\gamma \in H(D, P)$. Uniqueness of γ is clear.

6.12. Let $\mathscr{K}$ be an arbitrary category, $S = \{a_j\}_{j\in J}$ a system of its objects. $c \in O(\mathscr{K})$ together with a system of morphisms $\alpha_j : a_j \to c$ is said to be a fastfree join of S if for every system of morphisms $\beta_j : a_j \to d$ there exists $\gamma : c \to d$ such that $\alpha_j\gamma = \beta_j$ for all $j \in J$. α_j are called injections.

Note. If $J = \emptyset$, then c is a fastfree (free) join of S if and only if c is a weakly initial (initial) object of $\mathscr{K}$.

6.13. *In $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_6$ every system $\{(A_j, \varrho_j)\}_{j\in J}$ possesses a free join.*

Proof. For $J = \emptyset$ clear.

Let $J \neq \emptyset$. Let P be a cardinal sum of A_j', A_j' provided with the relation $\tilde{\varrho}_j$ isomorphic to ϱ_j. For $\mathscr{K}_1$ and $\mathscr{K}_2$ P is a free join of $\{(A_j', \tilde{\varrho}_j)\}$. For $\mathscr{K}_6$ one must identify in P all distinguished elements of A_j in one element and to consider it to be the distinguished element (relation is then defined as the transitive hull).

6.14. *Let $\{A_j\}_{j\in J}$ be a system of objects in $\mathscr{K}_i$ ($i = 3, 4, 5$) containing two non-empty sets A_1, A_2 (so $1, 2 \in J$). Then a fastfree join of $\{A_j\}_{j\in J}$ does not exist.*

Proof. Admit that C is a fastfree join of $\{A_j\}_{j\in J}$, α_j the corresponding injection. Let $\sum_{j\in J-\{1,2\}} A_j$ denote the cardinal sum (one can suppose that A_j are mutualy disjunct), $\oplus$ means the ordinal summation. Put $D_1 = A_1 \oplus A_2 \oplus \sum_{j\in J-\{1,2\}} A_j$, $D_2 = A_2 \oplus A_1 \oplus \sum_{j\in J-\{1,2\}} A_j$. Let β_j^1 be the mapping of A_j into D_1 induced by identity mapping of A_j. Let $x \in A_1$, $y \in A_2$. We have $(x)\,\beta_1^1 < (y)\,\beta_2^1$. Let γ be a mapping from the definition of the fastfree join. Then $(x)\,\alpha_1\gamma = (x)\,\beta_1^1$, $(y)\,\alpha_2\gamma = (y)\,\beta_2^1$. As γ cannot map incomparable elements into distinct comparable, we have $(x)\,\alpha_1 < (y)\,\alpha_2$. Considering D_2 instead of D_1 we get $(x)\,\alpha_1 < (y)\,\alpha_2$, a contradiction.

6.15. *If a system* $\{A_j\}_{j\in J}$, $A_j \in O(\mathscr{K}_i)$ $(i = 3, 4, 5)$, *contains an object* A_{j_1} *such that* $j \neq j_1 \Rightarrow A_j = \emptyset$, *then* A_j *is a fastfree join of* $\{A_j\}_{j\in J}$. Clear.

From the theorem on p. 77 in [9] and 5.2., 5.4., 6.4. and 6.13. one gets

6.16. *Every functor from a small category to* $\mathscr{K}_1$ *(*$\mathscr{K}_2$ *or* $\mathscr{K}_6$*, respectively) has a left and right roots, so that* $\mathscr{K}_1$, $\mathscr{K}_2$ *and* $\mathscr{K}_6$ *are complete.*

6.17. *Let* $\mathscr{K}$ *be a subcategory of* $\mathscr{K}_2$ *with the following properties.*

1. $O(\mathscr{K}) = O(\mathscr{K}_2)$.
2. *If* X, $Y \in O(\mathscr{K}_2)$, $\varphi\colon X \to Y$ *isomorphism, then* $\varphi \in M(\mathscr{K})$.
3. *If* $\varphi\colon X \to Y$, $Y \subset Z$, $\varphi \in M(\mathscr{K})$, ι *inclusion mapping* $Y \to Z$ *in* $\mathscr{K}_2$ *then* $\varphi\iota \in M(\mathscr{K})$.
4. *Let* A *be two-point chain,* B *one-point set. Then* $\varphi\colon A \to B$ *is an element of* $M(\mathscr{K})$.

Let $\mathscr{K}'$ *be a right complete subcategory in* $\mathscr{K}_2$ (see [9] p. 26), *for which* $M(\mathscr{K}) \subset M(\mathscr{K}') \subset M(\mathscr{K}_2)$. *Then* $\mathscr{K}' = \mathscr{K}_2$.

The proof will be accomplished by means of several lemmas.

Lemma 1. The free join $B + B$ in $\mathscr{K}'$ is a two-point antichain.

Proof. Let $B = \{b\}$. Let $C = \{c_1, c_2\}$ be two-point antichain, $\varphi_1\colon B \to C$, $(b)\,\varphi_1 = c_1$, $\varphi_2\colon B \to C$, $(b)\,\varphi_2 = c_2$. $\varphi_1, \varphi_2 \in M(\mathscr{K}')$ by 1. and 3. There exists exactly one $\varphi \in M(\mathscr{K}')$, $\varphi\colon B + B \to C$ so that $\alpha_j\varphi = \varphi_j$ $(j = 1, 2)$, where α_j means the corresponding injection. Hence $(b)\,\alpha_1 \neq (b)\,\alpha_2$ and $(b)\,\alpha_1 \parallel (b)\,\alpha_2$. Let $D = B + B - \{(b)\,\alpha_1, (b)\,\alpha_2\}$. Admit $D \neq \emptyset$. Let ψ mean the isomorphic mapping of C on $\{(b)\,\alpha_1, (b)\,\alpha_2\}$, $(c_1)\,\psi = (b)\,\alpha_1$, $(c_2)\,\psi = (b)\,\alpha_2$. Clearly $\alpha_1\varphi\psi\iota = \alpha_1$, $\alpha_2\varphi\psi\iota = \alpha_2$, where ι is an inclusion map of $\{(b)\,\alpha_1, (b)\,\alpha_2\}$ in $B + B$. Simultaneously $\varphi\psi\iota \in M(\mathscr{K}')$ by 2. and 3., $\varphi\psi\iota \neq 1_{B+B}$. We have $\alpha_1 1_{B+B} = \alpha_1$, $\alpha_2 1_{B+B} = \alpha_2$, too. This is a contradiction to the definition of the free join.

Lemma 1. impies that a mapping of two-point antichain to one-point set is an element of $M(\mathscr{K}')$.

Lemma 2. Let $C = \{c_1, c_2\}$ be a two-point antichain. Let $A = \{a_1, a_2\}$, $a_1 < a_2$. Let $(c_1)\,\varphi = a_1$, $(c_2)\,\varphi = a_2$. Then $\varphi \in M(\mathscr{K}')$.

Proof. $C = B + B$ by lemma 1, α_1, α_2 being again the corresponding injections, $(b)\,\alpha_1 = c_1$, $(b)\,\alpha_2 = c_2$, $(b)\,\varphi_1 = a_1$, $(b)\,\varphi_2 = a_2$, where $\varphi_j: B \to A$ for $j = 1, 2$. By 2 and 3 $\varphi_j \in M(\mathscr{K}')$. Let $\alpha_j\varphi = \varphi_j$ for $j = 1, 2$. Then $(c_1)\,\varphi = a_1$, $(c_2)\,\varphi = a_2$.

Lemma 3. Let $X \in O(\mathscr{K}')$. Let the category $\mathscr{X}$ consist of one-point and two-point subsets of X, morphisms of $\mathscr{X}$ are inclusion mappings. Let $F_{\mathscr{X}}$ be the identity functor of the category $\mathscr{X}$ in $\mathscr{K}'$. Then X is a right root of the functor $F_{\mathscr{X}}$ and inclusion maps are injections.

Proof. Let $Y \in O(\mathscr{X})$ and $\varphi(Y)$ be the inclusion map of Y in X. Let C be a right root of $F_{\mathscr{X}}$, $\alpha(Y)$ the corresponding injection and $\varphi: C \to X$ the mapping corresponding to $\varphi(Y)$. So $\varphi(Y) = \alpha(Y)\,\varphi$. For $Y \subset Z$, $Y, Z \in O(\mathscr{X})$ it is $\alpha(Y) = (\iota)F_{\mathscr{X}}\alpha(Z)$ where ι is the inclusion map of Y in Z.

As $\bigcup_{Y \in O(\mathscr{X})} [(Y)\,[\varphi(Y)]] = X$, φ is a mapping of C on X and as $\varphi(Y)$ is one-to-one mapping, i.e. monomorphism in $\mathscr{K}_2$, so $\alpha(Y)$ is a monomorphism in $\mathscr{K}_2$, i.e. one-to-one, for $Y \in O(\mathscr{X})$. If $Y \in O(\mathscr{X})$ is an antichain, so is $(Y)\,[\varphi(Y)]$ and $(Y)\,[\alpha(Y)]$. So, if $Y = \{y_1, y_2\}$, then $(y_1)\,[\alpha(Y)] \parallel (y_2)\,[\alpha(Y)] \equiv y_1 \parallel y_2$ and $(y_1)\,(\alpha(Y)] < (y_2)\,[\alpha(Y)] \equiv y_1 < y_2$ Hence φ maps isomorphically $\bigcup_{Y \in O(\mathscr{X})} [(Y)\,[\alpha(Y)]]$ on X. Let $E = C - \bigcup_{Y \in O(\mathscr{X})} [(Y)\,[\alpha(Y)]]$ and admit $E \neq \emptyset$. By foregoing arguments φ induces an isomorphic mapping $\psi: \bigcup_{Y \in O(\mathscr{X})} [(Y)\,[\alpha(Y)]] \to X$, so $\varphi\psi^{-1} \in M(\mathscr{K}')$, which is not onto, so different from 1_C. For all $Y \in O(\mathscr{X})$ we have $\varphi(Y)\,\psi^{-1} = \alpha\,(Y) = \alpha(Y)\,1_C = \alpha(Y)\,\varphi\psi^{-1}$. As C is a right root of $F_{\mathscr{X}}$, there exists unique χ such that $\alpha(Y) = \alpha(Y)\,\chi$ and it is 1_C. So $1_C = \varphi\psi^{-1}$, a contradiction.

Lemma 4. Let X, $V \in O(\mathscr{K}_2)$, $\varphi^*: X \to V$, $\varphi^* \in M(\mathscr{K}_2)$. Then $\varphi^* \in M(\mathscr{K}')$.

Proof. Let $F_{\mathscr{X}}$ be the same as in lemma 3. Let $Y \in O(\mathscr{X})$, $\varphi^*(Y)$ the mapping $Y \to V$ equal to $\varphi^* \mid Y$. $\varphi^*(Y) \in M(\mathscr{K}')$ by lemmas 1. and 2. and suppositions 2. and 4. on $\mathscr{K}'$. Then $\psi: X \to V$ exists in $M(\mathscr{K}')$ for which $\varphi^*(Y) = [\alpha(Y)]\,\psi$ for all $Y \subset X$. As $\alpha(Y)$ is the inclusion map of Y into X, $\psi = \varphi^*$.

The assertion 6.17 has been proved.

6.18. *Let $\mathscr{K}$ possess the difference cokernels, $M(\mathscr{K}_4) \subset M(\mathscr{K}) \subset M(\mathscr{K}_2)$. Then $\mathscr{K}$ satisfies* 4. *from* 6.17.

Proof. Let F be a three- point chain $\{f_1, f_2, f_3\}$, $f_1 < f_2 < f_3$. Let $A = \{a_1, a_2\}$, $a_1 < a_2$, $(a_1)\,\varphi_1 = f_2$, $(a_2)\,\varphi_1 = f_3$, $(a_1)\,\varphi_2 = f_1$, $(a_2)\,\varphi_2 = f_2$, φ be a difference cokernel of φ_1, φ_2, $\varphi: F \to C$. We shall prove that C is one-point set. Admit $C - \{x\} \neq \emptyset$, where $x = (f_1)\,\varphi = (f_2)\,\varphi = (f_3)\varphi$. Let ψ be an inversible mapping of object C_1 on C, $C_1 \cap C = \emptyset$. Let D

be constructed from $C_1 \cup C$ by identifying x and $(x)\ \psi^{-1}$ and completening the relation to the transitive hull. Let ι' be an inclusion map $C \to D$, i'' inclusion map $C_1 \to D$, $\psi' = \psi^{-1}\iota''$.

It is $\varphi_1\varphi\iota' = \varphi_2\varphi\psi'$ and $\varphi\iota' = \varphi\psi'$. Simultaneously $\iota' \neq \psi'$, which contradicts the definition of the difference cokernel.

So C is one-point set and $\varphi_1\varphi$ is the demanded map.

6.19. *Let i be one of the number $i = 3, 4, 5$. Let $\mathscr{K}$ be a complete category, $M(\mathscr{K}_i) \subset M(\mathscr{K}) \subset M(\mathscr{K}_2)$. Then $\mathscr{K} = \mathscr{K}_2$.*

6.19 follows from 6.17 and 6.18.

7. ORDERING OF THE SYSTEM OF FULL RELATION SUBOBJECTS OF AN OBJECT

7.1. In the paper [21] the orderings of the set exp (A) -the system of all subset of a given set A—are studied, which are invariant to all permutations of the set A. That means, such orderings $\leqq$ (called "topological orderings") have been studied that for all permutations f of the set A it is $X \leqq Y \Rightarrow (X)f \leqq (Y)f$. It was proved (theorem 8 p. 295) that the ordering by inclusion is a maximal lattice topological ordering.

Now, we shall deal with similar questions for the category $\mathscr{K}_2$ in a slightly more general way. A full-relation subobject of (A, ϱ) has been defined in 1.2. If e.g. A is a well-ordered set, identity mapping is the only isotone mapping of A in A, which is inversible. So every ordering of the set of all full relation subobjects of A is invariant to all inversible mappings of A to A.

7.2. Let $\mathscr{K}_2'$ be the category, which originates from $\mathscr{K}_2$ by putting $O(\mathscr{K}_2') = O(\mathscr{K}_2)$ and morphisms of $\mathscr{K}_2'$ are one-to-one isotone mappings.

Let $\mathscr{K}$ be a subcategory of $\mathscr{K}_2$. For $A \in O(\mathscr{K})$ ϱ_A be an order of exp (A). Say that $\{\varrho_A\}_{\mathscr{K}}$ is stable on $\mathscr{K}$, if $X\varrho_A Y \Rightarrow (X)f\varrho_B(Y)f$ for every $f: A \to B, f \in M(\mathscr{K})$.

7.3. *Let $A \in O(\mathscr{K}_2')$, σ_A be the ordering of* exp (A) *by inclusion (the elements of* exp (A) *are taken as full-relation subobjects of A). Let $\sigma_A \subset \varrho_A$ for all $A \in O(\mathscr{K}_2')$ and* (exp (A), ϱ_A) *be a lower semilattice. If $\{\varrho_A\}_{\mathscr{K}_2'}$ is stable on $\mathscr{K}_2'$ then $\sigma_A = \varrho_A$ for all A.*

Proof. Admit that $A \in O(\mathscr{K}_2')$ exists such that $\varrho_A \supsetneqq \sigma_A$. Let $X \subset A$, $Y \subset A$ be such full-relation subobjects that $X \not\subset Y$, $X\varrho_A Y$. Let $x \in X$ be such an element that x non $\in Y$. Then $\{x\}\ \varrho_A A - \{x\}$. Let the ordering of A be completed to a total order. The object gained in such a way will be denoted by A'. So the inclusion map ι of A in A' (as for set point of view-the identity) is a morphism in $\mathscr{K}_2'$, so $\{x\}\ \varrho_{A'} A' - \{x\}$.

By similar arguments one proves that there exists a chain $B \in O(\mathscr{K}_2')$ and $b \in B$ with the following properties.

1. $\{b\}\ \varrho_B B - \{b\}$.

2. B is homogenous, i.e. for all $x, y \in B$ there exists an isomorphism f of B onto B such that $(x) f = y$.

Let B_i be mutualy disjoint copies of B for $i = \ldots, -1, 0, 1, \ldots$. Put $T = \ldots \oplus B_{-1} \oplus B_0 \oplus B_1 \oplus \ldots$, $T' = \bigcup_i B_{2i}$, $T'' = \bigcup_i B_{2i+1}$. By 1. and by stability of $\{\varrho_A\}_{\mathscr{K}'_2}$ $x \in T \Rightarrow \{x\} \varrho_T T'$, $\{x\} \varrho_T T''$. Put $P = T' \wedge T''$ (infimum in ϱ_T). As $\{x\} \varrho_T P$ for all $x \in T$, $P \neq \emptyset$. Let $y \in P$. Suppose $y \in B_{i_0}$. If $z \in B_{i_0}$, too, define $f: T \to T$ in the following way: On B_{i_0} f is an isomorphism mapping y into z. Otherwise f is an identity. So $(T') f = T'$, $(T'') f = T''$ and we have $(P) f = P$, So $B_{i_0} \subset P$.

Let g be an isomorphism $T \to T$ such that $(B_i) g = B_{i+1}$ Then $(T') g = T''$, $(T'') g = T'$. So $(P) g = P$. As $\bigcup_{n=\ldots-1,0,1\ldots} (B_{i_0}) g^n = P$. It results $P = T$. But this contradicts $T' \subset P$, $T' \neq P$.

7.4. Now, we shall define an ordering ν_A for $A \in O(\mathscr{K}'_2)$. For $X, Y \subset A$ $X \nu_A Y$ means one of the following conditions:

1. X, Y infinite and $X \subset Y$.
2. X finite, Y infinite.
3. X, Y finite and for every $x \in X - Y z \in Y - X$ exists such that $x < z$.

7.5. ν_A *is an ordering of* exp (A).

Proof. a) Clearly $X \nu_A X$.

b) Let $X \nu_A Y$, $Y \nu_A X$.

Let X or Y be infinite. Then the other of them is infinite, too, and clearly $X = Y$.

Let X, Y be finite. Suppose x is maximal in $X - Y$. Then $z \in Y - X$ exists such that $x < z$, $v \in X - Y$ exists such that $v > z$, so $v > x$, a contradiction. So $X = Y$.

c) Let $X \nu_A Y$, $Y \nu_A Z$. If X is infinite or Z is infinite, then $X \nu_A Z$. So we can suppose X, Y, Z to be finite. Let $x \in X - Z$, we can suppose x to be maximal in $X - Z$. It is $x \in (X - Y) - Z$ or $x \in X \cap Y - Z$.

c$_1$) Let $x \in (X - Y) - Z$. There exists $y \in Y - X$ (suppose it to be maximal) such that $y > x$. Then $y \in (Y - X) - Z$ or $y \in (Y - X) \cap Z$.

c$_{11}$) $y \in (Y - X) - Z \Rightarrow z$ exists such that $z \in Z - Y$, $z > y$. If $z \in X$ then $z \in X - Y$, which contradicts $X \nu_A Y$, so z non $\in X$ and $z \in Z - - X$. Simultaneously $z > x$.

c$_{12}$) $y \in (Y - X) \cap Z \Rightarrow y \in Z - X$.

c$_2$) $x \in X \cap Y - Z \Rightarrow x \in Y - Z \Rightarrow y$ exists in $Z - Y$ such that $y > x$. If $y \in X$, then $y \in X - Y$, so y_1, maximal in $Y - X$, exists with $y_1 > y$. If $y_1 \in Z$, the proof is finished. If $y_1 \in Y - Z$ then $y_2 \in Z - Y$ exists with $y_2 > y_1$. It cannot be $y_2 \in X$ because y_1 is maximal in $Y - X$ and $X \nu_A Y$. So y_2 non $\in X$, i.e. $y_2 \in Z - X$ and it is $y_2 > x$.

7.6. $\{\nu_A\}_{\mathscr{K}'_2}$ *is stable in* $\mathscr{K}'_2$ *and* $\sigma_A \subset \nu_A$ *for all* $A \in O(\mathscr{K}'_2)$.

Clear.

7.7. Let X, Y be finite subsets of $A \in O(\mathscr{K}'_2)$. We shall use the following notation.

$$M = \{x\colon x \in (X - Y) \cup (Y - X),\ x \text{ maximal}\},\quad M_X = M \cap X,$$
$$M_Y = M \cap Y,$$
$$(M_X] = \{x\colon x \in A \text{ and there exists } y \in M_X \text{ such that } x \leqq y\}.$$

Similarly $(M_Y]$ is defined. $X \vee Y = [X - (M_Y]] \cup [Y - (M_X]] \cup [X \cap Y - (M_X] \cap (M_Y]]$. Clearly $M_X \subset X - (M_Y]$, $M_Y \subset Y - (M_X]$.

7.8. *$X \vee Y$ is supremum of X and Y in ν_A.*

Proof. Let $X\nu_A Z$, $Y\nu_A Z$. If Z is infinite, so $(X \vee Y)\, \nu_A Z$. Suppose Z finite. Let x be maximal in $X \vee Y - Z$.

1. Let $x \in (X - (M_Y]) - Z$. There exists $y \in Z - X$, $x < y$. Admit $y \in X \vee Y$. Then $y \in Y - (M_X]$. So $y \in Y - X$, a contradiction.

2. Let $x \in (Y - (M_X]) - Z$. Similarly.

3. $x \in [X \cap Y - (M_X] \cap (M_Y]] - Z$. There exist $y_1 \in Z - X$, $y_2 \in Z - Y$, $y_1, y_2 > x$. Suppose that both of them are elements of $X \vee Y$. Then $y_1 \in Y - (M_X]$, $y_2 \in X - (M_Y]$, a contradiction. So e.g. y_1 non $\in X \vee Y$. We get $(X \vee Y)\, \nu_A Z$.

Now we prove $X\nu_A(X \vee Y)$. Let $x \in X - X \vee Y$. x non $\in X \vee Y \Rightarrow$ $\Rightarrow x$ non $\in X - (M_Y]$, i.e. $y \in M_Y$ exists such that $y > x$. As y non $\in X$, $y \in X \vee Y - X$.

Similarly $Y\nu_A(X \vee Y)$.

7.9. *Let A be finite. Then*

$$X \wedge Y = (X \cap Y) \cup (X \cap (M_Y]) \cup (Y \cap (M_X]) \cup ((M_X] \cap$$
$$\cap\ (M_Y]) \text{ is infimum of } X \text{ and } Y \text{ in } \nu_A.$$

Proof. $X \wedge Y \nu_A X$.

Let $x \in X \wedge Y - X$. Then $x \in (Y \cap (M_X]) \cup ((M_X] \cap (M_Y])$. So $x \in (M_X]$. Hence y exists in M_X such that $x \leqq y$. So $y \in X$. Clearly (by definition of M_X) y non $\in Y$ and y non $\in (M_Y]$. Hence y non $\in X \wedge Y$.

Simirarly $X \wedge Y\nu_A Y$.

Let $Z\nu_A X$, $Z\nu_A Y$. Let $x \in Z - X \wedge Y$ (we may suppose it to be maximal).

a) Suppose x non $\in X$, x non $\in Y$. Then $x \in Z - X$. So $y \in X - Z$ exists so that $y > x$. If $y \in X \cap Y$, it is $y \in X \wedge Y - Z$. Let $y \in X - Y$. If $y \in (M_Y]$, then $y \in X \wedge Y - Z$, too. If y non $\in (M_Y]$, then we construct $y' \in Y - Z$ in similar way as y taking Y instead of X. So suppose that y' non $\in (M_X]$ and $y' \in Y - X$. We have now $y \in X - Y$, y non $\in (M_Y]$, $y' \in Y - X$, y' non $\in (M_X)$, $x \leqq y$, $x \leqq y'$. Hence $x \in (M_X] \cap (M_Y]$, so $x \in X \wedge Y$, a contradiction.

b) Suppose $x \in X$. Then x non $\in Y$. $y \in Y - Z$ exists such that $y > x$. Suppose y non $\in X \wedge Y$. So $y \in (M_Y] - (M_X]$. Then $x \in X \cap (M_Y]$. So $x \in X \wedge Y$, a contradiction. So $y \in X \wedge Y - Z$. Similarly for $x \in Y$. So $Z \nu_A X \wedge Y$.

7.10. *Let $\mathscr{K}_2''$ be the category of all finite ordered sets with one-to-one isotone mappings as morphisms. Then $\{\nu_A\}_{\mathscr{K}_2''}$ is a maximal stable system in $\mathscr{K}_2''$, i.e. if $\{\varrho_A\}_{\mathscr{K}_2''}$ is stable and $\varrho_A \supset \nu_A$ for all $A \in O(\mathscr{K}_2'')$, then $\varrho_A = \nu_A$ for all A.*

Proof. Let there exist such a stable system $\{\varrho_A\}_{\mathscr{K}_2''}$ described in the theorem which is different from $\{\nu_A\}_{\mathscr{K}_2''}$. Let B be such an object of $\mathscr{K}_2''$ that $\nu_B \subsetneqq \varrho_B$. Hence such $X, Y \subset B$ exist that $X \varrho_B Y$ and X non ν_B Y. So $x_1 \in X - Y$ exists such that $y \in Y - X \Rightarrow y$ non $\geqq x_1$. As Y non ν_B X, $y_1 \in Y - X$ exists for which $x_1 \parallel y_1$. Add to the ordering $\leqq$ of $B \langle y_1, x_1 \rangle$ and construct the transitive hull to this new relation. We get an ordering of $B \leqq_1$ such that in $B' = (B, \leqq_1)$ X non $\nu_{B'}$ Y, as there exists for $x_1 \in X - Y$ no $y \in Y - X$ with $x_1 \leqq_1 y$. Simultaneously $X \varrho_{B'} Y$. So after finite number of steps we get an ordered set $B^{(n)} = (B, \leqq_n)$ with X non $\nu_{B^{(n)}}$ Y, $X \varrho_{B^{(n)}} Y$ and with no $y \in Y - X$ for which $x_1 \parallel y$ in $\leqq_n$ and this is imposible.

8. AUTOMORPHISM CLASS GROUPS OF SUBCATEGORIES IN $\mathscr{K}_2$

8.1. The most of fundamental definitions of this section are taken from [9] p. 28. A slightly different point of view see [18] p. 61. Let us recall that it is necessary in 8.19 to restrict oneself to the small categories when one wants to remain in Gödel—Bernays theory.

8.2. Let $\mathscr{K}$ be a category. Say that a functor F from the category $\mathscr{K}$ to the same category $\mathscr{K}$ is almost identical if F is naturally equivalent to the identical functor of $\mathscr{K}$ to $\mathscr{K}$.

8.3. *Let F be an almost identical functor of $\mathscr{K}$ to $\mathscr{K}$. Then F induces a one-to -one mapping of $H(x, y)$ onto $H((x) F, (y) F)$ for all $x, y \in O(\mathscr{K})$.*

Proof. Let φ_x: $x \to (x) F$ be an inversible mapping corresponding to a natural equivalence between identity functor and F. Let $\alpha \in H(x, y)$. Then $\alpha\varphi_y = \varphi_x[(\alpha) F]$. Hence $\alpha = \varphi_x[(\alpha) F] \varphi_y^{-1}$. So F is one-to -one on $H(x, y)$. If $\beta \in H((x) F, (y) F)$, it is $\varphi_x \beta \varphi_y^{-1} \in H(x, y)$ and $(\varphi_x \beta \varphi_y^{-1}) F = \beta$. So F induces a mapping of $H(x, y)$ onto $H((x) F, (y) F)$.

8.4. A functor F mapping $\mathscr{K}$ into $\mathscr{K}$ will be called an equivalence, if there exists a functor G mapping $\mathscr{K}$ into $\mathscr{K}$ such that FG and GF are almost identical functors.

8.5. *Let F be an equivalence. Then F induces a one-to-one mapping of $H(x, y)$ onto $H((x) F, (y) F)$ for all $x, y \in O(\mathscr{K})$.*

Proof follows from the fact that FG and GF induce one-to-one mapping of $H(x, y)$ onto $H((x)FG, (y)FG)$ or $H((x)GF, (y)GF)$, respectively.

8.6. *If α is an inversible map in $H(x, y)$, $(\alpha)F$ is inversible in $H((x)F, (y)F)$.*

Proof. If α is inversible, then $\alpha\alpha^{-1} = 1_x$, $\alpha^{-1}\alpha = 1_y$. Hence $(\alpha)F(\alpha^{-1})F = 1_{(x)F}$, $(\alpha^{-1})F(\alpha)F = 1_{(y)F}$.

8.7. We say that $\alpha \in M(\mathscr{K})$, $\alpha: x \to y$ is one-pointed morphism if for all $c \in O(\mathscr{K})$ and $\beta, \gamma: c \to x$ it is $\beta\alpha = \gamma\alpha$ (see [15] p. 254).

8.8. *Let $\alpha^*: (x)F \to (y)F$ be one-pointed, F an equivalence of a category $\mathscr{K}$. Let $\alpha: x \to y$ satisfy $(\alpha)F = \alpha^*$. Then α is one-pointed.*

Proof. Let $\gamma, \beta: c \to x$, $\gamma\alpha \neq \beta\alpha$. Then $(\gamma)F(\alpha)F \neq (\beta)F(\alpha)F$, a contradiction.

8.9. Let $\mathscr{K}^*$ be a full subcategory of $\mathscr{K}_2$, F an equivalence on $\mathscr{K}^*$, then α^* will be occasionally used instead of $(\alpha)F$.

8.10. *A morphism $\alpha: A \to B$, $A \neq \emptyset$ of $\mathscr{K}^*$ is one pointed exactly when it maps A on one single element of B.*

Proof. Let α be one-pointed, $x \in A$ and define φ_x as $(z)\varphi_x = x$ for all $z \in A$. Then $(z)\varphi_{x_1}\alpha = (x_1)\alpha$, $(z)\varphi_{x_2}\alpha = (x_2)\alpha$. As $\varphi_{x_1}\alpha = \varphi_{x_1}\alpha$ it is $(x_1)\alpha = (x_2)\alpha$. The converse is clear.

8.11. *Let $\alpha \in M(K^*)$ be one-pointed, F an equivalence. Then $(\alpha)F$ is one-pointed.*

Proof. Let $\alpha: A \to B$. If $A = \emptyset$, then α is zero map $[\emptyset, B, \emptyset]$. Then $(A)F = \emptyset$, so $(\alpha)F = [\emptyset, (B)F, \emptyset]$, which is clearly one-pointed. Let $A \neq \emptyset$. Then $(A)F \neq \emptyset$, too. Admit $(\alpha)F$ is not one-pointed. There exists $a_1, a_2 \in (A)F$ so that $(a_1)[(\alpha)F] \neq (a_2)[(\alpha)F]$. Let $\alpha_1^*, \alpha_2^*: (A)F \to (A)F$, for which $[(A)F]\alpha_1^* = \{a_1\}$, $[(A)F]\alpha_2^* = \{a_2\}$. Clearly $\alpha_1^*(\alpha)F \neq \alpha_2^*(\alpha)F$. α_1 and α_2 exist such that $\alpha_1, \alpha_2: A \to A$ and $(\alpha_1)F = \alpha_1^*$, $(\alpha_2)F = \alpha_2^*$ (so the notation agrees with 8.9). It is $\alpha_1\alpha = \alpha_2\alpha$ as α is one-pointed. So $\alpha_1^*(\alpha)F = \alpha_2^*(\alpha)F$, a contradiction.

Let A be a non-void object of $\mathscr{K}^*$. In the sequel we fix one of such A. Let $\alpha: A \to B$ be one-pointed, let $|\alpha|$ denote the common value of α (so $|\alpha| \in B$). Let $\varphi_{F,B}$ be a mapping $B \to (B)F$ given by the formula

$$(|\alpha|)\varphi_{F,B} = |(\alpha)F|.$$

From 8.5., 8.8., 8.11. we get

8.12. *$\varphi_{F,B}$ is one-to-one map of B onto $(B)F$.*

Instead of $|(\alpha)F|$ $|\alpha|^*$ will be used, i.e. in a more general way $x^* = (x)\varphi_{F,B}$ for $x \in B$.

8.13. *Let $\beta: X \to Y$. Let $(x)\beta = y$. Then $(x^*)[(\beta)F] = y^*$.*

Proof. Let $\alpha: A \to X$ be one-pointed, $|\alpha| = x$. Then $(\alpha)F.(\beta)F$ is one pointed with the value $|\alpha\beta|^*$. Simultaneously we have $(x^*)[(\beta)F] = (|(\alpha)F|)[(\beta)F] = (z)[(\alpha)F(\beta)F]$ where z is quite arbitrary element of $(A)F$. Further on $(z)[(\alpha)F.(\beta)F] = (z)[(\alpha\beta)F] = |\alpha\beta|^*$. On the other hand $|\alpha\beta| = y$.

8.14. *Let $a, b \in A$, $a < b$. Let $a^* < b^*$, $x, y \in X$. Then $x^* < y^*$ exactly when $x < y$.*

Proof. Let $x < y$. There exists a mapping $\beta : A \to X$ such that $(A)\,\beta = \{x, y\}$ and $(a)\,\beta = x$, $(b)\,\beta = y$. Then $(a^*)\,\beta^* = x^*$, $(b^*)\,\beta^* = y^*$ (it follows from 8.13), so $x^* < y^*$.

Let $x^* < y^*$. There exists $\beta: A \to X$ so that $(a^*)\,\beta^* = x^*$, $(b^*)\,\beta^* = y^*$ and $[(A)\,F]\,\beta^* = \{x^*, y^*\}$. Again, from 8.13 we get $(a)\,\beta = x$, $(b)\,\beta = y$, so $x < y$.

8.15. *Let $a, b \in >$, $a < b$. Let $a^* \, A \, b^*$. Let $x, y \in X \in O(\mathscr{K}^*)$. Then $x < y$ exactly when $x^* > y^*$.*

The proof is similar to that of 8.14.

8.16. *Let $a, b \in A$, $a < b$. Then a^*, b^* are comparable.*

Proof. Admit $a^* \parallel b^*$ $\{a^*\}$, $\{b^*\}$ are components in $(A)\,F$. Then γ: $A \to A$ exists such that $(a^*)\,\gamma^* = b^*$, $(b^*)\,\gamma^* = a^*$. Then $(a)\,\gamma = b$, $(b)\,\gamma = a$, which is impossible.

8.14, 8.15, 8.16 imply.

8.17. *Do not let A be an antichain. Then $\varphi_{F,B}$ is simultaneously for all B a relation-isomorphism or antiisomorphism.*

8.18. *Do not let B be an antichain, F, G two equivalences of $\mathscr{K}^*$ such that $\varphi_{F,B}$ and $\varphi_{G,B}$ are simultaneously relation-isomorphisms or antiisomorph isms. Then F and G are naturally equivalent.*

Proof. Define for every $B \in O(\mathscr{K}^*)$ $(B)\,\psi$: $(B)\,F \to (B)\,G$ as follows. Let $x \in (B)\,F$. According to 8.12 $\varphi_{F,B}$ is one-to-one mapping of B onto $(B)\,F$. Let $(x')\,\varphi_{F,B} = x$. Put $(x)\,[(B)\,\psi] = (x')\,\varphi_{G,B}$. $(B)\,\psi$ is by assumption on F and G a relation-isomorphism. It can be easily see that the diagram

$$\begin{array}{ccc} (B)\,F & \xrightarrow{(\alpha)\,F} & (C)\,F \\ {\scriptstyle (B)\,\psi}\downarrow & & \downarrow{\scriptstyle (C)\,\psi} \\ (B)\,G & \xrightarrow{(\alpha)\,G} & (C)\,G \end{array}$$

commutes for all α: $B \to C$

8.19. Let $\mathscr{K}$ be a category, $\mathfrak{K}$ a system of classes of naturally equivalent equivalences. In $\mathscr{K}$ there can be defined a multiplication by means of composition of representatives. In regard to this multiplication $\mathfrak{K}$ satisfies the axioms for group multiplication and it is called automorphism class group (see [9], p. 28).

8.20. *Every full subcategory $\mathscr{K}^*$ of $\mathscr{K}_2$ has the trivial or the two-point automorphism class group.*

Proof. If $\mathscr{K}^*$ consists only of $\emptyset$, the assertion is clear. So let $\mathscr{K}^*$ contain a non-void set. If $\mathscr{K}^*$ consists of antichains, then by 8.12 and 8.13 every two equivalences are naturally equivalent, so $\mathfrak{K}^*$ consists of one class. If $\mathscr{K}^*$ contains an object, which is not an antichain, we get the assertion by 8.18.

From 8.20 following theorem (see [9] p. 30) can be proved.

8.21. *The automorphism class group* $\mathfrak{K}_2$ possesses two elements.

Proof. Let F be the identity functor on $\mathscr{K}_2$, G the functor defined as follows $(A, \leqq) G = (A, \prec)$, where $s \leqq b = a \succ b$, $(\alpha) G = \alpha$. F and G are equivalences in $\mathscr{K}_2$, which are not naturally equivalent.

One can see from following examples that in 8.20. "full" cannot be omitted and "the category of small categories" cannot be put in the place of $\mathscr{K}_2$ (compare with the considerations in [9] p. 29).

8.22. Let K_1 and K_2 be two different objects from $\mathscr{K}_2$. Let $H(K_j, K_j)$ $(j = 1, 2)$ consists only of the identity map, $H(K_1, K_2) = H_{\mathscr{K}_2}(K_1, K_2)$, $H(K_2, K_1) = \emptyset$. The obtained category will be denoted by $\mathscr{K}$. So $\mathscr{K}$ is a subcategory in $\mathscr{K}_2$. Let π be an arbitrary permutation of $H(K_1, K_2)$. Then the functor F_π, for which $(K_j) F_\pi = K_j$ and $(\alpha) F_\pi = (\alpha) \pi$ for $\alpha \in H_{\mathscr{K}}(K_1, K_2)$ is an equivalence on $\mathscr{K}$, as $F_\pi \cdot F_{\pi-1}$ and $F_{\pi-1} F_\pi$ are equal to the identity functor. Simultaneously, functors belonging to the different permutation are not equivalent as on K_1 and K_2 in $\mathscr{K}$ there exist only the identity morphisms. So the automorphism class group for $\mathscr{K}$ is isomorphic to the permutation group on the set $H_{\mathscr{K}}(K_1, K_2)$.

8.23. Let $\mathscr{K}$ be a category with one object a and with morphisms $H_{\mathscr{K}}(a, a) = F$, where F is a free non cyclic group. Let $\{\xi_j\}_{j \in J}$ be some complete system of free generators in F, ξ', ξ'' two of them. Define $(\alpha) F_\xi = (\xi')^{-1} \alpha \xi'$, $(\alpha) F_{\xi''} = (\xi'')^{-1} \alpha \xi''$ for $\alpha \in F$, $(a) F_{\xi'} = (a) F_{\xi''} = a$. $F_{\xi'}$, $F_{\xi''}$ are clearly equivalences. Let us admit that φ is a natural equivalence carrying $F_{\xi'}$ into $F_{\xi''}$, i.e. for every $\alpha \in F$ it holds $(\xi')^{-1} \alpha \xi' \varphi = \varphi (\xi'')^{-1} \alpha \xi''$. Let $\alpha = \xi'$. Then $\varphi (\xi'')^{-1} \xi' \xi'' = \xi' \varphi$, so $\xi' \xi'' = \xi'' \varphi^{-1} \xi' \varphi$. Let $\varphi = \xi_1^{s_1} \ldots \xi_{s_t}^{s_t}$ be the noncancelable form for φ. Then $\xi'' \xi_t^{-s_t} \ldots \xi_1^{-s_1} \xi' \xi_t^{s_1} \ldots \xi_t^{s_t} = \xi' \xi''$ hence $\xi_1 = \xi'$. In the same way we get $\xi_2 = \ldots = \xi_t = \xi'$, so $t = 0$ (i.e. φ is a unit for F) or $t = 1$. In both cases $\xi' \xi'' = \xi'' \xi'$, which is impossible. So card $\mathfrak{K} \geqq$ card J.

8.24. Now, we construct a category $\mathscr{K}$ with two-element automorphism class group which cannot be embedded in $\mathscr{K}_2$ as a full subcategory.

$O(\mathscr{K}) = \{a, b\}$, $a \neq b$, $H(a, a) = \{1_a\}$, $H(b, b) = \{1_b\}$, $H(a, b) = \{\varphi_1, \varphi_2\}$, $\varphi_1 \neq \varphi_2$, $H(b, a) = \emptyset$. The identity functor is the only equivalence in $\mathscr{K}$. No full embedding of $\mathscr{K}$ in $\mathscr{K}_2$ exists, as $X, Y \in O(\mathscr{K}_2)$, $X \neq \emptyset \neq Y \Rightarrow H_{\mathscr{K}_2}(X, Y) \neq \emptyset$.

9. CATEGORIES WITH THE ORDERED SETS OF MORPHISMS

9.1. A category $\mathscr{K}$ is said to have the ordered sets of morphisms, if

1. $A, B \in O(\mathscr{K}) \Rightarrow H_{\mathscr{K}}(A, B) \in O(\mathscr{K}_2)$.

2. $\alpha \leqq \beta \Rightarrow \alpha\gamma \leqq \beta\gamma$, if the compositions are defined.

3. $\alpha \leqq \beta \Rightarrow \gamma\alpha \leqq \gamma\beta$, if the compositions are defined.

Briefly $\mathscr{K}$ is called an o- category.

9.2. A subcategory $\mathscr{K}'$ in an o-category $\mathscr{K}$, which is provided with the restrictions of orders of $\mathscr{K}$ is said to be a good subcategory in $\mathscr{K}$.

9.3. *Let* $A, B \in O(\mathscr{K}_2)$. *Order* $H(A, B)$ *as the cardinal power, i.e.* $\alpha, \beta \in H(A, B)$, $\alpha \leqq \beta \equiv (x)\,\alpha \leqq (x)\,\beta$ *for all* $x \in A$. $\mathscr{K}_2$ *is then an o-category.*

9.4. Let $\mathscr{K}$ and $\mathscr{K}'$ be o-categories. Let F be isomorphic mapping of the category $\mathscr{K}$ onto the category $\mathscr{K}'$, for which $\alpha \leqq \beta \Leftrightarrow (\alpha)\,F \leqq (\beta)\,F$. Then F is called o-isomorphic mapping. $\mathscr{K}'$ is said to be o-isomorphic to $\mathscr{K}$. This relation is clearly symmetrical.

9.5. *Every small o-category* $\mathscr{K}$ *is o-isomorphic to a certain good subcategory of* $\mathscr{K}_2$. *There exists such an o-isomorphic mapping that the monomorphisms of* $\mathscr{K}$ *are carried into one-to-one mappings.*

The proof runs as in Eilenberg—MacLane theorem. Let $(x)\,T = \sum_{u \in O(\mathscr{K})} H(u, x)$ for $x \in O(\mathscr{K})$ (Σ means here the cardinal sum of ordered sets). Let $\alpha : x \to y$. $(\alpha)\,T$ is defined as follows; for β: $u \to x$ it is $(\beta)\,[(\alpha)\,T] = \beta\alpha$.

a) $(\alpha)\,T$ is an isotone mapping of $(x)\,T$ into $(y)\,T$.

Proof. From $\gamma \leqq \beta$ in $H(x, y)$ we get $\gamma\alpha \leqq \beta\alpha \Rightarrow (\gamma)\,[(\alpha)\,T] \leqq (\beta)\,[(\alpha)\,T]$.

b) Clearly $(1_x)\,T = 1_{(x)T}$

c) $(\alpha)\,T\,(\beta)\,T = (\beta\alpha)\,T$.

Proof. Let α: $x \to y$, β: $y \to z$, $\gamma \in (x)\,T$. $\gamma \in H(u, x) \Rightarrow (\gamma)\,[(\alpha)\,T] = \gamma\alpha$. $(\gamma)\,[(\alpha)\,T(\beta)\,T] = (\gamma\alpha)\,[(\beta)\,T] = \gamma\alpha\beta = (\gamma)\,[(\alpha\beta)\,T]$.

d) Let α, $\beta \in H(x, y)$, $\alpha \neq \beta$. Then $(\alpha)\,T \neq (\beta)\,T$.

Proof. $(1_x)[(\alpha)\,T] = 1_x\alpha = \alpha \neq \beta = 1_x\beta = (1_x)\,[(\beta]\,T]$.

e) Let $(\alpha)\,T$, $(\beta)\,T \in H_{\mathscr{K}_2}[(x)\,T, (y)\,T]$. Then

$$(\alpha)\,T \leqq (\beta)\,T \equiv \alpha \leqq \beta.$$

Proof. 1. Let $\alpha \leqq \beta$. Then for $\gamma : u \to x$ we have $\gamma\alpha \leqq \gamma\beta$. So $(\gamma)\,[(\alpha)\,T] \leqq (\gamma)\,[(\beta)\,T]$ for $\gamma \in (x)\,T$.

2. Let $(\alpha)\,T \leqq (\beta)\,T$. Then $\gamma\alpha \leqq \gamma\beta$ for all $\gamma \in (x)\,T$. So, in particular $1_x\alpha = \alpha \leqq \beta = 1_x\beta$.

f) Let α be a monomorphism in $\mathscr{K}$. Then $(\alpha)\,T$ is a monomorphism in $\mathscr{K}_2$.

Clear.

So the proof of 9.5 is finished.

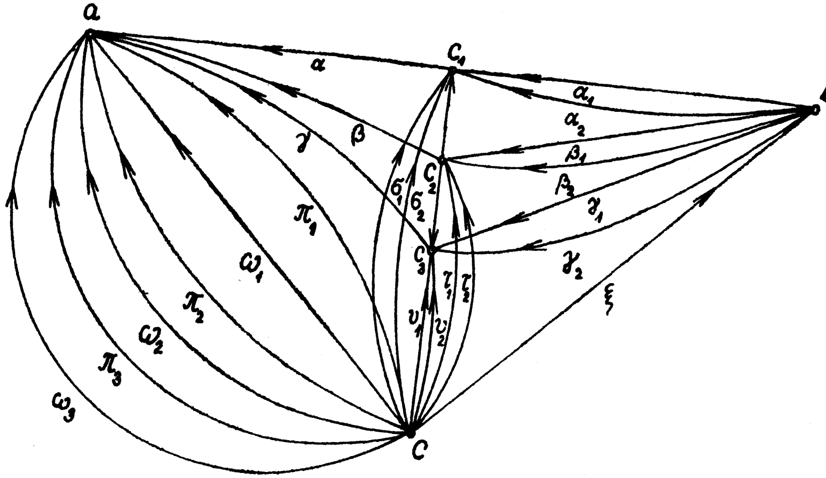
9.6. To the foregoing theorem let us add following notes.

1. There exists an o-category which is not isomorphic to any subcategory in $\mathscr{K}_2$. It can be seen from the fact that every category can be

regarded as o-category ($H(a, b)$ are antichains) and there exist categories non embeddable in the category of the sets ([9], p. 108, [7]).

2. It is an open problem, what is the analogon for o-categories of the theorem 1 of [23], p. 14. To this problem see also [25].

3. The property to be o-category is not amalgamic (definition of an amalgamic property see [24] p. 148, 1. Metadefinition). Let us prove it by an example. Let a category $\mathcal{K}^*$ possess objects a, c, c_1, c_2, c_3, $\mathcal{K}^{**}$ objects b, c, c_1, c_2, c_3 and let morphisms be as on figure (identities are not displayed) $\mu : c_2 \to c_1$, $\nu : c_2 \to c_3$.



Let the following relations hold: $\alpha_1 = \beta_1\mu$, $\alpha_2 = \beta_2\mu$. $\gamma_1 = \beta_1\nu$, $\gamma_2 = \beta_2\nu$, $\mu\alpha = \beta$, $\nu\gamma = \beta$, $\sigma_1 = \xi\alpha_1$, $\sigma_2 = \xi\alpha_2$, $\tau_1 = \xi\beta_1$, $\tau_2 = \xi\beta_2$, $\upsilon_1 = \xi\gamma_1$, $\upsilon_2 = \xi\gamma_2$, $\tau_1\mu = \sigma_1$, $\tau_2\mu = \sigma_2$, $\tau_1\nu = \upsilon_1$, $\tau_2\nu = \upsilon_2$, $\omega_1 = \sigma_1\alpha$, $\pi_1 = \sigma_2\alpha$, $\omega_2 = \tau_1\beta$, $\pi_2 = \tau_2\beta$, $\omega_3 = \upsilon_1\gamma$, $\pi_3 = \upsilon_2\gamma$.

Let $\alpha_1 > \alpha_2$, $\gamma_2 > \gamma_1$, $\beta_1 \parallel \beta_2$. So also $\sigma_1 > \sigma_2$, $\upsilon_2 > \upsilon_1$, $\omega_1 > \pi_1$, $\omega_3 < \pi_3$. Suppose now that this amalgam (consisting of the categories $\mathcal{K}^*$ and $\mathcal{K}^{**}$ with the amalgamated subcategory possesing as objects c, c_1, c_2, c_3, and morphisms μ, ν, σ_1, σ_2, τ_1, τ_2, υ_1, υ_2) is contained as a good subcategory in an o-category $\mathcal{K}$. As $\omega_2 \neq \pi_2$ and $\omega_2 = \tau_1\beta = \xi\beta_1\beta$, $\pi_2 = \tau_2\beta = \xi\beta_2\beta$ one gets in $\mathcal{K}$ $\beta_1\beta \neq \beta_2\beta$. Put $\delta_1 = \beta_1\beta$, $\delta_2 = \beta_2\beta$. It is $\delta_1 = \gamma_1\gamma = \beta_1\beta = \alpha_1\alpha$, $\delta_2 = \gamma_2\gamma = \beta_2\beta = \alpha_2\alpha$.

We have implications

$$\delta_1 = \alpha_1\alpha, \qquad \delta_2 = \alpha_2\alpha, \qquad \alpha_1 > \alpha_2 \Rightarrow \delta_1 > \delta_2,$$
$$\delta_1 = \gamma_1\gamma, \qquad \delta_2 = \gamma_2\gamma, \qquad \gamma_1 < \gamma_2 \Rightarrow \delta_1 < \delta_2,$$

which contradicts one the other.

4. In consequence of 3. the existence of a universal o-category in the sence of [24] p. 143 remains to be an open problem.

9.7. Let $\mathcal{K}$ be a category, in which for all $x, y \in O(\mathcal{K})$ $H(x, y)$ is an ordered set. Such category will be called a *wo*-category.

9.8. An example of a *wo*-category, which is not an o-category (even neither 1. nor 2. is valid) is the category $\mathcal{K}_2^+$ defined in the following way: $O(\mathcal{K}_2^+) = O(\mathcal{K}_2)$, set of morphisms $H_{\mathcal{K}_2^+}(X, Y)$ is the set of all mappings of the set X into Y ordered as $\exp_Y X$ (see [22]).

9.9. *For every small wo-category there exists one-to-one mapping T' of $O(\mathcal{K})$ into $O(\mathcal{K}_2^+)$ and $M(\mathcal{K})$* into *$M(\mathcal{K}_2^+)$ such that*

1'. $(\alpha\beta)\, T' = (\alpha)\, T'\, (\beta)\, T'$

2'. $\alpha \leqq \beta \Leftrightarrow (\alpha)\, T' \leqq (\beta)\, T'$.

(*1' is fullfilled, whenever $\alpha\beta$ is defined*).

Proof. Let $(x)\, T$, $(\alpha)\, T$ be defined as in 9.5. Let for every $x \in O(\mathcal{K})$ $(x)\, T' = \{x\} \oplus (x)\, T$, $(x)\, [(\alpha)\, T'] = \alpha$ for all $\alpha \in H(x, y)$, $(\beta)\, T' = (\beta)\, T$ for $\beta \in (x)\, T$. From the definition of $\exp_Y X$ one gets immediately 2'. Let us prove 1'. for $x\,.\,(x)\, [(\alpha\beta)\, T'] = \alpha\beta$, $(x)\, [(\alpha)\, T'(\beta)\, T'] = (\alpha)\, [(\beta)\, T'] = \alpha\beta$. For $\gamma \neq x$ 1'. is clear.

9.10. One cannot demand in general $(1_x)\, T' = 1_{(x)T'}$ in 9.9. Let us give an example.

Let $\mathcal{K}$ be a category with one object a and $H(a, a)$ be a cyclic two element group $\{1_a, \alpha\}$. Let e.g. $1 > \alpha$. Admit that $A \in O(\mathcal{K}_2^+)$ and morphism $\alpha_1 \in H_{\mathcal{K}_2^+}(A, A)$ exist so that the subcategory in $\mathcal{K}_2^+$ with the object A and the set of morphisms $\{1_A, \alpha_1\}$ is isomorphic to $\mathcal{K}$ even as for order, so $1_A > \alpha_1$. Let z be a minimal element of A, for which $z \neq (z)\, \alpha_1$. Then $z > (z)\, \alpha_1$, α_1 is one-to-one map onto A, as $\alpha_1^2 = 1_A$. So $(z)\, \alpha_1\alpha_1 \neq (z)\, \alpha_1$ which contradicts the minimality of z.

10. THE PRODUCTS IN $\mathcal{K}_6$

Now, we shall be interested in studying of the category $\mathcal{K}_6$, which is the only category in our considerations possessing a zero object. A zero morphism will be denoted by ω. It is routine to check the existence of kernels and cokernels of morphisms of $\mathcal{K}_6$.

10.1. *Let (N, n, ν), $(P, p, \pi) \in O(\mathcal{K}_6)$ Let $\mu\colon N \to P$ be a monomorphism. Then μ is a normal monomorphism (see [16]) iff it is a relation-isomorphic mapping of N into P and $(N)\, \mu$ is a convex subset in P ($X \subset P$ is convex, if $x, z \in X$ and $x < y < z$ implies $y \in X$).*

Proof. Let μ possess the demanded properties. Let $\mu\gamma = \omega$, $\gamma\colon P \to S$. Let s be the distinguished element of S. Then $(N)\, \mu \subset (s)\, \gamma^{-1}$. Let $x \in P - (N)\, \mu$.

Admit $z \in (N)\, \mu$ exists such that $x > z$. Then $v \parallel x$ or $v < x$ for all

$v \in (N)\,\mu$ (consequence of the convexity of $(N)\,\mu$ in P). Let $R = \{x^*, y^*\}$, $x^* > y^*$, y^* will be the distinguished element of R. Let γ_1: $P \to R$ be defined as follows: $(y)\,\gamma_1 = x^*$ for $y \geqq x$, $(y)\,\gamma_1 = y^*$ otherwise. Let α: $K \to P$ be a morphism for which $\alpha\beta = \omega$ for all morphisms with $\mu\beta = \omega$. As $\mu\gamma_1 = \omega$, $\alpha\gamma_1 = \omega$, too. The procedure is similar also in the case when for $x \in P - (N)\,\mu$ there exists in $(N)\,\mu$ an element greater then x or all elements of $(N)\,\mu$ are incomparable with x.

If we notice that x non $\in (y^*)\,\gamma_1^{-1}$, we get $(K)\,\alpha \subset (N)\,\mu$. As μ is a relation-isomorphic mapping, it induces an inversible mapping μ_1 of N onto $(N)\,\mu$ (p being the distinguished element of $(N)\,\mu$). Then $\alpha = \alpha_1\mu_1^{-1}\mu$, where α_1 is the map of K onto $(N)\,\mu$ induced by the mapping α So $\alpha = \alpha'\mu$ for $\alpha' = \alpha_1\mu_1^{-1}$.

On the contrary, let μ be a normal monomorphism. Admit that $(N)\,\mu$ is not convex in P. So $x \in P - (N)\,\mu$ exists such that $(x)\,\gamma = s$ for all γ: $P \to (S, s, \sigma)$ for which $\mu\gamma = \omega$. Let P' be the convex hull of $(N)\,\mu$ in P (i.e. the least convex subset in P containing $(N)\,\mu$). Let α be the mapping of P' in P induced by the identity $1_{P'}$. Then $\alpha\gamma = \omega$ and α cannot be written as $\alpha'\mu$. So $P' = (N)\,\mu$. Let μ_1 be the induced mapping of N onto P' induced by μ. Then $1_{P'} = \alpha'\mu_1$ and so μ_1 is an inversible mapping. So μ is an relation-isomorphic mapping of N into P.

10.2. In the sequel following commonly known proposition will be used without any reference.

Let α: $N \to P$, $x \in P$. Then $(x)\,\alpha^{-1}$ is a convex set in N.

10.3. Let R be a decomposition of $N \in O(\mathscr{K}_6)$, elements of which are one-element subsets with one possible exception and this exceptional element, when exists, is a convex subset of N. In R one defines the relation ϱ in an usual way, i.e. $X\varrho Y \equiv$ there exist $x \in X$, $y \in Y$ so that $x \leqq y$. The transitive hull of ϱ will be noted as $\leqq$ and it is clearly an order of R. The distinguished element of R will be defined as that containing the distinguished elément of N.

10.4. *μ: $N \twoheadrightarrow (P, p, \pi)$ is a normal epimorphism, iff $x \in P$, $x \neq p \Rightarrow$ $\Rightarrow (x)\,\mu^{-1}$ is one-pointed and the mapping $\bar{\mu}$ of the decomposition R belonging to μ is a relation-isomorphism of R on P (for order in R see* 10.3).

Proof. Let μ have the demanded properties. Let $\alpha \in M(\mathscr{K}_6)$, $\alpha : N \to U$ and $\gamma\mu = \omega \Rightarrow \gamma\alpha = \omega$. Let $x \in (p)\,\mu^{-1}$, γ_1: $\{x, n\} \to N$ be inclusion mapping. It is $\gamma_1\mu = \omega$. So $\gamma_1\alpha = \omega$ and $(x)\,\alpha = u$. Therefore $(u)\,\alpha^{-1} \supset (p)\,\mu^{-1}$. Let T be the decomposition on the set N belonging to the mapping α. The relation $\leqq$ on T will be defined as in 10.3 for R. As α is an isotone mapping, $\leqq$ is an order. T is a covering of R, let δ be the mapping of R onto T given by incidence of elements. δ is clearly a relation-homomorphism of R onto T. $\bar{\mu}$ is by assumption relation-isomorphism of R onto P. It holds $\alpha = \mu(\bar{\mu})^{-1}\delta\alpha'$, where α' is the mapping of N onto $(N)\,\alpha$ induced by α.

On the contrary, let μ be a normal epimorphism. Let R be the decomposition on N, elements of which are $(p)\,\mu^{-1}$ and one-point sets $\{x\}$ for $x \in N - (p)\,\mu^{-1}$. Let $(p)\,\mu^{-1}$ be the distinguished element of R ordered according to 8.3. Let α be the canonical mapping of N onto R. Clearly $\alpha \in M(\mathscr{K}_6)$. If $\xi\mu = \omega$, so $\xi\alpha = \omega$, too. Hence $\alpha = \mu\alpha'$. Let $x, y \in N - (p)\,\mu^{-1}$, $x \neq y$. Then $(x)\,\alpha \neq (y)\,\alpha$ and so $(x)\,\mu \neq (y)\,\mu$. That proves that the decomposition belonging to μ possesses the demanded properties. Let $(X)\,\bar{\mu} \geqq (Y)\,\bar{\mu}$, $x \in X$, $y \in Y$. Then $(x)\,\mu = (X)\,\bar{\mu} \geqq (Y)\,\bar{\mu} = (y)\,\mu$, i.e. $(x)\,\mu\alpha' \geqq (y)\,\mu\alpha'$, i.e. $X = (x)\,\alpha \geqq (y)\,\alpha = Y$. So $\bar{\mu}$ is a relation-isomorphism.

10.5. By [5] a regular product is defined in such a way:

Let $\mathscr{K}$ be a category with zero morphisms. An object a is called a regular product of objects a_j, $j \in J \neq \emptyset$, if for every $j \in J$ $\sigma_j : a_j \to a$; π_j: $a \to a_j$ are given and it holds:

I. $\sigma_j\pi_j = 1_{a_j}$, $\sigma_j\pi_{j'} = \omega$ for $j \neq j'$.

II. For each $b \in O(\mathscr{K})$ and $\alpha, \beta \in H_{\mathscr{K}}(a, b)$ the equalities $\sigma_j\alpha = \sigma_j\beta$ for all $j \in J$ imply $\alpha = \beta$. Put $a = \prod^R_{j \in J} a_j$.

Let us remark that (a) from [5] is not satisfied in $\mathscr{K}_6$ as it can be seen from 10.4. (i.e. there exists a morphism not possessing a normal image).

Following theorem is valid.

10.6. $\prod^R_{j \in J} A_j = \sum^*_{j \in J} A_j$($\Sigma^*$ means a free join in $\mathscr{K}_6$).

Proof. Let $C = \prod^R_{j \in J} A_j$ be a regular product of objects $A_j \in O(\mathscr{K}_6)$, σ_j, π_j being the corresponding morphisms. Let $T = \bigcup_{j \in J} (A_j)\,\sigma_j$. It is $T \subset C$. Admit $x \in C - T$. Put $T_1 = T \cup \{x\}$. Construct T_2 from T_1 by taking an embedded antichain $\{y, z\}$ in the place of x. The distinguished element of T_2 be that of C. It belongs to T. Let $\varphi_1(\varphi_2)$ be the mapping of T_1 in T_2, which is the identity on T and $(x)\,\varphi_1 = y$ $[(x)\,\varphi_2 = z]$. It is $\varphi_1 \neq \varphi_2$ and $\sigma_j\varphi_1 = \sigma_j\varphi_2$ for $j \in J$. So x does not exist, in other words, $C = T$. Let $j_1, j_2 \in J$, $j_1 \neq j_2$. Let c be the distinguished element of C. Let $x \in (A_{j_1})\,\sigma_{j_1} \cap (A_{j_2})\,\sigma_{j_2}$. Let $(x)\,\pi_{j_1} = x_1$, $(x)\,\pi_{j_2} = x_2$. It is $\sigma_{j_1}\pi_{j_2} = \omega$ and $\sigma_{j_1}\pi_{j_1} = 1_{A_{j_1}}$. Hence $(x_1)\,\sigma_{j_1} = x$ and x_2 is the distinguished element of A_{j_2}. From $\sigma_{j_2}\pi_{j_2} = 1_{A_{j_2}}$ it follows that σ_{j_2} is a monomorphism and $(x_2)\,\sigma_{j_2} = x$. So $x = c$. We see that $(A_j)\,\sigma_j$ have pairwise only the element c in common. As σ_j are monomorphisms, we can take $\sum^*_{j \in J} A_j$ as a carrier of C and the relation $\leqq_1$ on C is greater or equal to the relation of $\sum^*_{j \in J} A_j$ (this relation will be denoted as $\leqq$). σ_j is injection in the free join. The equality $\sigma_j\pi_j = 1_{A_j}$ implies that σ_j is a relation-isomorphism of A_j onto $(A_j)\,\sigma_j$, for $j \neq j'$ $\sigma_j\pi_{j'} =$

$= \omega \Rightarrow (A_j)\, \sigma_j \pi_{j'} = \{a_{j'}\}$, where $a_{j'}$ is the distinguished element of $A_{j'}$.
Let $x \in A_{j_1}$, $y \in A_{j_2}$, $(x)\, \sigma_{j_1} \leqq_1 (y)\, \sigma_{j_2}$.

1. $j_1 \neq j_2$. Then $(x)\, \sigma_{j_1} \pi_{j_2} \leqq (y)\, \sigma_{j_2} \pi_{j_2} \Rightarrow a_{j_2} \leqq y$. Analogously we get $x \leqq a_1$. Hence $(x)\, \sigma_{j_1} \leqq (y)\, \sigma_{j_2}$.

2. Let $j_1 = j_2$. Then $x \leqq y$ in A_{j_1} and so $(x)\, \sigma_{j_1} \leqq (y)\, \sigma_{j_2}$. This implies that the relations $\leqq_1$ and $\leqq$ are equal.

10.7. The assertion 10.6 can be deduced from the results of [5], in the fact from 4.3, 4.4, 4.5., 4.6 of [5], as those results have been mostly proved under weaker assumptions as a) or b) (b): every bimorphism is inversible).

10.8. An object a of a category $\mathscr{K}$ will be called a special subdirect sum of objects a_j, $j \in J$, $J \in \emptyset$, if for each $j \in J$ the mappings σ_j: $a_j \to a$, π_j: $a \to a_j$ are given and it holds:

I′) $\sigma_j \pi_j = 1_{a_j}$, $\sigma_j \pi_{j'} = \omega$ for $j \neq j'$.

II′) If $\beta \pi_j = \gamma \pi_j$ for $b \in O(\mathscr{K})$, $\beta, \gamma \in H(b, a)$ and each $j \in J$, then $\beta = \gamma$.

We shall write $a = \sum\limits_{j \in J}^{S} a_j$.

10.9. Let $\{(A_j, a_j, \alpha_j)\}_{j \in J}$, $J \neq \emptyset$ be a system of objects in $\mathscr{K}_6$. Let C be its special subdirect sum. Similarly as in 10.7 σ_j are relation-isomorphic mappings of A_j in C. Let $x \in C$. Put $(x)\, \pi_j = x_j$. One can easily see that a one-to-one mapping φ of C into c'rtesian product $\prod\limits_{j \in J} A_j$ is defined, namely $(x)\, \varphi = (\ldots, x_j, \ldots)$. So the carrier of C can be considered to be equal to $(C)\, \varphi \subset \prod\limits_{j \in J} A_j$. The corresponding order of $(C)\, \varphi$ will be denoted as $\leqq^*$. From $\sigma_j \pi_j = 1_{A_j}$, $\sigma_j \pi_{j'} = \omega$ for $j \neq j'$ it follows for $x \in A_j$ that $(x)\, \sigma_j = (\ldots, a_{j_1}, \ldots, x, \ldots, a_{j_2}, \ldots)$, where x stands for j-coordinate of $(x)\, \sigma_j$. Further, if $x \geqq y$ in A_j, it is $(x)\, \sigma_j \geqq (y)\, \sigma_j$. So C contains all elements of the form $(\ldots, a_{j_1}, \ldots, x, \ldots, a_{j_2}, \ldots)$ for $x \in A_j$, $j \in J$ and the ordering $\leqq_2$ on this subset (for a given j) defined coordinatewise is contained in $\leqq^*$.

Define the relation ϱ on C in the following manner; $x, y \in C$, $x \varrho y$ iff there exists j such that j'-coordinate of the elements x and y, $j' \neq j$, is equal to a_j and $x_j \leqq y_j$. Let $\leqq_1$ be the transitive hull of ϱ. $\leqq_1$ is clearly an ordering of C. Evidently $x \leqq {}_1 y \Rightarrow x \leqq^* y$.

Let $x \leqq^* y$. Then $(x)\, \pi_j \leqq (y)\, \pi_j$ in A_j, i.e. $x \leqq_2 y$ ($\leqq_2$ coordinatewise order).

The foregoing properties of C in $\prod\limits_{j \in J} A_j$ are characteristic for special subdirect sum. Let C^* be now a subset in $\prod\limits_{j \in J} A_j$ with the following properties.

α) C^* contains all elements all coordinates of which with possible exception of one are equal to the distinguished elements of the corresponding A_j.

β) Let $\leqq^+$ be an ordering of C^* containing $\leqq_1$ and contained in $\leqq_2$. Then C^* is a special subdirect sum of $\{A_j\}_{j\in J}$.

Proof is evident.

So we can conclude.

10.10. *A set D is a special subdirect sum of $A_j, j \in J \neq \emptyset$, iff it is isomorphic to $C^* \subset \prod_{j\in J} A_j$ with the properties* α), β).

10.11. Now we shall describe all natural transformations between direct and free joins of two objects ($+$ denotes free join, $\times$ direct join, A, B in $A + B$ are supposed to be disjunct).

Define two functors F_1, F_2: $\mathcal{K}_6 \times \mathcal{K}_6 \to \mathcal{K}_6$ as follows:

F_1: Let $A_j, B_j \in O(\mathcal{K}_6)$, $j = 1, 2$, $\alpha : A_1 \to A_2$, β: $B_1 \to B_2$. Then $(A_1, B_1) F_1 = A_1 + B_1$, $(x)(\alpha, \beta) F_1 = (x)\alpha$ for $x \in A_1$, $x \neq a_1$; $(x)(\alpha, \beta) F_1 = (x)\beta$ for $x \in B_1$, $x \neq b_1$; if x is the distinguished element of $A_1 + B_1$ (see 6.13) then $(x)(\alpha, \beta) F_1$ is the distinguished element of $A_2 + B_2$.

F_2: Under the same assumptions as for F_1

$$(A_1, B_1) F_2 = A_1 \times B_1, \qquad (\langle x, y\rangle) [(\alpha, \beta) F_2] = \langle (x)\alpha, (x)\beta\rangle.$$

Following mappings φ and φ', φ'', φ''' are evidently natural transformations of F_1 to F_2.

$\varphi_{\langle A,B\rangle} : A + B \to A \times B$, $(x)\varphi_{\langle A,B\rangle} = \langle x, b\rangle$ for $x \in A$, $x \neq a$, $(y)\varphi_{\langle A,B\rangle} = \langle a, y\rangle$ for $y \in B$, $y \neq b$ and if c is the distinguished element of $A + B$, then $(c)\varphi_{\langle A,B\rangle} = \langle a, b\rangle$.

$\varphi'_{\langle A,B\rangle}$: $A + B \to A \times B$, $\quad (z)\varphi'_{\langle A,B\rangle} = \langle a, b\rangle$ for $z \in A + B$.

$\varphi''_{\langle A,B\rangle}$: $A + B \to A \times B$, $\quad (x)\varphi''_{\langle A,B\rangle} = \langle x, b\rangle$ for $x \in A$, $x \neq a$,
$(x)\varphi''_{\langle A,B\rangle} = \langle a, b\rangle$ otherwise.

$\varphi'''_{\langle A,B\rangle}$: $A + B \to A \times B$ $\quad (x)\varphi'''_{\langle A,B\rangle} = \langle a, x\rangle$ for $x \in B$, $x \neq b$,
$(x)\varphi'''_{\langle A,B\rangle} = \langle a, b\rangle$ otherwise.

We shall prove that no other natural transformation of F_1 to F_2 exists. Let χ be a natural transformation of F_1 to F_2. Let A_j, B_j have the above meaning. Admit that there exists $z \in A_1 + B_1$, $z \in A$ so that $(z)\chi_{\langle A_1,B_1\rangle} = \langle x, y\rangle$, $y \neq b_1$.

Let α: $A_1 \to A_1$ be the identity, β: $B_1 \to B_1$ the zero map. Then $[(z)[(\alpha, \beta) F_1]]\ \chi_{\langle A_1,B_1\rangle} = \langle x, y\rangle \neq \langle x, b_1\rangle = [(z)\chi_{\langle A_1,B_1\rangle}](\alpha, \beta) F_2$, a contradiction.

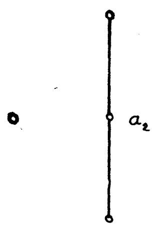
So $\chi_{\langle A_1, B_1\rangle}$ induces a mapping of the set A_1 into the set of the elements of the form $\langle x, b_1\rangle$. Let $(z)\,\chi_{\langle A_1,B_1\rangle} = \langle x, b_1\rangle$ for $z \in A_1$, $z \neq a_1$. Put $(z)\,\pi_{A_1} = x$, $(a_1)\,\pi_{A_1} = a_1$. We shall prove that π_{A_1} is independent on B_1. Let π'_{A_1} be an analogical mapping defined by means of $A_1 + B_2$. Let $\alpha = 1_{A_1}$, β: $B_1 \to B_2$ be the zero map. Then, if $z \in A_1 + B_1$, $z \in A_1$, we get $[(z)\,\chi_{\langle A_1,B_1\rangle}]\,(\alpha, \beta)\; F_2 = \langle (z)\,\pi_{A_1}, b_2\rangle$, $[(z)\,[(\alpha, \beta)\,F_1]]\,\chi_{\langle A_1,B_2\rangle} = (z)\,\chi_{\langle A_1,B_2\rangle} = \langle (z)\,\pi'_{A_1}, b_2\rangle$; hence $(z)\,\pi_{A_1} = (z)\,\pi'_{A_1}$.

Let α: $A_1 \to A_2$, $z \in A_1$. Then $(z)\,\alpha\pi_{A_2} = (z)\,\pi_{A_1}\alpha$. So π_{A_1} gives a natural transformation of the identity functor I on $\mathscr{K}_6$ to I.

Suppose there exist $A_1 \in O(\mathscr{K}_6)$ and $x \in A_1$ such that $x \neq (x)\,\pi_{A_1} \neq a_1$. Let $A = \{x, (x)\,\pi_{A_1}, a_1\}$, as for order let it be an antichain with a_1 as the distinguished element. Let α: $A \to A_1$, $(x)\,\alpha = x$, $((x)\,\pi_{A_1})\,\alpha = (a_1)\,\alpha = a_1$. If $(x)\,\pi_A = x$, then $(x)\,\pi_A\alpha = x \neq (x)\,.\,\alpha\pi_{A_1} = (x)\,\pi_{A_1}$, a contradiction. If $(x)\,\pi_A \neq x$, then $(x)\,\pi_A\alpha = a_1 \neq (x)\,\alpha\pi_{A_1} = (x)\,\pi_{A_1}$.

So always $(x)\,\pi_A = x$ or a. Suppose there exist $A_1 \in O(\mathscr{K}_6)\; x, y \in A_1$, $x \neq a_1 \neq y$ such that $(x)\,\pi_{A_1} = a_1$, $(y)\,\pi_{A_1} = y$. Let $A = \{x, y, a_1\}$ be an antichain with a_1 as the distinguished element. One immediately sees that $(y)\,\pi_A = y$. Let α: $A \to A_1$, $(x)\,\alpha = y$, $(y)\,\alpha = x$, $(a_1)\,\alpha = a_1$. Then $x = (y)\,\pi_A\alpha \neq (y)\,\alpha\pi_{A_1} = (x)\,\pi_{A_1} = a_1$.

So π_A is identity map or zero map. We shall prove that π_A is simultaneously for all A identity or zero map. Let $A' \in O(\mathscr{K}_6)$, card $A' > 1$, ω' the zero map $A' \to A'$, $\pi_{A'} = \omega'$. Let $A_1 \in O(\mathscr{K}_6)$. Let α: $A' \to A_1$ be such that $(A')\,\alpha$ is a two-point set. Then $\alpha\pi_{A_1} = \omega'\alpha = \omega$, so $\pi_{A_1} = \omega$ as $\pi_{A_1} \neq 1_{A_1}$. Let A_2 have Hasse diagram



According to the previous considerations $\pi_{A_2} = \omega$. There exists α: $A_2 \to A$ with card $A_2 = 2$, $\alpha \in M(\mathscr{K}_6)$ for all $A \in O(\mathscr{K}_6)$ with card $A \geqq 2$, As above on gets $\pi_A = \omega$.

Combining these results with similar ones for B we get the assertion.

Remark. In $\mathscr{K}_2$ and the more in $\mathscr{K}_1$ no natural transformation for functors analogous to F_1 and F_2 exists.

10.12. It is easy to see that the only natural transformations of F_2 in F_1 are induced by zero maps or the projections (to A or to B).

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