Jiří Karásek

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ON GENERAL ALGEBRAS

JIŘÍ KARÁSEK, BRNO

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The paper concerns general algebras, i.e. sets with a system of generalized operations. Some fundamental properties of homomorphisms of general algebras, of congruence relations on them and of factor-algebras are studied. As special cases of general algebras are obtained partial algebras, algebras (see [1]) and r-systems which are a generalization of sets with relations.

1. General algebras

Definition. Let A be a non-void set, K a set. A mapping

$$\boldsymbol{a} = (a_\varkappa)_{\varkappa \in K}$$

of the set K into the set A is called a *sequence of type K in A* or shortly a *K-sequence in A*. The family of all K-sequences $\boldsymbol{a}$ in A is denoted by A^K. In the case of K being finite (card $K = k$) we identify the K-sequences in A with the ordered k-tuples $(a_0, a_1, \ldots, a_{k-1})$ of elements of A. A mapping f of the family A^K of all K-sequences in A into the family 2^A of all subsets of the set A is called an *operation of type K on A* (a *K-operation on A*). Such an operation f ascribes to each sequence $\boldsymbol{a} = (a_\varkappa)_{\varkappa \in K}$ a subset

$$f(\boldsymbol{a}) = f(a_\varkappa \mid \varkappa \in K)$$

of the set A. In the case of finite type we write

$$f(\boldsymbol{a}) = f(a_0, a_1, \ldots, a_{k-1}).$$

Definition. Let I be a set, $(f_\iota)_{\iota \in I}$ a family of operations f_ι on A, $(K_\iota)_{\iota \in I}$ the system of corresponding types. Then the ordered pair $(A, (f_\iota)_{\iota \in I})$ is called a *general algebra of type* $(K_\iota)_{\iota \in I}$. Two general algebras $(A, (f_\iota)_{\iota \in I})$ $(B, (g_\iota)_{\iota \in I})$ of the same type $(K_\iota)_{\iota \in I}$ are called *similar*.

Definition. Let $(A, (f_\iota)_{\iota \in I})$, $(B, (g_\iota)_{\iota \in I})$ be similar general algebras. Let φ be a mapping of A onto B fulfilling the condition

$$\varphi[f_\iota(a_\varkappa \mid \varkappa \in K_\iota)] \subseteq g_\iota(\varphi(a_\varkappa) \mid \varkappa \in K_\iota)$$

for all $\iota \in I$ and for all K_ι-sequences $(a_\varkappa)_{\varkappa \in K_\iota}$ in A. Then φ is called a *weak homomorphism* (respectively, a *homomorphism*). A one-to-one homomorphism is called an *isomorphism*. If there exists a (weak) homomorphism (respectively, an isomorphism) of a general algebra $(A, (f_\iota)_{\iota \in I})$ onto a general algebra $(B, (g_\iota)_{\iota\in I})$ then we say $(B, (g_\iota)_{\iota\in I})$ is a *(weakly) homomorphic image* (respectively, an *isomorphic image*) of $(A, (f_\iota)_{\iota\in I})$.

Remark. The mapping inverse to an isomorphism is again an isomorphism. A one-one weak homomorphism such that its inverse mapping is also a weak homomorphism is an isomorphism.

1.1. *Let $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$, $(C, (h_\iota)_{\iota\in I})$ be similar general algebras, φ a mapping of A onto B, ψ a mapping of B onto C.*

(1) If the mappings φ and ψ are (weak) homomorphisms, then their composite $\varphi\circ\psi$) is also a (weak) homomorphism.*

(2) If the mappings φ and $\varphi\circ\psi$ are homomorphisms, then the mapping ψ is also a homomorphism.

Definition. Let $(A, (f_\iota)_{\iota\in I})$ be a general algebra, $\bar{A}$ a decomposition of A. For arbitrary $\iota \in I$ define an operation $\bar{f}_\iota$ on $\bar{A}$ as follows:
$\bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota) = \{\bar{x} \mid \bar{x} \in \bar{A}$, *there exists a K_ι-sequence $(a_\varkappa)_{\varkappa\in K_\iota}$ in A such that $a_\varkappa \in \bar{a}_\varkappa$ and $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \cap \bar{x} \neq 0\}$.*
Then the general algebra $(\bar{A}, (\bar{f}_\iota)_{\iota\in I})$ is called a *factor-algebra on* $(A, (f_\iota)_{\iota\in I})$.**)

Remark. A general algebra and its arbitrary factor-algebra are clearly similar.

Definition. Let A be a set, $\bar{A}$ a decomposition of A, $\bar{\bar{A}}$ a decomposition of $\bar{A}$. For each $\bar{\bar{a}} \in \bar{\bar{A}}$ put $\psi(\bar{\bar{a}}) \bigcup\limits_{\bar{a}\in\bar{\bar{a}}} \bar{a}$. Then the decomposition $\{\psi(\bar{\bar{a}}) \mid \bar{\bar{a}} \in \bar{\bar{A}}\}$ of A is denoted by $\bar{\bar{A}} \rhd \bar{A}$. The mapping ψ of $\bar{\bar{A}}$ onto $\bar{\bar{A}} \rhd \bar{A}$ is called *natural.*

1.2. *Let $(A, (f_\iota)_{\iota\in I})$ be a general algebra, $\bar{A}$ a decomposition of A, $\bar{\bar{A}}$ a decomposition of $\bar{A}$, ψ the natural mapping of $\bar{\bar{A}}$ onto $\bar{\bar{A}} \rhd \bar{A}$. Then ψ is an isomorphism of the factor-algebra $(\bar{\bar{A}}, (\bar{\bar{f}}_\iota)_{\iota\in I})$ onto the factor-algebra $(\bar{\bar{A}} \rhd \bar{A}, (\bar{f}_\iota^{\rhd})_{\iota\in I})$.*

Proof. The mapping ψ is a one-one mapping of $\bar{\bar{A}}$ onto $\bar{\bar{A}} \lhd \bar{A}$. Therefore it is sufficient to show that ψ is a homomorphism. Le $\iota \in I$, let $(\bar{\bar{a}}_\varkappa)_{\varkappa\in K_\iota}$ be a K_ι-sequence in $\bar{\bar{A}}$.

I. Let $\bar{y} \in \psi[\bar{\bar{f}}_\iota(\bar{\bar{a}}_\varkappa \mid \varkappa \in K_\iota)]$. Then there exists $\bar{\bar{x}} \in \bar{\bar{f}}_\iota(\bar{\bar{a}}_\varkappa \mid \varkappa \in K_\iota)$ such that $\psi(\bar{\bar{x}}) = \bar{y}$. Consequently there exists a K_ι-sequence $(\bar{a}_\varkappa)_{\varkappa\in K_\iota}$ in $\bar{A}$ such that $\bar{a}_\varkappa \in \bar{\bar{a}}_\varkappa$ for $\varkappa \in K_\iota$ and $\bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota) \cap \bar{\bar{x}} \neq 0$. Let $\bar{x} \in \bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota) \cap \bar{\bar{x}}$. Since $\bar{x} \in \bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota)$, there exists a K_ι-sequence $(a_\varkappa)_{\varkappa\in K_\iota}$ in A such that $a_\varkappa \in \bar{a}_\varkappa$ for $\varkappa \in K_\iota$ and $\bar{x} \cap f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \neq 0$. But $\bar{y} \supseteq \bar{x}$, therefore also $\bar{y} \cap f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \neq 0$. Since $a_\varkappa \in \bar{a}_\varkappa \subseteq \psi(\bar{\bar{a}}_\varkappa)$ for $\varkappa \in K_\iota$, it is evidently $\bar{y} \in \bar{f}_\iota^{\rhd}(\psi(\bar{\bar{a}}_\varkappa) \mid \varkappa \in K_\iota)$.

*) $\varphi\circ\psi$ is defined in the following way: $(\varphi\circ\psi)(x) = \psi(\varphi(x))$, where $x \in A$.
**) [5], the construction of the hull of a subset in a decomposition.

II. Conversely, let $\bar{y} \in \bar{f}_\iota^{\triangleright}(\psi(\bar{\bar{a}}_\varkappa) \mid \varkappa \in K_\iota)$. Next, let $\bar{\bar{x}} \in \bar{\bar{f}}_\iota(\bar{\bar{a}}_\varkappa \mid \varkappa \in K_\iota)$ be such that $\psi(\bar{\bar{x}}) = \bar{y}$. There exists a K_ι-sequence $(a_\varkappa)_{\varkappa \in K_\iota}$ in A such that $a_\varkappa \in \psi(\bar{\bar{a}}_\varkappa)$ for $\varkappa \in K_\iota$ and $\bar{y} \cap f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \neq 0$. But then there exists an element $\bar{x} \in \bar{\bar{x}}$ such that also $\bar{x} \cap f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \neq 0$. Let $(\bar{a}_\varkappa)_{\varkappa \in K_\iota}$ be the K_ι-sequence in $\bar{A}$ for which $a_\varkappa \in \bar{a}_\varkappa$ for all $\varkappa \in K_\iota$. Indeed, then $\bar{x} \in \bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota)$. Since also $\bar{x} \in \bar{\bar{x}}$, we have $\bar{\bar{x}} \cap \bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota) \neq 0$ and consequently $\bar{\bar{x}} \in \bar{\bar{f}}_\iota(\bar{\bar{a}}_\varkappa \mid \varkappa \in K_\iota)$, for $\bar{a}_\varkappa \in \bar{\bar{a}}_\varkappa (\varkappa \in K_\iota)$. Hence $\bar{y} \in \psi[\bar{\bar{f}}_\iota(\bar{\bar{a}}_\varkappa \mid \varkappa \in K_\iota)]$.

Definition. Let $(A, (f_\iota)_{\iota \in I})$ be a general algebra, Θ an equivalence relation on A. Θ is called a *congruence relation on A* if and only if for each $\iota \in I$ and for arbitrary K_ι-sequences $(a_\varkappa)_{\varkappa \in K_\iota}$, $(b_\varkappa)_{\varkappa \in K_\iota}$ in A with the property $a_\varkappa \Theta b_\varkappa$ for $\varkappa \in K_\iota$ there exists to each $x \in f_\iota(a_\varkappa \mid \varkappa \in K_\iota)$ such $y \in f_\iota(b_\varkappa \mid \varkappa \in K_\iota)$ and to each $y' \in f_\iota(b_\varkappa \mid \varkappa \in K_\iota)$ such $x' \in f_\iota(a_\varkappa \mid \varkappa \in K_\iota)$ that $x\Theta y$, $x'\Theta y'$.

1.3. *Let $(A, (f_\iota)_{\iota \in I})$, $(B, (g_\iota)_{\iota \in I})$ be similar general algebras, φ a mapping of A onto B. Define an equivalence relation Θ on A as follows: $a\Theta b$ if and only if $\varphi(a) = \varphi(b)$. Further define a mapping ω of the set A/Θ onto the set B by $\omega(\bar{x}) = \varphi(x)$, where $x \in \bar{x}$.*

(1) Let φ be a weak homomorphism. Then ω is a one-one weak homomorphism of $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$ onto $(B, (g_\iota)_{\iota \in I})$.

(2) Let φ be a homomorphism. Then Θ is a congruence relation on A and ω is an isomorphism of $(A/\omega, (\bar{f}_\iota)_{\iota \in I})$ onto $(B, (g_\iota)_{\iota \in I})$.

Proof. The mapping ω is clearly one-one. Let $\iota \in I$, let $(\bar{a}_\varkappa)_{\varkappa \in K_\iota}$ be a K_ι-sequence in A/Θ.

I. Let φ be a weak homomorphism, $y \in \omega[\bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota)]$. Let $\bar{x} \in \bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota)$ be such that $\omega(\bar{x}) = y$. Then there exists a K_ι-sequence $(a_\varkappa)_{\varkappa \in K_\iota}$ in A such that $a_\varkappa \in \bar{a}_\varkappa$ for $\varkappa \in K_\iota$ and $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \cap \bar{x} \neq 0$. Let $x \in f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \cap \bar{x}$. Clearly $\varphi(x) = y$, therefore $y \in \varphi[f_\iota(a_\varkappa \mid \varkappa \in K_\iota)] \subseteq g_\iota(\varphi(a_\varkappa) \mid \varkappa \in K_\iota) = g_\iota(\omega(\bar{a}_\varkappa) \mid \varkappa \in K_\iota)$ and (1) holds.

II. To prove (2) it suffices to show $g_\iota(\omega(\bar{a}_\varkappa) \mid \varkappa \in K_\iota) \subseteq \omega[\bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota)]$. Let $y \in g_\iota(\omega(\bar{a}_\varkappa) \mid \varkappa \in K_\iota)$. Let $(a_\varkappa)_{\varkappa \in K_\iota}$ be such a K_ι-sequence in A that $a_\varkappa \in \bar{a}_\varkappa$ for all $\varkappa \in K_\iota$. Then $g_\iota(\omega(\bar{a}_\varkappa) \mid \varkappa \in K_\iota) = g_\iota(\varphi(a_\varkappa) \mid \varkappa \in K_\iota) = \varphi[f_\iota(a_\varkappa \mid \varkappa \in K_\iota)]$. Therefore there exists $x \in A$ such that $\varphi(x) = y$ and $x \in f_\iota(a_\varkappa \mid \varkappa \in K_\iota)$. Let $\bar{x} \in A/\Theta$ be such that $x \in \bar{x}$. Since $x \in \bar{x} \cap f_\iota(a_\varkappa \mid \varkappa \in K_\iota)$, we have $\bar{x} \in \bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota)$, consequently $\omega(\bar{x}) = y \in \omega[\bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota)]$.

Definition. Let A be a set, $\bar{A}$ a decomposition of A. Let $a \in A$, $\bar{a} \in \bar{A}$ be such that $a \in \bar{a}$. The mapping of A onto $\bar{A}$ whose value at a is $\bar{a}$ is called *canonical*.

Definition. Let A be a set, $\bar{A}$, $\bar{B}$ decompositions of A such that

$\bar{A} \leqq \bar{B}$.*) Let $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$ be such that $\bar{a} \subseteq \bar{b}$. The mapping of $\bar{A}$ onto $\bar{B}$ whose value at $\bar{a}$ is $\bar{b}$ is called *canonical.*

1.4. *Let $(A, (f_\iota)_{\iota \in I})$ be a general algebra, Θ an equivalence relation on A, φ the canonical mapping of A onto A/Θ. Then φ is a weak homomorphism of the general algebra $(A, (f_\iota)_{\iota \in I})$ onto the factor-algebra $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$. φ is a homomorphism of the general algebra $(A, (f_\iota)_{\iota \in I})$ onto the factor-algebra $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$ if and only if Θ is a congruence relation on A.*

Proof. Let $(a_\varkappa)_{\varkappa \in K_\iota}$ be a K_ι-sequence in A, $(\bar{a}_\varkappa)_{\varkappa \in K_\iota}$ the K_ι-sequence in A/Θ such that $a_\varkappa \in \bar{a}_\varkappa$ for all $\varkappa \in K_\iota$.

I. If $\bar{x} \in \varphi[f_\iota(a_\varkappa \mid \varkappa \in K_\iota)]$, then there exists $x \in f_\iota(a_\varkappa \mid \varkappa \in K_\iota)$ such that $\varphi(x) = \bar{x}$. Therefrom $\bar{x} \cap f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \neq 0$, consequently $\bar{x} \in \bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota) = \bar{f}_\iota(\varphi(a_\varkappa) \mid \varkappa \in K_\iota)$ and φ is a weak homomorphism.

II. (a) Let Θ be a congruence relation on A. By the preceding, φ is a weak homomorphism and it suffices to show $\bar{f}_\iota(\varphi(a_\varkappa) \mid \varkappa \in K_\iota) \subseteq \varphi[f_\iota(a_\varkappa \mid \varkappa \in K_\iota)]$. Let $\bar{x} \in \bar{f}_\iota(\varphi(a_\varkappa) \mid \varkappa \in K_\iota)$. Then there exists a K_ι-sequence $(a'_\varkappa)_{\varkappa \in K_\iota}$ in A such that $a'_\varkappa \in \varphi(a_\varkappa)$ for all $\varkappa \in K_\iota$ and $\bar{x} \cap f_\iota(a'_\varkappa \mid \varkappa \in K_\iota) \neq 0$. Since $a_\varkappa \Theta a'_\varkappa$ for all $\varkappa \in K_\iota$ and Θ is a congruence relation, there exists to each $x' \in f_\iota(a'_\varkappa \mid \varkappa \in K_\iota)$ such $x \in f_\iota(a_\varkappa \mid \varkappa \in K_\iota)$ that $x \Theta x'$. Let $x' \in \bar{x} \cap f_\iota(a'_\varkappa \mid \varkappa \in K_\iota)$. Then $x \in \bar{x} \cap f_\iota(a_\varkappa \mid \varkappa \in K_\iota)$ and therefore $\varphi(x) = \bar{x} \in \varphi[f_\iota(a_\varkappa \mid \varkappa \in K_\iota)]$. Consequently φ is a homomorphism.

(b) Let φ be a homomorphism. For $x, y \in A$ we have $x \Theta y$ if and only if $\varphi(x) = \varphi(y)$, for φ is the canonical mapping of A onto A/Θ. Therefore Θ is a congruence relation on A according to 1.3.

1.5. *Let $(A, (f_\iota)_{\iota \in I})$ be a general algebra whose all operations are of finite type. Then the family $C(A)$ of all congruence relations on A is a complete lattice (with respect to the inclusion of relations) and for $M \subseteq C(A)$ we have $\sup_{C(A)} M = \sup_{E(A)} M$, where $E(A)$ is the family of all equivalence relations on A.*

Proof. The family $E(A)$ of all equivalence relations on A is a complete lattice (see [2], *p.* 146). Let $M \subseteq C(A)$ and denote $\Theta = \sup_{E(A)} M$. We shall prove that $\Theta \in C(A)$. If $M = 0$, then Θ equals the least equivalence relation on A, which is a congruence relation on A. Let $M \neq 0$. For $x, y \in A$ we have $x \Theta y$ if and only if there exist a natural number m, elements $x = t_0, t_1, \ldots, t_m = y \in A$ and elements $\Phi_1, \ldots, \Phi_m \in M$ such that $t_0 \Phi_1 t_1, \ldots, t_{m-1} \Phi_m t_m$ (see[2]). Let $\iota \in I$, card $K_\iota = k$. Let $(a_0, a_1, \ldots, a_{k-1})$, $(b_0, b_1, \ldots, b_{k-1})$ be such K_ι-sequences in A that $a_i \Theta b_i$ for $i = 0, 1, \ldots, k-1$. Since, $a_0 \Theta b_0$, there exist a natural number m_0, elements $a_0 = t_0^{(0)}, t_0^{(1)}, \ldots, t_0^{(m_0)} = b_0 \in A$ and $\Phi_0^{(1)}, \ldots, \Phi_0^{(m_0)} \in M$ such that $t_0^{(0)} \Phi_0^{(1)} t_0^{(1)}, \ldots, t_0^{(m_0-1)} \Phi_0^{(m_0)} t_0^{(m_0)}$. Further since $\Phi_0^{(1)}$ is a congruence relation and $a_0 = t_0^{(0)} \Phi_0^{(1)} t_0^{(1)}$, $a_i \Phi_0^{(1)} a_i$ $(i = 1, \ldots, k-1)$, there exists

*) $\bar{A} \leqq \bar{B}$ denotes that the decomposition $\bar{A}$ is a refinement of the decomposition $\bar{B}$. (See[5].)

to each $x \in f_\iota(a_0, a_1, \ldots, a_{k-1})$ such $x_0^{(1)} \in f_\iota(t_0^{(1)}, a_1, \ldots, a_{k-1})$ and to each $x_0'^{(1)} \in f_\iota(t_0^{(1)}, a_1, \ldots, a_{k-1})$ such $x' \in f_\iota(a_0, a_1, \ldots, a_{k-1})$ that $x\Phi_0^{(1)}x_0^{(1)}$, $x'\Phi_0^{(1)}x_0'^{(1)}$. Analogically, there exists to each $x_0^{(1)} \in f_\iota(t_0^{(1)}, a_1, \ldots, a_{k-1})$ such $x_0^{(2)} \in f_\iota(t_0^{(2)}, a_1, \ldots, a_{k-1})$ and to each $x_0'^{(2)} \in f_\iota(t_0^{(2)}, a_1, \ldots, a_{k-1})$ such $x_0'^{(1)} \in f_\iota(t_1^{(1)}, a_1, \ldots, a_{k-1})$ that $x_0^{(1)}\Phi_0^{(2)}\Phi x_0^{(2)}$, $x_0'^{(1)}\Phi_0^{(2)}x_0'^{(2)}$ etc. Therefore there exists to each $x \in f_\iota(a_0, a_1, \ldots, a_{k-1})$ such $x_0^{(m_0)} \in f_\iota(b_0, a_1, \ldots, a_{k-1})$ and to each $x_0'^{(m_0)} \in f_\iota(b_0, a_1, \ldots, a_{k-1})$ such $x' \in f_\iota(a_0, a_1, \ldots, a_{k-1})$ that $x\Theta x_0^{(m_0)}$, $x'\Theta x_0'^{(m_0)}$. Therefrom we obtain after k steps that there exists to each $x \in f_\iota(a_0, a_1, \ldots, a_{k-1})$ such $y \in f_\iota(b_0, b_1, \ldots, b_{k-1})$ and to each $y' \in f_\iota(b_0, b_1, \ldots, b_{k-1})$ such $x' \in f_\iota(a_0, a_1, \ldots, a_{k-1})$ that $x\Theta y$, $x'\Theta y'$. Consequently $\Theta \in C(A)$, so that $\Theta = \sup_{C(A)} M$. To show the existence of the infimum denote by N the family of all $\Psi \in C(A)$ such that $\Psi \subseteq \Phi$ for all $\Phi \in M$. According to the preceding part of the proof there exists $\sup_{C(A)} N$. Evidently $\inf_{C(A)} M = \sup_{C(A)} N$.

2. Partial algebras

Definition. Let $(A, (f_\iota)_{\iota \in I})$ be a general algebra. This general algebra is called a *partial algebra if* and only if

$$\operatorname{card} f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \leqq 1$$

for all $\iota \in I$ and for all K_ι-sequences $(a_\varkappa)_{\varkappa \in K_\iota}$ in A.

Definition. Let $(A, (f_\iota)_{\iota \in I})$ be a partial algebra, $\bar{A}$ a decomposition of A. If there exists for each $\iota \in I$ and for each K_ι-sequence $(\bar{a}_\varkappa)_{\varkappa \in K_\iota}$ in $\bar{A}$ an element $\bar{a} \in \bar{A}$ such that for all K_ι-sequences $(a_\varkappa)_{\varkappa \in K_\iota}$ in A for which $a_\varkappa \in \bar{a}_\varkappa$ $(\varkappa \in K_\iota)$ we have $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \subseteq \bar{a}$, then $\bar{A}$ is called a *generating decomposition of A.**)

2.1. *Let $(A, (f_\iota)_{\iota \in I})$ be a partial algebra, $\bar{A}$ a decomposition of A. Then the following statements are equivalent:*

(A) $\bar{A}$ is a generating decomposition of A.

(B) The factor-algebra $(\bar{A}, (\bar{f}_\iota)_{\iota \in I})$ is a partial algebra.

Proof. I. Let $\bar{A}$ be a generating decomposition. Let $\iota \in I$, let $(\bar{a}_\varkappa)_{\varkappa \in K_\iota}$ be a K_ι-sequence in $\bar{A}$. Then there exists to the K_ι-sequence $(\bar{a}_\varkappa)_{\varkappa \in K_\iota}$ an element $\bar{a} \in \bar{A}$ such that we have $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \subseteq \bar{a}$ for all K_ι-sequences $(a_\varkappa)_{\varkappa \in K_\iota}$ fulfilling the condition (a) $a_\varkappa \in \bar{a}_\varkappa (\varkappa \in K_\iota)$. If we have $f_\iota(a_\varkappa) \varkappa \in K_\iota) = 0$ for all such K_ι-sequences, then also $\bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota) = 0$. But if there exists a K_ι-sequence $(a_\varkappa)_{\varkappa \in K_\iota}$ fulfilling the condition (a) such that $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \neq 0$, then clearly $\bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota) = \{\bar{a}\}$. Therefore the factor-algebra $(\bar{A}, (\bar{f}_\iota)_{\iota \in I})$ is a partial algebra.

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*) [5].

II. Let $\bar{A}$ fail to be a generating decomposition. Consequently, let there exist to $\iota \in I$ and to a K_ι-sequence $(\bar{a}_\varkappa)_{\varkappa \in K_\iota}$ no element $\bar{a} \in \bar{A}$ such that $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \subseteq \bar{a}$ for all K_ι-sequences $(a_\varkappa)_{\varkappa \in K_\iota}$ fulfilling the condition (a). Then there exist elements $\bar{a}', \bar{a}'' \in \bar{A}$ and K_ι-sequences $(a'_\varkappa)_{\varkappa \in K_\iota}$, $(a''_\varkappa)_{\varkappa \in K_\iota}$ fulfilling the condition (a) such that $f_\iota(a'_\varkappa \mid \varkappa \in K_\iota) \subseteq \bar{a}'$, $f_\iota(a''_\varkappa \mid \varkappa \in K_\iota) \subseteq \bar{a}''$. But then $\bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota) \supseteq \{\bar{a}', \bar{a}''\}$, so that the factor-algebra $(\bar{A}, (\bar{f}_\iota)_{\iota \in I})$ cannot be a partial algebra.

Definition. Let $(A, (f_\iota)_{\iota \in I})_\iota$ be a partial algebra, Θ an equivalence relation on A. Θ is called a *weak congruence relation on A* if and only if for arbitrary $\iota \in I$ and for arbitrary K_ι-sequences $(a_\varkappa)_{\varkappa \in K_\iota}$, $(b_\varkappa)_{\varkappa \in K_\iota}$ in A with the property $a_\varkappa \Theta b_\varkappa$ for $\varkappa \in K_\iota$ either at least one of the sets $f_\iota(a_\varkappa \mid \varkappa \in K_\iota)$, $f_\iota(b_\varkappa \mid \varkappa \in K_\iota)$ is void or (if $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = \{a\}$, $f_\iota(b_\varkappa \mid \varkappa \in K_\iota) = \{b\}$) $a \Theta b$.

Remark. Every congruence relation on a partial algebra is a weak congruence relation.

2.2. *Let $(A, (f_\iota)_{\iota \in I})$ be a partial algebra, Θ an equivalence relation on A. Then the following statements are equivalent:*

(A) Θ is a weak congruence relation on A.

(B) A/Θ is a generating decomposition of A.

Proof. I. Let Θ be a weak congruence relation. Let $\iota \in I$, let $(\bar{a}_\varkappa)_{\varkappa \in K_\iota}$ be a K_ι-sequence in A/Θ. If we have $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = 0$ for all K_ι-sequences $(a_\varkappa)_{\varkappa \in K_\iota}$ in A such that $a_\varkappa \in \bar{a}_\varkappa (\varkappa \in K_\iota)$, it suffices to choose arbitrarily $\bar{a} \in A/\Theta$ and $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \subseteq \bar{a}$ for all such K_ι-sequences. Consequently, let there exist a K_ι-sequence $(a_\varkappa)_{\varkappa \in K_\iota}$ in A such that $a_\varkappa \in \bar{a}_\varkappa$ for all $\varkappa \in K_\iota$ and $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = \{a\}$. Let $\bar{a} \in A/\Theta$ be such that $a \in \bar{a}$. Let $(a'_\varkappa)_{\varkappa \in K_\iota}$ be an arbitrary K_ι-sequence in A such that $a'_\varkappa \in \bar{a}_\varkappa$ for all $\varkappa \in K_\iota$. Then $a_\varkappa \Theta a'_\varkappa$ for all $\varkappa \in K_\iota$ and therefore either $f_\iota(a'_\varkappa \mid \varkappa \in K_\iota) = 0$ or $f_\iota(a'_\varkappa \mid \varkappa \in K_\iota) = \{a'\}$ and $a \Theta a'$, for Θ is a weak congruence relation. Thence we have in both cases $f_\iota(a'_\varkappa \mid \varkappa \in K_\iota) \subseteq \bar{a}$.

II. Let A/Θ be a generating decomposition. Let $\iota \in I$, let $(a_\varkappa)_{\varkappa \in K_\iota}$, $(b_\varkappa)_{\varkappa \in K_\iota}$ be two K_ι-sequences in A such that $a_\varkappa \Theta b_\varkappa$ for all $\varkappa \in K_\iota$. There exists a K_ι-sequence $(\bar{a}_\varkappa)_{\varkappa \in K_\iota}$ in A/Θ such that $a_\varkappa, b_\varkappa \in \bar{a}_\varkappa$ for $\varkappa \in K_\iota$. But then there exists an element $\bar{a} \in A/\Theta$ for which $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \subseteq \bar{a}$, $f_\iota(b_\varkappa \mid \varkappa \in K_\iota) \subseteq \bar{a}$ hold. Therefrom it follows that either at least one of the sets $f_\iota(a_\varkappa \mid \varkappa \in K_\iota)$, $f_\iota(b_\varkappa \mid \varkappa \in K_\iota)$ is void or (in the case that $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = \{a\}$, $f_\iota(b_\varkappa \mid \varkappa \in K_\iota) = \{b\}$) $a \Theta b$.

2.3. *Let $(A, (f_\iota)_{\iota \in I})$ be a partial algebra, $(B, (g_\iota)_{\iota \in I})$ a general algebra similar with $(A, (f_\iota)_{\iota \in I})$, φ a homomorphism of $(A, (f_\iota)_{\iota \in I})$ onto $(B, (g_\iota)_{\iota \in I})$. Then $(B, (g_\iota)_{\iota \in I})$ is also a partial algebra.*

2.4. *Let $(A, (f_\iota)_{\iota \in I})$ be a partial algebra, $\bar{A}$ a generating decomposition of A, $\bar{\bar{A}}$ a generating decomposition of $\bar{A}$, ψ the natural mapping of $\bar{\bar{A}}$ onto $\bar{\bar{A}} \rhd \bar{A}$. Then the factor-algebras $(\bar{\bar{A}}, (\bar{\bar{f}}_\iota)_{\iota \in I})$, $(\bar{\bar{A}} \rhd \bar{A}, (\bar{f}_\iota^{\rhd})_{\iota \in I})$ are* *partial algebras and ψ is an isomorphism of $(\bar{\bar{A}}, (\bar{\bar{f}}_\iota)_{\iota \in I})$ onto $(\bar{\bar{A}} \rhd \bar{A}, (\bar{f}_\iota^{\rhd})_{\iota \in I})$.*

Proof. The factor-algebra $(\bar{\bar{A}}, (\bar{\bar{f}}^\iota)_{\iota \in I})$ is a partial algebra by 2.1. The mapping ψ is an isomorphism by 1.2. According to 2.3 the factor-algebra $(\bar{\bar{A}} \rhd \bar{A}, (\bar{f}_\iota^{\rhd})_{\iota \in I})$ is also a partial algebra.

2.5. *Let $(A, (f_\iota)_{\iota \in I})$, $(B, (g_\iota)_{\iota \in I})$ be similar partial algebras, φ a mapping of A onto B. Let Θ be the equivalence relation on A and ω the mapping of A/Θ onto B defined in 1.3.*

(1) *Let φ be a weak homomorphism. Then Θ is a weak congruence relation on A, $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$ is a partial algebra and ω is a one-one weak homomorphism of $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$ onto $(B, (g_\iota)_{\iota \in I})$.*

(2) *Let φ be a homomorphism. Then Θ is a congruence relation on A and ω is an isomorphism of $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$ onto $(B, (g_\iota)_{\iota \in I})$.*

Proof. We shall show only that Θ is a weak congruence relation in the case (1). Let $(a_\varkappa)_{\varkappa \in K_\iota}$, $(b_\varkappa)_{\varkappa \in K_\iota}$ be K_ι-sequences in A such that $a_\varkappa \Theta b_\varkappa$ for all $\varkappa \in K_\iota$. Then $\varphi(a_\varkappa) = \varphi(b_\varkappa)$ for all $\varkappa \in K_\iota$ and since φ is a weak homomorphism, we have $\varphi[f_\iota(a_\varkappa \mid \varkappa \in K_\iota)] \subseteq g_\iota(\varphi(a_\varkappa) \mid \varkappa \in K_\iota) = g_\iota(\varphi(b_\varkappa) \mid \varkappa \in K_\iota) \supseteq \varphi[f_\iota(b_\varkappa \mid \varkappa \in K_\iota)]$. If $g_\iota(\varphi(a_\varkappa) \mid \varkappa \in K_\iota) = 0$, we have $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = f_\iota(b_\varkappa \mid \varkappa \in K_\iota) = 0$. If $g_\iota(\varphi(a_\varkappa) \mid \varkappa \in K_\iota) = \{c\}$, then either at least one of the sets $f_\iota(a_\varkappa \mid \varkappa \in K_\iota)$, $f_\iota(b_\varkappa \mid \varkappa \in K_\iota)$ is void or (if $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = \{a\}$, $f_\iota(b_\varkappa \mid \varkappa \in K_\iota) = \{b\}$) $\varphi(a) = \varphi(b)$, i.e. $a \Theta b$. The remaining part of the theorem follows from 1.3, 2.2 and 2.1.

2.6. *Let $(A, (f_\iota)_{\iota \in I})$ be a partial algebra. Then the family $C_w(A)$ of all weak congruence relations on A is a complete lattice (with respect to the inclusion of relations) and for $M \subseteq C_w(A)$ we have $\inf_{C_w(A)} M = \inf_{E(A)} M$.*

Proof. Let $M \subseteq C_w(A)$ and denote $\Theta = \inf_{E(A)} M$. We shall prove that $\Theta \in C_w(A)$. If $M = 0$, then Θ equals the greatest equivalence relation on A which is a weak congruence relation on A. Let $M \neq 0$. For $x, y \in A$ we have $x \Theta y$ if and only if $x \Phi y$ for all $\Phi \in M$ (see [2]). Let $\iota \in I$, let $(a_\varkappa)_{\varkappa \in K_\iota}$, $(b_\varkappa)_{\varkappa \in K_\iota}$ be K_ι-sequences in A such that $a_\varkappa \Theta b_\varkappa$ for all $\varkappa \in K_\iota$. Since $a_\varkappa \Theta b_\varkappa (\varkappa \in K_\iota)$, we have $a_\varkappa \Phi b_\varkappa (\varkappa \in K_\iota)$ for all $\Phi \in M$. Therefore either at least one of the sets $f_\iota(a_\varkappa \mid \varkappa \in K_\iota)$, $f_\iota(b_\varkappa \mid \varkappa \in K_\iota)$ is void or (if $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = \{a\}$, $f_\iota(b_\varkappa \mid \varkappa \in K_\iota) = \{b\}$) $a \Phi b$ for all $\Phi \in M$. i.e. $a \Theta b$. Consequently $\Theta \in C_w(A)$ and $\Theta = \inf_{C_w(A)} M$. To show the existence of the supremum denote by N the family of all $\Psi \in C_w(A)$ such that $\Psi \supseteq \Phi$ for all $\Phi \in M$. According to the preceding part of the proof there exists $\inf_{C_w(A)} N$. Evidently $\sup_{C_w(A)} M = \inf_{C_w(A)} N$.

2.7. *Let $(A, (f_\iota)_{\iota \in I})$ be a partial algebra whose all operations are of finite type. Then the family $C(A)$ of all congruence relations on A is a complete lattice (with respect to the inclusion of relations) and for $M \subseteq C(A)$ we have $\sup_{C(A)} M = \sup_{E(A)} M$ and in the case that $M \neq 0$ also $\inf_{C(A)} M = \inf_{E(A)} M$.*

Proof. $C(A)$ is a complete lattice and $\sup_{C(A)}M = \sup_{E(A)}M$ by 1.5. Let $0 \neq M \subseteq C(A)$ and denote $\Theta = \inf_{E(A)}M$. We shall prove that $\Theta \in C(A)$. Let $\iota \in I$, let $(a_\varkappa)_{\varkappa \in K_\iota}$, $(b_\varkappa)_{\varkappa \in K_\iota}$ be K_ι-sequences in A such that $a_\varkappa \Theta b_\varkappa$ for all $\varkappa \in K_\iota$. Since $a_\varkappa \Theta b_\varkappa$ $(\varkappa \in K_\iota)$, we have $a_\varkappa \Phi b_\varkappa$ $(\varkappa \in K_\iota)$ for all $\Phi \in M$. Therefore either $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = f_\iota(b_\varkappa \mid \varkappa \in K_\iota) = 0$ or (if $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = \{a\}, f_\iota(b_\varkappa \mid \varkappa \in K_\iota) = \{b\}$) $a\Phi b$ for all $\Phi \in M$, i.e. $a\Theta b$. Consequently in both cases $\Theta \in C(A)$ and $\Theta = \inf_{C(A)}M$.

3. Algebras.

Definition. Let $(A, (f_\iota)_{\iota \in I})$ be a partial algebra. This partial algebra is called an *algebra* if and only if

$$\operatorname{card} f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = 1$$

for all $\iota \in I$ and for all K_ι-sequences $(a_\varkappa)_{\varkappa \in K_\iota}$ in A. (See [1].)

Remark. On the study of algebras every weak congruence relation is a congruence relation and every weak homomorphism is a homomorphism.

3.1. *Let $(A, (f_\iota)_{\iota \in I})$ be an algebra, $\bar{A}$ a decomposition of A. Then the following statements are equivalent:*

(A) $\bar{A}$ is a generating decomposition of A.

(B) The factor-algebra $(\bar{A}, (\bar{f}_\iota)_{\iota \in I})$ is an algebra.

Proof. I. Let $\bar{A}$ be a generating decomposition. Let $\iota \in I$. Then there exists to an arbitrary K_ι-sequence $(\bar{a}_\varkappa)_{\varkappa \in K_\iota}$ in $\bar{A}$ an element $\bar{a} \in \bar{A}$ such that $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \subseteq \bar{a}$ for all K_ι-sequences $(a_\varkappa)_{\varkappa \in K_\iota}$ in A fulfilling the condition $a_\varkappa \in \bar{a}_\varkappa$ $(\varkappa \in K_\iota)$. But then obviously $\bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota) = \{\bar{a}\}$. Therefore the factor-algebra $(\bar{A}, (\bar{f}_\iota)_{\iota \in I})$ is an algebra.

II. Let the factor-algebra $(\bar{A}, (\bar{f}_\iota)_{\iota \in I})$ be an algebra. Then $(\bar{A}, (\bar{f}_\iota)_{\iota \in I})$ is also a partial algebra and $\bar{A}$ is a generating decomposition by 2.1.

3.2. *Let $(A, (f_\iota)_{\iota \in I})$ be an algebra, Θ an equivalence relation on A. Then following statements are equivalent:*

(A) Θ is a congruence relation on A.

(B) A/Θ is a generating decomposition of A.

3.3. *Let $(A, f_\iota)_{\iota \in I})$ be an algebra, $\bar{A}$ a generating decomposition of A, $\bar{\bar{A}}$ a generating decomposition of $\bar{A}$, ψ the natural mapping of $\bar{\bar{A}}$ onto $\bar{\bar{A}} \rhd \bar{A}$. Then the factor-algebras $(\bar{\bar{A}}, (\bar{\bar{f}}_\iota)_{\iota \in I})$, $(\bar{\bar{A}} \rhd \bar{A}, (\bar{f}_\iota^{\rhd})_{\iota \in I})$ are algebras and ψ is an isomorphism of $(\bar{\bar{A}}, (\bar{\bar{f}}_\iota)_{\iota \in I})$ onto $(\bar{\bar{A}} \rhd \bar{A}, (\bar{f}_\iota^{\rhd})_{\iota \in I})$.*

3.4. *Let $(A, (f_\iota)_{\iota \in I})$, $(B, (g_\iota)_{\iota \in I})$ be similar algebras, φ a homomorphism of $(A, (f_\iota)_{\iota \in I})$ onto $(B, (g_\iota)_{\iota \in I})$. Let Θ be the equivalence relation on A and ω the mapping of A/Θ onto B defined in 1.3. Then Θ is a congruence relation on A, the factor-algebra $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$ is an algebra and ω is an isomorphism of $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$ onto $(B, (g_\iota)_{\iota \in I})$.*

3.5. *Let $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$ be similar algebras, Θ an equivalence relation on A. If $(B, (g_\iota)_{\iota\in I})$ is an isomorphic image of the factor-algebra $(A/\Theta, (\bar{f}_\iota)_{\iota\in I})$, then $(B, (g_\iota)_{\iota\in I})$ is a homomorphic image of $(A, (f_\iota)_{\iota\in I})$.*

Proof. Since $(B, (g_\iota)_{\iota\in I})$ is an isomorphic image of $(A/\Theta, (\bar{f}_\iota)_{\iota\in I})$, $(A/\Theta, (\bar{f}_\iota)_{\iota\in I})$ is a partial algebra according to 2.3. By 2.1, A/Θ is a generating decomposition and consequently Θ is a congruence relation according to 3.2. But from 1.4 it follows that the canonical mapping of A onto A/Θ is a homomorphism of $(A, (f_\iota)_{\iota\in I})$ onto $(A/\Theta, (\bar{f}_\iota)_{\iota\in I})$. The homomorphism of $(A, (f_\iota)_{\iota\in I})$ onto $(B, (g_\iota)_{\iota\in I})$ is obtained according to 1.1 by the composition of this homomorphism of $(A, (f_\iota)_{\iota\in I})$ onto $(A/\Theta, (\bar{f}_\iota)_{\iota\in I})$ and the isomorphism of $(A/\Theta, (\bar{f}_\iota)_{\iota\in I})$ onto $(B, (g_\iota)_{\iota\in I})$.

3.6. *Let $(A, (f_\iota)_{\iota\in I})$ be an algebra. Then the family $C(A)$ of all congruence relations on A is a complete lattice (with respect to the inclusion of relations) and for $M \subseteq C(A)$ we have $\inf_{C(A)}M = \inf_{E(A)}M$. If all operations of the algebra $(A, (f_\iota)_{\iota\in I})$ are of finite type, we have also $\sup_{C(A)}M = \sup_{E(A)}M$.*

Remark. If card $I = 1$, card $K = 2$, we get from the preceding the well known theorems on the isomorphism of groupoids. (See [5].)

4. r-systems

Definition. Let $(A, (f_\iota)_{\iota\in I})$ be a partial algebra, e an arbitrary element in A. Let the following conditions be fulfilled for arbitrary $\iota \in I$ and for an arbitrary K_ι-sequence $(a_\varkappa)_{\varkappa\in K_\iota}$ in A:

(a) $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = 0$ or $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = \{e\}$;

(b) $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = 0$, if $a_\varkappa = e$ for some $\varkappa \in K_\iota$.

Then this partial algebra is called an *r-system*, the element e its *marked element.*

Definition. Let $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$ be similar r-systems, φ a (weak) homomorphism of $(A, (f_\iota)_{\iota\in I})$ onto $(B, (g_\iota)_{\iota\in I})$, e_A the marked element in $(A, (f_\iota)_{\iota\in I})$, e_B the marked element in $(B, (g_\iota)_{\iota\in I})$. If $\varphi(e_A) = e_B$, $\varphi[A - \{e_A\}] = B - \{e_B\}$, we say φ is a *(weak) r-homomorphism.* A one-one r-homomorphism is called an *r-isomorphism.*

4.1. *Let $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$ be similar r-systems, φ an r-homomorphism of $(A, (f_\iota)_{\iota\in I})$ onto $(B, (g_\iota)_{\iota\in I})$. Then $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = X$ is equivalent with $g_\iota(\varphi(a_\varkappa) \mid \varkappa \in K_\iota) = \varphi[X]$ for arbitrary $\iota \in I$ and for an arbitrary K_ι-sequence $(a_\varkappa)_{\varkappa\in K_\iota}$ in A.*

4.2. *Let $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$, $(C, (h_\iota)_{\iota\in I})$ be similar r-systems, φ a mapping of A onto B, ψ a mapping of B onto C.*

(1) *If the mappings φ and ψ are weak r-homomorphisms, then also their composite $\varphi^\circ\psi$ is a weak r-homomorphism.*

(2) *If any two of the mappings φ, ψ, $\varphi\circ\psi$ are r-homomorphisms, then the third mapping is also an r-homomorphism.*

Proof. I. If φ and ψ are (weak) r-homomorphisms, $\varphi\circ\psi$ is a (weak) homomorphism according to 1.1. If we denote by e_A, e_B, e_C the marked elements of the r-systems $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$, $(C, (h_\iota)_{\iota\in I})$, then we have $(\varphi\circ\psi)(e_A) = \psi(e_B) = e_C$, $(\varphi\circ\psi)[A - \{e_A\}] = \psi[B - \{e_B\}] = C - \{e_C\}$. Thereby it is shown that $\varphi\circ\psi$ is a (weak) r-homomorphism.

II. If φ, $\varphi\circ\psi$ are r-homomorphisms, ψ is a homomorphism according to 1.1. If we admitted that $\psi(e_B) \neq e_C$ or $\psi[B - \{e_B\}] \neq C - \{e_C\}$, we should obtain either $(\varphi\circ\psi)(e_A) = \psi(e_B) \neq e_C$ or $(\varphi\circ\psi)[A - \{e_A\}] = \psi[B - \{e_B\}] \neq C - e_C$, and that is a contradiction in both cases. Therefore ψ is an r-homomorphism.

III. Let ψ, $\varphi\circ\psi$ be r-homomorphisms. Let $\iota\in I$, let $(a_\varkappa)_{\varkappa\in K_\iota}$ be a K_ι-sequence in A. By 4.1, $f_\iota(a_\varkappa \mid \varkappa\in K_\iota) = X$ is equivalent with $h_\iota((\varphi\circ\psi)(a_\varkappa) \mid \varkappa\in K_\iota) = (\varphi\circ\psi)[X]$, for $\varphi\circ\psi$ is an r-homomorphism. By the same theorem, $h_\iota((\varphi\circ\psi)(a_\varkappa) \mid \varkappa\in K_\iota) = (\varphi\circ\psi)[X]$ is equivalent with $g_\iota(\varphi(a_\varkappa) \mid \varkappa\in K_\iota) = \varphi[X]$, so that φ is a homomorphism. If we admit that $\varphi(e_A) \neq e_B$ or $\varphi[A - \{e_A\}] \neq B - \{e_B\}$, we obtain either $(\varphi\circ\psi)(e_A) \neq e_C$ or $(\varphi\circ\psi)[A - \{e_A\}] \neq C - \{e_C\}$, and that is again a contradiction in both cases. Therefore φ is even an r-homomorphism.

4.3. *Let $(A, (f_\iota)_{\iota\in I})$ be an r-system. Then every decomposition $\bar{A}$ of A is generating.*

Definition. Let $(A, (f_\iota)_{\iota\in I})$ be an r-system with the marked element e, $\bar{A}$ a decomposition of A. If $\{e\} \in \bar{A}$, then the decomposition $\bar{A}$ is called an *r-decomposition* of A.

4.4. *Let $(A, (f_\iota)_{\iota\in I})$ be an r-system with the marked element e, $\bar{A}$ an r-decomposition of A. Then the factor-algebra $(\bar{A}, (\bar{f}_\iota)_{\iota\in I})$ is an r-system whose marked element is $\{e\}$.*

Proof. $\bar{A}$ is a generating decomposition according to 4.3, so that the factor-algebra $(\bar{A}, (\bar{f}_\iota)_{\iota\in I})$ is a partial algebra by 2.1. Let $\iota\in I$, let $(\bar{a}_\varkappa)_{\varkappa\in K_\iota}$ be a K_ι-sequence in $\bar{A}$. If $f_\iota(a_\varkappa \mid \varkappa\in K_\iota) = 0$ for all K_ι-sequences $(a_\varkappa)_{\varkappa\in K_\iota}$ in A such that $a_\varkappa \in \bar{a}_\varkappa (\varkappa\in K_\iota)$, then also $\bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa\in K_\iota) = 0$. If $f_\iota(a_\varkappa \mid \varkappa\in K_\iota) \neq 0$ for a K_ι-sequence $(a_\varkappa)_{\varkappa\in K_\iota}$ in A such that $a_\varkappa \in \bar{a}_\varkappa$ $(\varkappa\in K_\iota)$, we have $f_\iota(a_\varkappa \mid \varkappa\in K_\iota) = \{e\}$ for all such K_ι-sequences, so that $\bar{f}_\iota(\bar{a}_\varkappa) \varkappa\in K_\iota) = \{\{e\}\}$. Further consider the case that $\bar{a}_{\varkappa'} = \{e\}$ for some $\varkappa'\in K_\iota$. Let $(a_\varkappa)_{\varkappa\in K_\iota}$ be a K_ι-sequence in A such that $a_\varkappa \in \bar{a}_\varkappa$ for all $\varkappa\in K_\iota$. Since $\bar{a}_{\varkappa'} = \{e\}$, we have necessarily $a_{\varkappa'} = e$ and $f_\iota(a_\varkappa \mid \varkappa\in K_\iota) = 0$. Consequently also $\bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa\in K_\iota) = 0$ and the factor-algebra $(\bar{A}, (\bar{f}_\iota)_{\iota\in I})$ is an r-system with the marked element $\{e\}$.

4.5. *Let $(A, (f_\iota)_{\iota\in I})$ be an r-system, $\bar{A}$ an r-decomposition of A, $\bar{\bar{A}}$ an r-decomposition of $\bar{A}$, ψ the natural mapping of $\bar{\bar{A}}$ onto $\bar{\bar{A}} \rhd \bar{A}$. Then the* *factor-algebras* $(\bar{\bar{A}}, (\bar{\bar{f}}_\iota)_{\iota \in I})$, $(\bar{A} \rhd \bar{A}, (\bar{f}_\iota^{\rhd})_{\iota \in I})$ *are r-systems and* ψ *is an r-isomorphism of* $(\bar{\bar{A}}, (\bar{\bar{f}}_\iota)_{\iota \in I})$ *onto* $(\bar{A} \rhd \bar{A}, (\bar{f}_\iota^{\rhd})_{\iota \in I})$.

Proof. By 4.4, $(\bar{\bar{A}}, (\bar{\bar{f}}_\iota)_{\iota \in I})$ is an r-system with the marked element $\{\{e\}\}$, where e is the marked element in $(A, (f_\iota)_{\iota \in I})$. $(\bar{A} \rhd \bar{A}, (\bar{f}_\iota^{\rhd})_{\iota \in I})$ is a partial algebra by 2.4. We shall show that it is an r-system with the marked element $\{e\}$. Let $\iota \in I$, let $(\bar{b}_\varkappa)_{\varkappa \in K_\iota}$ be a K_ι-sequence in $\bar{A} \rhd \bar{A}$. For an arbitrary K_ι-sequence $(a_\varkappa)_{\varkappa \in K_\iota}$ in A such that $a_\varkappa \in \bar{b}_\varkappa$ for all $\varkappa \in K_\iota$ we have $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) \subseteq \{e\}$. Since $\{\{e\}\} \in \bar{\bar{A}}$, we have $\{e\} \in \bar{A} \rhd \bar{A}$ by the definition of the decomposition $\bar{A} \rhd \bar{A}$. From the preceding it follows $\bar{f}_\iota^{\rhd}(\bar{b}_\varkappa \mid \varkappa \in K_\iota) \subseteq \{\{e\}\}$. Further assume that $\bar{b}_{\varkappa'} = \{e\}$ for some $\varkappa' \in K_\iota$. Again let $(a_\varkappa)_{\varkappa \in K_\iota}$ be a K_ι-sequence in A such that $a_\varkappa \in \bar{b}_\varkappa$ for all $\varkappa \in K_\iota$. Since $\bar{b}_{\varkappa'} = \{e\}$, we have necessarily $a_{\varkappa'} = e$ and $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = 0$. Therefore also $\bar{f}_\iota^{\rhd}(\bar{b}_\varkappa \mid \varkappa \in K_\iota) = 0$ and the partial algebra $(\bar{A} \rhd \bar{A}, (\bar{f}_\iota^{\rhd})_{\iota \in I})$ is an r-system with the marked element $\{e\}$. By 1.2, the mapping ψ is an isomorphism. It remains to show that it is also an r-homomorphism. It was already shown that $\{\{e\}\}$ is the marked element of the r-system $(\bar{\bar{A}}, (\bar{\bar{f}}_\iota)_{\iota \in I})$ and $\{e\}$ the marked element of the r-system $(\bar{A} \rhd \bar{A}, (\bar{f}_\iota^{\rhd})_{\iota \in I})$. Clearly $\psi(\{\{e\}\}) = \{e\}$. Since ψ is a one-one mapping, we have $\psi[\bar{\bar{A}} - \{\{\{e\}\}\}] = \bar{A} \rhd \bar{A} - \{\{e\}\}$ and ψ is an r-isomorphism.

Definition. Let $(A, (f_\iota)_{\iota \in I})$ be an r-system with the marked element e, Θ an equivalence relation (respectively, a congruence relation) on A. If a non Θ e for all $a \in A - \{e\}$, then Θ is called an *r-equivalence relation* (respectively, an *r-congruence relation*) on A.

4.6. *Let* $(A, (f_\iota)_{\iota \in I})$ *be an r-system,* Θ *an equivalence relation on* A. *Then the equivalence relation* Θ *is an r-equivalence relation if and only if* A/Θ *is an r-decomposition of* A.

4.7. *Let* $(A, (f_\iota)_{\iota \in I})$, $(B, (g_\iota)_{\iota \in I})$ *be similar r-systems,* φ *a mapping of* A *onto* B. *Let* Θ *be the equivalence relation on* A *and* ω *the mapping of* A/Θ *onto* B *defined in 1.3.*

(1) *Let* φ *be a weak r-homomorphism. Then* Θ *is an r-equivalence relation on* A, $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$ *is an r-system and* ω *is a one-one weak homomorphism of* $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$ *onto* $(B, (g_\iota)_{\iota \in I})$.

(2) *Let* φ *be an r-homomorphism. Then* Θ *is an r-congruence relation on* A *and* ω *is an r-isomorphism of* $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$ *onto* $(B, (g_\iota)_{\iota \in I})$.

4.8. *Let* $(A, (f_\iota)_{\iota \in I})$ *be an r-system,* Θ *an r-equivalence relation on* A, φ *the canonical mapping of* A *onto* A/Θ. *Then* φ *is a weak r-homomorphism of* $(A, (f_\iota)_{\iota \in I})$ *onto* $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$. φ *is an r-homomorphism of* $(A, (f_\iota)_{\iota \in I})$ *onto* $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$ *if and only if* Θ *an r-congruence relation on* A.

Proof. I. By 1.4, φ is a weak homomorphism of $(A, (f_\iota)_{\iota \in I})$ onto $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$. Since Θ is an r-equivalence relation on A, we have $\varphi(e) = \{e\}$, $\varphi(a) \neq \{e\}$ for $a \neq e$, so that φ is a weak r-homomorphism.

II. (a) Let Θ be an r-conguence relation. Then φ is a homomorphism of $(A, (f_\iota)_{\iota \in I})$ onto $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$ by 1.4. But by the preceding, φ is even an r-homomorphism.

(b) If φ is an r-homomorphism of $(A, (f_\iota)_{\iota \in I})$ onto $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$, then Θ is an r-congruence relation by 4.7.

Definition. Let A be an ordered set, $X \subseteq A$. If there exists an element $a \in A$ such that $X = \{x \mid x \in A, x \leq a\}$, then X is called a *principal ideal* of the ordered set A.

4.9. *Let $(A, (f_\iota)_{\iota \in I})$ be an r-system. Then the family $E_r(A)$ of all r-equivalence relations on A is a principal ideal of the complete lattice $E(A)$. If all operations of the r-system $(A, (f_\iota)_{\iota \in I})$ are of finite type, then also the family $C_r(A)$ of all r-congruence relations on A is a principal ideal of $E(A)$.* (See [3].)

Proof. I. Let e be the marked element in $(A, (f_\iota)_{\iota \in I})$. Define an equivalence relation Ξ on A as follows: $e \,\Xi\, e$ and $a \,\Xi\, b$ if and only if $a, b \in A - \{e\}$. Clearly $\Xi \in E_r(A)$ and for arbitrary $\Phi \subseteq \Xi$ we have $\Phi \in E_r(A)$. Consequently $E_r(A)$ is a principal ideal of $E(A)$.

II. Denote $\Theta = \sup_{E(A)} C_r(A)$. By 1.5, we have $\Theta \in C(A)$. We shall show that even $\Theta \in C_r(A)$. Let $a \in A$, $a\Theta e$. Therefore there exist a natural number m, elements $a = t_0, t_1, \ldots, t_m = e \in A$ and elements $\Phi_1, \ldots, \Phi_m \in C_r(A)$ such that $t_0 \Phi_1 t_1, \ldots, t_{m-1} \Phi_m t_m$ (see [2]). But since $\Phi_1, \ldots, \Phi_m$ are r-congruence relations, we have $e = t_m = t_{m-1} = \ldots = t_1 = t_0 = a$, so that $\Theta \in C_r(A)$ and $\Theta = \sup_{C_r(A)} C_r(A)$. Let $\Phi \in E(A)$, $\Phi \subseteq \Theta$. Let $\iota \in I$, card $K_\iota = k$. If $(a_0, a_1, \ldots, a_{k-1})$, $(b_0, b_1, \ldots, b_{k-1})$ are K_ι-sequences in A such that $a_i \Phi b_i$ for $i = 0, 1, \ldots, k-1$, then we have also $a_i \Theta b_i$ for $i = 0, 1, \ldots, k-1$, so that there exists to each $x \in f_\iota(a_0, a_1, \ldots, a_{k-1})$ such $y \in f_\iota(b_0, b_1, \ldots, b_{k-1})$ and to each $y' \in f_\iota(b_0, b_1, \ldots, b_{k-1})$ such $x' \in f_\iota(a_0, a_1, \ldots, a_{k-1})$ that $x\Theta y$, $x'\Theta y'$. Thence we have either $f_\iota(a_0, a_1, \ldots, a_{k-1}) = f_\iota(b_0, b_1, \ldots, b_{k-1}) = 0$ or $f_\iota(a_0, a_1, \ldots, a_{k-1}) = f_\iota(b_0, b_1, \ldots, b_{k-1}) = \{e\}$, so that there exists to each $x \in f_\iota(a_0, a_1, \ldots, a_{k-1})$ such $y \in f_\iota(b_0, b_1, \ldots, b_{k-1})$ and to each $y' \in f_\iota(b_0, b_1, \ldots, b_{k-1})$ such $x' \in f_\iota(a_0, a_1, \ldots, a_{k-1})$ that $x\Phi y$, $x'\Phi y'$. Consequently $\Phi \in C(A)$ and from the first part of the proof we have $\Phi \in C_r(A)$. Therefore $C_r(A)$ is a principal ideal of $E(A)$.

5. Derived r-systems.

Everywhere in this paragraph we suppose that all operations on r-systems are of finite type. The greatest element in the family $C_r(A)$ of all r-congruence relations on A will be denoted by Θ_A.

5.1. *Let $(A, (f_\iota)_{\iota \in I})$ be an r-system, Θ an equivalence relation on A. Then the following statements are equivalent:*

(A) Θ *is an r-congruence relation on A.*
(B) $\Theta \subseteq \Theta_A$.

Definition. Let $(A, (f_\iota)_{\iota \in I})$ he an r-system, $\bar{A}$ an r-decomposition of A. Let $\bar{\bar{A}}$ be such an r-decomposition of $\bar{A}$ that $\bar{\bar{A}} = \bar{A}/\Theta_{\bar{A}}$. Then the r-decomposition $\bar{A}' = \bar{\bar{A}} \rhd A$ is called *derived* from the r-decomposition $\bar{A}$, the r-system $(\bar{A}', (\bar{f}'_\iota)_{\iota \in I})$ *derived* from the r-system $(\bar{A}, (\bar{f}_\iota)_{\iota \in I})$. (See [3].)

5.2. *Let* $(A, (f_\iota)_{\iota \in I})$ *be an r-system,* $\bar{A}$, $\bar{B}$ *r-decompositions of A. Let* $\bar{A} \leqq \bar{B}$ *and let* χ *be the canonical mapping of* $\bar{A}$ *onto* $\bar{B}$. *Then the following statements are equivalent:*

(A) χ *is an r-homomorphism of* $(\bar{A}, (\bar{f}^A_\iota)_{\iota \in I})$ *onto* $(B, (\bar{f}^B_\iota)_{\iota \in I})$.
(B) $\bar{B} \leqq \bar{A}'$.

Proof. Let $\bar{\bar{A}}$, $\bar{\bar{\bar{B}}}$ be the decompositions of $\bar{A}$ for which we have $\bar{A}' = \bar{\bar{A}} \rhd \bar{A}$, $\bar{B} = \bar{\bar{\bar{B}}} \rhd \bar{A}$. Further let Θ be the equivalence relation on $\bar{A}$ such that $\bar{\bar{\bar{B}}} = \bar{A}/\Theta$. We have $\bar{\bar{A}} = \bar{A}/\Theta_{\bar{A}}$. Let φ be the canonical mapping of $\bar{A}$ onto $\bar{\bar{\bar{B}}}$, ψ the natural mapping of $\bar{\bar{\bar{B}}}$ onto $\bar{B}$. Then the following statements are equivalent:

(B) $\bar{B} \leqq \bar{A}'$.
(C) $\bar{\bar{\bar{B}}} \leqq \bar{\bar{A}}$.
(D) $\Theta \subseteq \Theta_{\bar{A}}$.
(E) Θ is an r-congruence relation on A.
(F) φ is an r-homomorphism of $(\bar{A}, (\bar{f}^A_\iota)_{\iota \in I})$ onto $(\bar{\bar{\bar{B}}}, (\bar{\bar{f}}_\iota)_{\iota \in I})$.
(A) χ is an r-homomorphism of $(\bar{A}, (\bar{f}^A_\iota)_{\iota \in I})$ onto $(\bar{B}, (\bar{f}^B_\iota)_{\iota \in I})$.

In fact, (B) is equivalent with (C) (see [3], Theorem 2.5) and (C) is equivalent with (D) (see [3], Theorem 2.3). (D) is equivalent with (E) by 5.1, (E) is equivalent with (F) by 4.8. Since $\chi = \varphi \circ \psi$ and according to 4.5 ψ is an r-isomorphism, (F) is equivalent with (A) by 4.2.

5.3. *Let* $(A, (f_\iota)_{\iota \in I})$ *be an r-system,* $\bar{A}$ *an r-decomposition of A. Then* $\bar{A}'' = \bar{A}'$.

Proof. From the definition of the derived r-decomposition it follows $\bar{A}' \leqq \bar{A}''$. Let φ be the canonical mapping of $\bar{A}$ onto $\bar{A}'$, ψ the canonical mapping of $\bar{A}'$ onto $\bar{A}''$. Since $\bar{A}' \leqq \bar{A}'$, $\bar{A}'' \leqq \bar{A}''$, the mapping φ is an r-homomorphism of $(\bar{A}, (\bar{f}_\iota)_{\iota \in I})$ onto $(\bar{A}', (\bar{f}'_\iota)_{\iota \in I})$ and the mapping ψ an r-homomorphism of $(\bar{A}', (\bar{f}'_\iota)_{\iota \in I})$ onto $(\bar{A}'', (\bar{f}''_\iota)_{\iota \in I})$ according to 5.2. Consequently $\varphi \circ \psi$ is an r-homomorphism of $(\bar{A}, (\bar{f}_\iota)_{\iota \in I})$ onto $(\bar{A}'', (\bar{f}''_\iota)_{\iota \in I})$ by 4.2. But by 5.2, we have $\bar{A}'' \leqq \bar{A}'$. Therefore $\bar{A}'' = \bar{A}'$.

5.4. *Let* $(A, (f_\iota)_{\iota \in I})$ *be an r-system,* $\bar{A}$, $\bar{B}$ *r-decompositions of A. Then the following statements are equivalent:*

(A) $\bar{A}' = \bar{B}'$.

(B) There exists an r-decomposition $\bar{C}$ of A such that $\bar{A} \leq \bar{C} \leq \bar{A}'$, $\bar{B} \leq \bar{C} \leq \bar{B}'$.

(C) $\bar{A} \vee \bar{B} \leq \bar{A}'$, $\bar{A} \vee \bar{B} \leq \bar{B}'$*).

Proof. I. Let (A) hold true. Then it suffices to put $\bar{C} = \bar{A}' = \bar{B}'$ and we have $\bar{A} \leq \bar{C} \leq \bar{A}'$, $\bar{B} \leq \bar{C} \leq \bar{B}'$. Therefore (B) holds true.

II. Let (B) hold true. Since $\bar{A} \leq \bar{C} \leq \bar{A}'$, $\bar{B} \leq \bar{C} \leq \bar{B}'$, we have also $\bar{A} \vee \bar{B} \leq \bar{C} \leq \bar{A}'$, $\bar{A} \vee \bar{B} \leq \bar{C} \leq \bar{B}'$, so that (C) holds true.

III. Let (C) hold true. Clearly $\bar{A} \vee \bar{B}$ is an r-decomposition of A. Denote $\bar{C} = \bar{A} \vee \bar{B}$. From (C) it follows $\bar{A} \leq \bar{C} \leq \bar{A}'$, $\bar{B} \leq \bar{C} \leq \bar{B}'$. Further denote by φ the canonical mapping of $\bar{A}$ onto $\bar{C}$, by ψ the canonical mapping of $\bar{C}$ onto $\bar{A}'$. Then $\varphi \circ \psi$ is the canonical mapping of $\bar{A}$ onto $\bar{A}'$. By 5.2, φ is an r-homomorphism of $(\bar{A}, (\bar{f}_\iota)_{\iota \in I})$ onto $(\bar{C}, (\bar{f}_\iota^\vee)_{\iota \in I}$. $\varphi \circ \psi$ an r-homomorphism of $(\bar{A}, (\bar{f}_\iota)_{\iota \in I})$ onto $(\bar{A}', (\bar{f}_\iota')_{\iota \in I})$, for $\bar{C} \leq \bar{A}'$, $\bar{A}' \leq \bar{A}'$. By 4.2, ψ is an r-homomorphism of $(\bar{C}, (\bar{f}_\iota^\vee)_{\iota \in I})$ onto $(\bar{A}', (\bar{f}_\iota')_{\iota \in I})$. We have $\bar{A}' \leq \bar{C}'$ again by 5.2. Similarly we should show that $\bar{B}' \leq \bar{C}'$. Let χ be the canonical mapping of $\bar{A}'$ onto $\bar{C}'$. Then $\psi \circ \chi$ is the canonical mapping of $\bar{C}$ onto $\bar{C}'$. By 5.2, $\psi \circ \chi$ is an r-homomorphism of $(\bar{C}, (\bar{f}_\iota^\vee)_{\iota \in I})$ onto $(\bar{C}, (\bar{f}_\iota^{\vee\prime})_{\iota \in I})$, for $\bar{C}' \leq \bar{C}'$. Consequently χ is an r-homomorphism of $(\bar{A}', (\bar{f}_\iota')_{\iota \in I})$ onto $(\bar{C}', (\bar{f}_\iota^{\vee\prime})_{\iota \in I})$ according to 4.2. By 5.2, we have $\bar{C}' \leq \bar{A}''$. Similarly we should show that $\bar{C}' \leq \bar{B}''$. Since $\bar{A}' \leq \bar{C}' \leq \bar{A}''$, $\bar{B}' \leq \bar{C}' \leq \bar{B}''$, we have $\bar{A}' = \bar{C}' = \bar{B}'$ by 5.3 and (A) holds true.

5.5. *Let $(A, (f_\iota)_{\iota \in I})$ be an r-system, $\bar{A}$, $\bar{B}$ r-decompositions of A. Let $\bar{A} \leq \bar{B}$. Then the following statements are equivalent:*

(A) $\bar{A}' = \bar{B}'$.

(B) $\bar{B} \leq \bar{A}'$.

Proof. (A) is equivalent with (C) from 5.4. Therefore we show that (C) from 5.4 is equivalent with (B).

I. Let (C) from 5.4 hold true. Then $\bar{A} \vee \bar{B} \leq \bar{A}'$ implies $\bar{B} \leq \bar{A}'$, so that (B) holds true.

II. Let (B) hold true. Since $\bar{A} \leq \bar{A}'$, $\bar{B} \leq \bar{A}'$, we have $\bar{A} \vee \bar{B} \leq \bar{A}'$. Further since $\bar{A} \leq \bar{B} \leq \bar{B}'$, we have $\bar{A} \vee \bar{B} \leq \bar{B}'$, so that (C) from 5.4 holds true.

5.6. *Let $(A, (f_\iota)_{\iota \in I})$ be an r-system, $\bar{A}$, $\bar{B}$ r-decompositions of A. Let $\bar{A}' \leq \bar{B}'$. Then $(\bar{A}' \vee \bar{B})' = \bar{B}'$.*

Proof. $\bar{A}' \vee \bar{B}$ is evidently an r-decomposition of A. Since $\bar{B} \leq \bar{A}' \vee \bar{B} \leq \bar{B}' \vee \bar{B} = \bar{B}'$, we have $(\bar{A}' \vee \bar{B})' = \bar{B}'$ by 5.5.

5.7. *Let $(A, (f_\iota)_{\iota \in I})$ be an r-system, $\bar{M}$ the least decomposition of A, $\bar{A}$ an r-decomposition of A, $\bar{P}$ an r-decomposition of A such that $\bar{P} \leq \bar{M}'$. Let α, π be such r-equivalence relations on A that $A/\alpha = \bar{A}$, $A/\pi = \bar{P}$.*

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*) $\bar{A} \vee \bar{B}$ denotes the least decomposition such that $\bar{A} \leq \bar{A} \vee \bar{B}$, $\bar{B} \leq \bar{A} \vee \bar{B}$.

Let $\alpha\pi = \pi\alpha$. Then the canonical mapping φ of the r-system $(\bar{A}, (\bar{f}_\iota)_{\iota\in I})$ onto the r-system $(\bar{A} \vee \bar{P}, (\bar{f}_\iota^\vee)_{\iota\in I})$ is an r-homomorphism.

Proof. Denote by e the marked element in $(A, (f_\iota)_{\iota\in I})$. By 4.4, $\{e\}$ is the marked element in $(A, (\bar{f}_\iota)_{\iota\in I})$ and $(\bar{A} \vee \bar{P}, (\bar{f}_\iota^\vee)_{\iota\in I})$. Let $\iota \in I$, card $K_\iota = k$, let $(\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{k-1})$ be a K_ι-sequence in $\bar{A}$. Let $\bar{f}_\iota(\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{k-1}) = \{\{e\}\}$. Then there exists a K_ι-sequence $(a_0, a_1, \ldots, a_{k-1})$ in A such that $a_i \in \bar{a}_i$ for $i = 0, 1, \ldots, k-1$ and $f_\iota(a_0, a_1, \ldots, a_{k-1}) = \{e\}$. But since $\bar{a}_i \subseteq \varphi(\bar{a}_i)$, we have $a_i \in \varphi(\bar{a}_i)$ for $i = 0, 1, \ldots, k-1$, so that $\bar{f}_\iota^\vee(\varphi(\bar{a}_0), \varphi(\bar{a}_1), \ldots, \varphi(\bar{a}_{k-1})) = \{\{e\}\} = \varphi[\{\{e\}\}]$. Let $\bar{f}_\iota^\vee(\varphi(\bar{a}_0), \varphi(\bar{a}_1), \ldots, \varphi(\bar{a}_{k-1})) = \{\{e\}\}$. Then there exists a K_ι-sequence $(a_0, a_1, \ldots, a_{k-1})$ in A such that $a_i \in \varphi(\bar{a}_i)$ for $i = 0, 1, \ldots, k-1$ and $f_\iota(a_0, a_1, \ldots, a_{k-1}) = \{e\}$. Let $(\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_{k-1})$ be such a K_ι-sequence in $\bar{P}$ that $a_i \in \bar{p}_i$ for $i = 0, 1, \ldots, k-1$. We have $\bar{p}_i \subseteq \varphi(\bar{a}_i)$, $\bar{a}_i \subseteq \varphi(\bar{a}_i)$ for $i = 0, 1, \ldots, k-1$. We shall show that $\bar{p}_i \cap \bar{a}_i \neq 0$. In fact, denote by β such an r-equivalence relation on A that $A/\beta = \bar{A} \vee \bar{P}$. Since $\alpha\pi = \pi\alpha$, $\alpha\pi$ is also an r-equivalence relation (see [2], p. 175) and $\alpha\pi = \alpha \vee \pi = \beta$. Let $b_i \in \bar{a}_i$. Since $a_i, b_i \in \varphi(\bar{a}_i)$, we have $a_i\beta b_i$. Consequently there exists $c_i \in A$ such that $a_i\pi c_i$, $c_i\alpha b_i$. But then $c_i \in \bar{p}_i \cap \bar{a}_i$. Let $\bar{\bar{P}}$ be the r-decomposition of $\bar{M}$ such that $\bar{P} = \bar{\bar{P}} \rhd \bar{M}$. Further let $\Theta^{\bar{M}}$ be such an r-equivalence relation on $\bar{M}$ that $\bar{\bar{P}} = \bar{M}/\Theta^{\bar{M}}$. Since $\bar{P} \leqq \bar{M}'$, we have $\Theta^{\bar{M}} \subseteq \Theta_{\bar{M}}$ (see [3]). Therefore by 5.1 $\Theta^{\bar{M}}$ is an r-congruence relation on $\bar{M}$. Define an r-equivalence relation Θ^A on A as follows: $a\Theta^A b$ if and only if $\{a\}\, \Theta^{\bar{M}}\{b\}$. Since the r-systems $(A, (f_\iota)_{\iota\in I})$ and $(\bar{M}, (f_\iota^M)_{\iota\in I})$ are r-isomorphic and $\Theta^{\bar{M}}$ is an r-congruence relation on $\bar{M}$, Θ^A is even an r-congruence relation on A. Considering that $a_i, c_i \in \bar{p}_i$ we have $\{a_i\}\, \Theta^{\bar{M}}\{c_i\}$ and therefore also $a_i\Theta^A c_i$ for $i = 0, 1, \ldots, k-1$. Since $f_\iota(a_0, a_1, \ldots, a_{k-1}) = \{e\}$, we have $f_\iota(c_0, c_1, \ldots, c_{k-1}) = \{e\}$, so that $\bar{f}_\iota(\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{k-1}) = \{\{e\}\}$. Since $(\bar{A}, (\bar{f}_\iota)_{\iota\in I})$ is an r-system, it is thereby shown that φ is a homomorphism. Finally, since φ is the canonical mapping of the r-system $(\bar{A}, (\bar{f}_\iota)_{\iota\in I})$ onto the r-system $(\bar{A} \vee \bar{P}, (\bar{f}_\iota^\vee)_{\iota\in I})$, we have $\varphi(\{e\}) = \{e\}$, $\varphi[A - \{\{e\}\}] = \bar{A} \vee \bar{P} - \{\{e\}\}$ and φ is even an r-homomorphism.

5.8. *Let $(A, (f_\iota)_{\iota\in I})$ be an r-system, $\bar{M}$ the least decomposition on A, $\bar{A}$ an r-decomposition on A, $\bar{P}$ an r-decomposition on A such that $\bar{P} \leqq \bar{M}'$. Let α, π be such r-equivalence relations on A that $\bar{A} = A/\alpha$, $\bar{P} = A/\pi$. Let $\alpha\pi = \pi\alpha$. Then $\bar{P} \leqq \bar{A}'$.*

Proof. By 5.7, the canonical mapping φ of the r-system $(\bar{A}, (\bar{f}_\iota)_{\iota\in I})$ onto the r-system $(\bar{A} \vee \bar{P}, (\bar{f}_\iota^\vee)_{\iota\in I})$ is an r-homomorphism. By 5.2, we have $\bar{A} \vee \bar{P} \leqq \bar{A}'$. Therefore $\bar{P} \leqq \bar{A} \vee \bar{P} \leqq \bar{A}'$.

6. Simple r-systems.

Definition. An r-system $(A, (f_\iota)_{\iota \in I})$ is called *simple*, if there exists a unique r-congruence relation on A. (See [4].)

6.1. *An r-system $(A, (f_\iota)_{\iota \in I})$ is simple if and only if the following condition (b) is fulfilled: To arbitrary elements $x, y \in A$, $x \neq y$ there exist $\iota_0 \in I$, K_{ι_0}-sequences $(a_\varkappa)_{\varkappa \in K_\iota}$, $(b_\varkappa)_{\varkappa \in K_\iota}$ and an index $\varkappa_0 \in K_{\iota_0}$ such that $a_{\varkappa_0} = x$, $b_{\varkappa_0} = y$, $a_\varkappa = b_\varkappa$ for $\varkappa \in K_{\iota_0} - \{\varkappa_0\}$, $f_{\iota_0}(a_\varkappa \mid \varkappa \in K_{\iota_0}) \neq f_{\iota_0}(b_\varkappa \mid \varkappa \in K_{\iota_0})$.*

Proof. I. Let the condition (b) fail to be fulfilled. Then there exist elements $x, y \in A, x \neq y$ such that we have $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = f_\iota(b_\varkappa \mid \varkappa \in K_\iota)$ for all $\iota \in I$, for all $\varkappa_0 \in K_\iota$ and for all K_ι-sequences $(a_\varkappa)_{\varkappa \in K_\iota}$, $(b_\varkappa)_{\varkappa \in K_\iota}$ with the property $a_{\varkappa_0} = x$, $b_{\varkappa_0} = y$, $a_\varkappa = b_\varkappa (\varkappa \in K_\iota - \{\varkappa_0\})$. Define an r-equivalence relation Θ on A as follows: $a\Theta b$ if and only if $a = b$ or $a = x$, $b = y$ or $a = y$, $b = x$. Θ is clearly an r-congruence relation on A different from the least one, so that $(A, (f_\iota)_{\iota \in I})$ is not simple.

II. Let the condition (b) be fulfilled. If we admit that $(A, (f_\iota)_{\iota \in I})$ is not simple, then there exists an r-congruence relation Θ on A which is not the least one. Consequently we have $x\Theta y$ for some $x, y \in A$, $x \neq y$. Let $\iota \in I$, $\varkappa_0 \in K_\iota$, let $(a_\varkappa)_{\varkappa \in K_\iota}$, $(b_\varkappa)_{\varkappa \in K_\iota}$ be such K_ι-sequences in A that $a_{\varkappa_0} = x$, $b_{\varkappa_0} = y$, $a_\varkappa = b_\varkappa$ for $\varkappa \in K_\iota - \{\varkappa_0\}$. Further let $(\bar{a}_\varkappa)_{\varkappa \in K_\iota}$ be such a K_ι-sequence in A/Θ that $a_\varkappa \in \bar{a}_\varkappa$ for all $\varkappa \in K_\iota$. Clearly $b_\varkappa \in \bar{a}_\varkappa$ for all $\varkappa \in K_\iota$. By 4.8, the canonical mapping of $(A, (f_\iota)_{\iota \in I})$ onto $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$ is an r-homomorphism and therefore we have $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = 0$ if and only if $\bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota) = 0$ and $\bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota) = 0$ if and only if $f_\iota(b_\varkappa \mid \varkappa \in K_\iota) = 0$, which is a contradiction with (b).

6.2. *Let $(A, (f_\iota)_{\iota \in I})$ be an r-system whose all operations are of finite type, Θ an r-congruence relation on A. Then the factor-algebra $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$ is simple if and only if $\Theta = \Theta_A$* (see Paragraph 5).

Proof. I. Let $(A/\Theta, (\bar{f}_\iota)_{\iota \in I})$ be a simple factor-algebra. By 5.1, we have $\Theta \subseteq \Theta_A$. Let Φ be an arbitrary r-congruence relation on A. Admit that there exist elements $x, y \in A$ such that $x\Phi y$, x non Θy. Let $\bar{x}, \bar{y} \in A/\Theta$ be such that $x \in \bar{x}$, $y \in \bar{y}$. Let $\iota \in I$, $\varkappa_0 \in K_\iota$, let $(\bar{a}_\varkappa)_{\varkappa \in K_\iota}$, $(\bar{b}_\varkappa)_{\varkappa \in K_\iota}$ be K_ι-sequences in A/Θ such that $\bar{a}_{\varkappa_0} = \bar{x}$, $\bar{b}_{\varkappa_0} = \bar{y}$, $\bar{a}_\varkappa = \bar{b}_\varkappa$ for $\varkappa \in K_\iota - \{\varkappa_0\}$, let $(a_\varkappa)_{\varkappa \in K_\iota}$, $(b_\varkappa)_{\varkappa \in K_\iota}$ be K_ι-sequences in A such that $a_{\varkappa_0} = x$, $b_{\varkappa_0} = y$, $a_\varkappa = b_\varkappa \in \bar{a}_\varkappa$ for $\varkappa \in K_\iota - \{\varkappa_0\}$. Finally let $(\bar{c}_\varkappa)_{\varkappa \in K_\iota}$ be a K_ι-sequence in A/Φ such that $x \in \bar{c}_{\varkappa_0}$, $a_\varkappa \in \bar{c}_\varkappa$ for $\varkappa \in K_\iota - \{\varkappa_0\}$. The following statements are clearly equivalent:

(A) $\bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota) = 0$;
(B) $f_\iota(a_\varkappa \mid \varkappa \in K_\iota) = 0$;
(C) $\bar{f}_\iota^\Phi(\bar{c}_\varkappa \mid \varkappa \in K_\iota) = 0$;
(D) $f_\iota(b_\varkappa \mid \varkappa \in K_\iota) = 0$;
(E) $\bar{f}(\bar{b}_\varkappa \mid \varkappa \in K_\iota) = 0$.

In fact, the canonical mapping of $(A, (f_\iota)_{\iota\in I})$ onto $(A/\Theta, (\bar{f}_\iota)_{\iota\in I})$ is an r-homomorphism (see 4.8), so that (A) holds if and only if (B) holds. Similarly we should show the equivalence of (B) and (C), (C) and (D), (D) and (E). Therefore (A) is equivalent with (E), which contradicts to the condition (b) of 6.1. Consequently $\Phi \subseteq \Theta$ holds for all r-congruence relations on A and hence $\Theta = \Theta_A$.

II. Let $\Theta = \Theta_A$. By 5.1, Θ is an r-congruence relation on A. If we admit that $(A/\Theta, (\bar{f}_\iota)_{\iota\in I})$ is not simple, then there exist elements $\bar{x}, \bar{y} \in A/\Theta$ such that for arbitrary $\iota \in I$, $\varkappa_0 \in K_\iota$ and for arbitrary K_ι-sequences $(\bar{a}_\varkappa)_{\varkappa\in K_\iota}$, $(\bar{b}_\varkappa)_{\varkappa\in K_\iota}$ in A/Θ ($\bar{a}_{\varkappa_0} = \bar{x}, \bar{b}_{\varkappa_0} = \bar{y}$, $\bar{a}_\varkappa = \bar{b}_\varkappa$ for all $\varkappa \in K_\iota -$ $- \{\varkappa_0\}$) we have $\bar{f}_\iota(\bar{a}_\varkappa \mid \varkappa \in K_\iota) = \bar{f}_\iota(\bar{b}_\varkappa \mid \varkappa \in K_\iota)$. Define an r-equivalence relation Φ on A as follows: $a\Phi b$ if and only if $a\Theta b$ or $a \in \bar{x}$, $b \in \bar{y}$ or $a \in \bar{y}$, $b \in \bar{x}$. Φ is clearly an r-congruence relation on A, but we have $\Phi \nsubseteq \Theta$, which is a contradiction.

6.3. *Let $(A, (f_\iota)_{\iota\in I})$ be an r-system whose all operations are of finite type, Θ an r-congruence relation on A. Then the simple factor-algebra $(\bar{A}, \bar{f}_\iota)_{\iota\in I})$ on $(A, (f_\iota)_{\iota\in I})$ is an r-homomorphic image of the factor-algebra $(A/\Theta, (\bar{f}_\iota^\Theta)_{\iota\in I})$.*

Proof. By 6. 2, we hawe $\Theta \subseteq \Theta_A$, so that $A/\Theta \leq \bar{A}$. Denute by φ (respectively, ψ, χ) the canonical mapping of A onto A/Θ (respectively, of A/Θ onto $\bar{A}$, of A onto $\bar{A}$. Clearly $\chi = \varphi^\circ\psi$. φ and χ are r-homomorphisms by 4. 8. Consequently ψ is also an r-homomorphism by 4.2.

6.4. *Let $(A, (f_\iota)_{\iota\in I})$ be an r-system whose all operations are of finite type. Then there exists a unique simple r-system (except for r-isomorphisms) which is similar to $(A, (f_\iota)_{\iota\in I})$ and which is its r-homomorphic image.*

Proof. Let $(\bar{A}, (\bar{f}_\iota)_{\iota\in I})$ be the simple factor-algebra on $(A, (f_\iota)_{\iota\in I})$ and let $(B, (g_\iota)_{\iota\in I})$ be a simple r-system which is similar to $(A, (f_\iota)_{\iota\in I})$ and which is its r-homomorphic image. Denote by φ the corresponding r-homomorphism. Define an r-equivalence relation Θ on A as follows: $a\Theta b$ if and only if $\varphi(a) = \varphi(b)$. By 4.7, Θ is an r-congruence relation on A and $(B, (g_\iota)_{\iota\in I})$ is an r-isomorphic image of the factor-algebra $(A/\Theta, (\bar{f}_\iota^\Theta)_{\iota\in I})$. Since $(B, (g_\iota)_{\iota\in I})$ is simple, $(A/\Theta, (\bar{f}_\iota^\Theta)_{\iota\in I})$ is clearly also simple. Therefore $\Theta = \Theta_A$ and $(B, (g_\iota)_{\iota\in I})$ is an r-isomorphic image of $(\bar{A}, (\bar{f}_\iota)_{\iota\in I})$.

Definition. Let $(A, (f_\iota)_{\iota\in I})$ be an r-system whose all operations are of finite type, $(B, (g_\iota)_{\iota\in I})$ a simple r-system which is similar to $(A, (f_\iota)_{\iota\in I})$ and which is its r-homomorphic image. Let φ be the corresponding r-homomorphism and let ϱ be a mapping defined on B as follows: $\varrho(b) =$ $=$ card $\varphi^{-1}[b]$. Then the ordered pair $[(B, (g_\iota)_{\iota\in I}), \varrho]$ is called an *h-characteristic* of the r-system $(A, (f_\iota)_{\iota\in I})$.

Remark. Let $(B, (g_\iota)_{\iota\in I})$ be a simple r-system whose all operations are of finite type, ϱ a mapping assigning to each $b \in B$ a cardinal number

$\varrho(b) \geqq 1$. Then there exists a unique r-system $(A, (f_\iota)_{\iota\in I})$ (except for r-isomorphisms) such that $[(B, (g_\iota)_{\iota\in I}), \varrho]$ is its h-characteristic.

6.5. *Let $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$ be similar r-systems whose all operations are of finite type. Let $[(C, (h_\iota)_{\iota\in I}), \varrho]$, $[(D, (k_\iota)_{\iota\in I}), \sigma]$ be their h-characteristics. Then $(B, (g_\iota)_{\iota\in I})$ is an r-homomorphic image of $(A, (f_\iota)_{\iota\in I})$ if and only if the following condition (c) is fulfilled: There exists an r-isomorphism φ of $(C, (h_\iota)_{\iota\in I})$ onto $(D, (k_\iota)_{\iota\in I})$ such that $\varrho(c) \geqq \sigma(\varphi(c))$ holds for all $c \in C$.*

Proof. Denote by χ the r-homomorphism of $(A, (f_\iota)_{\iota\in I})$ onto $(C, (h_\iota)_{\iota\in I})$ and by τ the r-homomorphism of $(B, (g_\iota)_{\iota\in I})$ onto $(D, (k_\iota)_{\iota\in I})$ and define an r-equivalence relation X on A and an r-equivalence relation T on B as follows: aXa' if and only if $\chi(a) = \chi(a')$; bTb' if and only if $\tau(b) = \tau(b')$.

I. Let $(B, (g_\iota)_{\iota\in I})$ be an r-homomorphic image of $(A, (f_\iota)_{\iota\in I})$. Denote by ψ the corresponding r-homomorphism of $(A, (f_\iota)_{\iota\in I})$ onto $(B, (g_\iota)_{\iota\in I})$. The mapping $\psi \circ \tau$ of $(A, (f_\iota)_{\iota\in I})$ onto $(D, (k_\iota)_{\iota\in I})$ is an r-homomorphism by 4.2. Define an r-equivalence relation Ψ on A as follows: $a\Psi a'$ if and only if $(\psi\circ\tau)(a) = (\psi\circ\tau)(a')$. By 4.7, X, Ψ are r-congruence relations on A, T is an r-congruence relation on B. By the same theorem, $(A/X, (\bar{f}_\iota^X)_{\iota\in I})$ is an r-isomorphic image of $(C, (h_\iota)_{\iota\in I})$ and $(A/\Psi, (\bar{f}_\iota^\Psi)_{\iota\in I})$, $(B/T, (\bar{g}_\iota)_{\iota\in I})$ are r-isomorphic images of $(D, (k_\iota)_{\iota\in I})$. $(A/X, (\bar{f}_\iota^X)_{\iota\in I})$, $(A/\Psi, (\bar{f}_\iota^\Psi)_{\iota\in I})$ and $(B/T\ (\bar{g}_\iota)_{\iota\in I})$ are clearly simple, for $(C, (h_\iota)_{\iota\in I})$ and $(D, (k_\iota)_{\iota\in I})$ are simple. Therefore we have $X = \Psi = \Theta_A$, $T = \Theta_B$ by 6.2. With regard to that we may define a mapping φ of C onto D as follows: $\varphi(c) = (\psi\circ\tau)(a)$, where a is an arbitrary element such that $a \in \chi^{-1}[c]$. Since $\chi\circ\varphi = \psi\circ\tau$, φ is an r-homomorphism of $(C, (h_\iota)_{\iota\in I})$ onto $(D, (k_\iota)_{\iota\in I})$ (according to 4.2) which is clearly one-one. But since ψ maps each class of the decomposition A/Ψ onto a class of the decomposition B/T, we have $\varrho(c) = \text{card}\ \chi^{-1}[c] = \text{card}\ (\psi\circ\tau)^{-1}[\varphi(c)] \geqq \text{card}\ \tau^{-1}[\varphi(c)] = \sigma(\varphi(c))$ for each $c \in C$.

II. Let the condition (c) be fulfilled. Let $c \in C$ and let $\bar{a}_c \in A/X$ be such that $\chi(a) = c$ for $a \in \bar{a}_c$, $\bar{b}_d \in B/T$ be such that $\tau(b) = d$ for $b \in \bar{b}_d$. Since $\varrho(c) \geqq \sigma(\varphi(c))$, there exists a mapping φ_c of $\bar{a}_c$ onto $\bar{b}_{\varphi(c)}$ for each $c \in C$. Define a mapping ψ of A onto B as follows: $\psi(a) = \varphi_c(a)$ for $a \in \bar{a}_c$. We have clearly $\chi\circ\varphi = \psi\circ\tau$ and therefore ψ is an r-homomorphism of $(A, (f_\iota)_{\iota\in I})$ onto $(B, (g_\iota)_{\iota\in I})$ by 4.2, for $\chi\circ\varphi$ is an r-homomorphism of $(A, (f_\iota)_{\iota\in I})$ onto $(D, (k_\iota)_{\iota\in I})$ by the same theorem.

6.6. *Let $(A, (f_\iota)_{\iota\in I})$, $(B, (g_\iota)_{\iota\in I})$ be similar r-systems whose all operations are of finite type. Let $[(C, (h_\iota)_{\iota\in I}), \varrho]$ be an h-characteristic of $(A, (f_\iota)_{\iota\in I})$. Then the following statements are equivalent:*

(A) $(A, (f_\iota)_{\iota\in I})$ is simple.

(B) $(B, (g_\iota)_{\iota\in I})$ is an r-homomorphic image of $(A, (f_\iota)_{\iota\in I})$ implies that $(B, (g_\iota)_{\iota\in I})$ is an r-isomorphic image of $(A, (f_\iota)_{\iota\in I})$.

(C) $\varrho(c) = 1$ for all $c \in C$.

Proof. We shall show only the equivalence of (A) and (B), for (A) is equivalent with (C) by the definition. Let φ be an r-homomorphism of $(A, (f_\iota)_{\iota\in I})$ onto $(B, (g_\iota)_{\iota\in I})$. Define an r-equivalence relation Θ on A as follows: $a\Theta a'$ if and only if $\varphi(a) = \varphi(a')$. Θ is an r-congruence relation on A by 4.7. But $(A, (f_\iota)_{\iota\in I})$ is simple if and only if Θ is the least r-congruence relation on A and this holds if and only if φ is one-one.

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