Archivum Mathematicum

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On real-time Turing machines

Archivum Mathematicum, Vol. 2 (1966), No. 2, 79--92

Persistent URL: http://dml.cz/dmlcz/104610

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ON REAL—TIME TURING MACHINES

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(Received March 4, 1966)

Generalizing Rabin's idea of the "bottleneck square" we give certain sufficient conditions for the non-recognition by one-tape one-head real-time Turing machines. The use of our result (Theorem 2) is illustrated in examples, especially there is constructed a set not recognizable by the above mentioned Turing machines but recognized by a one-twodimensional tape one-head real-time Turing machine and by a two-(onedimensional) tape one-head real-time Turing machine, too.

1. BASIC CONCEPTS

1.0. card X means the cardinal number of a set X. $N = \{1, 2, \ldots\}$, $N_0 = N \cup \{0\}$.

Σ^∞ is the set of all words consisting of symbols from a set Σ including the empty word Λ, e.g. $\emptyset^\infty = \{\Lambda\}$. w^{-1} denotes the converse word of word w. $l(w)$ is the *length* of a word w.

Everywhere in the following we shall consider only words from N_0^∞. For $T \subseteq N_0^\infty$ let

$\mathfrak{B}(T) = \{\sigma \mid \sigma \in N_0$, there exist $w_1, w_2 \in N_0^\infty$ such that $w_1 \sigma w_2 \in T\}$. ($\mathfrak{B}(T)$ is the set of all letters contained in words from T.) If $\mathfrak{B}(T)$ is finite, then T is called an *event*.

1.1. Definition. Let $T \subseteq N_0^\infty$. We say that words $w_1, w_2 \in N_0^\infty$ are *distinguished on* T by $\tilde{w} \in N_0^\infty$ when one of the words $w_1\tilde{w}$, $w_2\tilde{w}$ is in T and the other is not. Words w_1, w_2 are called *distinguishable* (*L-distinguishable*) *on* T if they are distinguished on T by some word $\tilde{w}$ ($l(\tilde{w}) \leqq L$).

(If w_1, w_2 are distinguished on T by $\tilde{w}$, then $\tilde{w} \in [\mathfrak{B}(T)]^\infty$ and w_1 or w_2 is also in $[\mathfrak{B}(T)]^\infty$.)

1.2. A one- (onedimensional) tape one-head real-time Turing machine (we will say in short a [1,1]-*Turing machine*) is a 5-tuple

$$\mathfrak{M} = [\Sigma, S, W, F, M]$$

where Σ, S, W are finite subsets of N_0, $0 \in S \cap W$, $F \subseteq S$ and M is a mapping, $M\colon \Sigma \times S \times W \to W \times P \times S$, where $P = \{-1, 0, 1\}$.

Interpretation: Σ is the *input alphabet*, S is the set of *states* (0 is the initial state), F is the set of *designated states*, W is the *working alphabet*

(0 is the "blank symbol"). Let $[\sigma, s, \alpha] \in \Sigma \times S \times W$, $M([\sigma, s, \alpha]) = [M_1, M_2, M_3]$; if $\mathfrak{M}$ is in state s, the head sees α and the input is σ, then on the observing square $\mathfrak{M}$ prints M_1, the head will move one square right (when $M_2 = 1$) or one square left (when $M_2 = -1$) or the head does not move (when $M_2 = 0$), and $\mathfrak{M}$ will go into the state M_3. The described action we call a *tact* (also an atomic move). $\mathfrak{M}$ performs the following tact under the successive letter of the input word or $\mathfrak{M}$ stops if the last tact was performed under the last letter of the input word. For every input $w \in \Sigma^\infty$, $\mathfrak{M}$ starts the whole work (the *computation under* w) in the initial state (i.e. in 0) and with only blank symbols (noughts) on the tape. $s(w)$ will denote the state in which $\mathfrak{M}$ is at the end of the computation under w, especially $s(\Lambda) = 0$.

Now we define

$$\mathbf{T}(\mathfrak{M}) = \{w \mid w \in \Sigma^\infty,\ s(w) \in F\}.$$

$\mathbf{T}(\mathfrak{M})$ is the *event recognized by* $\mathfrak{M}$.

1.3. By a *covering of square A* (of the tape of a [1,1] — Turing machine in a computation) we mean every tact after which the head is on A; moreover, we assign one covering more to the initial square. (Thus, during the computation under $w \in \Sigma^\infty$ $l(w) + 1$ coverings occur on the whole.) The set of all squares covered during the computation under an input w we denote $t(w)$ and we call this set the *work space on input* w. Of course, $t(w)$ is a segment*); its length we denote $\lambda(w)$. Evidently, $1 \leqq \lambda(w) \leqq l(w) + 1$. The position of the head in $t(w)$ in the end of the computation under an input w (the first left square of $t(w)$ we regard as the first, the first right square of $t(w)$ we regard as the $\lambda(w)$th we denote $\pi(w)$, the word printed on $t(w)$ in the end of this computation we shall denote $\tau(w)$.

If we want to call attention to that Σ, S, t (or W, $s(w)$ and similar) belong to a [1, 1]-Turing machine $\mathfrak{M}$, we write also $\Sigma_{\mathfrak{m}}$, $S_{\mathfrak{m}}$, $t_{\mathfrak{m}}$ (or $W_{\mathfrak{m}}$, $s_{\mathfrak{m}}(w)$ and similar).

1.4. A triplet $\varkappa(w) = [\tau(w), \pi(w), s(w)]$ (for $w \in \Sigma^\infty$) we call the *coding of* w. The number of codings with the length of the work spaces not exceeding number $k \in N$ is evidently less than $k(1 + \text{card } W)^k \text{ card } S$. If words $w_1, w_2 \in \Sigma^\infty$ are distinguishable on $\mathbf{T}(\mathfrak{M})$, then $\varkappa(w_1) \neq \varkappa(w_2)$.

1.5. We define

$$\mathscr{T}_{[1,1]} = \{\mathbf{T}(\mathfrak{M}) \mid \mathfrak{M} \text{ is a [1,1]-Turing machine}\}.$$

Thus, $\mathscr{T}_{[1,1]}$ consists of all *events recognizable by* [1,1]-*Turing machines*.

1.6. Analogically as above we may define also other types of real-time Turing machines, e.g. machines having p (onedimensional) tapes with h

*) By a segment we mean a nonempty set of squares (of the tape) which with every two squares contains also all the squares lying between them.

heads on each of them (we denote these machines as $[p, h]$-*Turing machines*) or machines having one twodimensional tape with one head (we denote them as $\square$-*Turing machines*) and similar ones, and to them we may define $\mathscr{T}_{[p,h]}$, $\mathscr{T}_{\square}$. Besides, also finite automata*) we can consider as real-time Turing machines (without tapes), we shall denote them as 0-*Turing machines*. Thus, $\mathscr{T}_0$ is the system of all regular events (letters of which are here only nonnegative integers).

1.7. If T_1 and T_2 are recognizable by any real-time Turing machines having the same type, then $T_1 \cup T_2$ (and also $T_1 \cap T_2$) need not be recognizable by Turing machines of this type (e.g. see Examples 5.2 and 5.3). (As is known the system of all regular events is closed with respect to operations $\cup$, $\cap$.) Nevertheless, there holds: if $T_1, \ldots, T_k \in \mathscr{T}_{[1,1]}$, then $T_1 \cup \ldots \cup T_k \in \mathscr{T}_{[k,1]}$ (and $T_1 \cap \ldots \cap T_k \in \mathscr{T}_{[k,1]}$, too). (It is easily possible to generalize this notion.)

1.8. If a 0-Turing machine (i.e. a finite automaton) $\mathfrak{M}$ has m states and every two (distinct) words from a set U are distinguishable on $\mathbf{T}(\mathfrak{M})$, then evidently card $U \leqq m$. Thus, if $T \subseteq N_0^\infty$ and U_∞ is an infinite event and its every two words are distinguishable on T**), then $T \notin \mathscr{T}_0$. (Compare e.g. with the well-known example $T = \{0^n 1 0^n \mid n \in N_0\}$ where we may take $U_\infty = \{0\}^\infty$.)

1.9. There holds $\mathscr{T}_0 \subseteq \mathscr{T}_{[1,1]} \subseteq \mathscr{T}_{[2,1]} \subseteq \mathscr{T}_{[1,2]}$***), $\mathscr{T}_{[1,1]} \subseteq \mathscr{T}_{\square}$. Nevertheless, $\mathscr{T}_{[1,1]} \neq \mathscr{T}_0$ (e.g. $\{0^n 1 0^n \mid n \in N_0\} \in \mathscr{T}_{[1,1]} - \mathscr{T}_0$), $\mathscr{T}_{[1,1]} \neq \mathscr{T}_{[2,1]}$ (for the first time it was proved by Rabin [1]), $\mathscr{T}_{[1,1]} \neq \mathscr{T}_{\square}$ (moreover $(\mathscr{T}_{[2,1]} \cap \mathscr{T}_{\square}) - \mathscr{T}_{[1,1]} \neq \emptyset$ — see Example 5.3).

2. $[d, f]$-BOTTLENECK SQUARES

Let in 2.0 — 2.7 $\mathfrak{M}$ be a $[1, 1]$-Turing machine with m states, let Σ and t belong to $\mathfrak{M}$.

2.0. Definition. A square B (of the tape of $\mathfrak{M}$) we call a $[d, f]$-*bottleneck square of* an (ordered) *pair* $[u, v]$ (on $\mathfrak{M}$) if $d \geqq 1$, f is a real function on N_0, $[u, v] \in \Sigma^\infty \times \Sigma^\infty$ and there holds:

(1) $B \in t(uv) - t(u)$,

(2) under input uv B is covered at most d times,

(3) if B lies between $t(u)$ and the end E of $t(uv)$, then the length of the segment with end squares B and E (including both) is greater than $f(l(u))$.

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*) See, e.g. [2].

**) Evidently, such set U_∞ exists if and only if the decomposition on $[\mathfrak{B}(T)]^\infty$ which is induced by the relation of equivalence "to be not distinguishable on T" has infinitely many classes.

***) Generally there is $\mathscr{T}_{[p,h]} \subseteq \mathscr{T}_{[1,ph]}$ (see [3], pp. 483—484).

2.1. If B is a $[d, f]$-bottleneck square of a pair $[u, v]$ (on $\mathfrak{M}$) and $d' \geqq [d]$, $g \leqq f$ (i.e. g is a real function on N_0 such that $g(k) \leqq f(k)$ for every $k \in N_0$), then evidently B is a $[d', g]$-bottleneck square of $[u, v]$.

2.2. Let $[u, v]$ have a $[d, f]$-bottleneck square B, let B be right (left) of $t(u)$ and let E be the right (left) end of $t(uv)$. Let C be the right (left) neighbouring square of B. A *passage* (of $\mathfrak{M}$) *through* B is every tact from B to C and also every tact from C to B (under uv), the state (of $\mathfrak{M}$) during the passage is the state in which $\mathfrak{M}$ is after the passage. Let s_i ($i = 1, \ldots, r$) be the states during all passages through B (s_1 corresponding to the first, s_2 to the second etc.). By the *scheme* of B (and of $[u, v]$) we mean the $(r + 1)$-tuple $[e, s_1, \ldots, s_n]$, where $e = 1$ ($e = -1$). (See [1].)

As B is (under uv) covered at every even passage and besides at least once (before the first passage), B is covered at least $\frac{r+1}{2}$ times and $\frac{r+2}{2}$ times for r odd and for r even, respectively; thus, $r \leqq 2[d] - 1$ and $r \leqq 2[d] - 2$, respectively.

2.3. Let $U, V \subseteq \Sigma^\infty$ and for every $u \in U$ let there exist $v_u \in V$ such that the pair $[u, v_u]$ has some $[d, f]$-bottleneck square B_u. Let B_u lie between $t(u)$ and the end E_u of $t(uv_u)$, let $\bar{v}_u$ be the beginning of v_u such that after input $u\bar{v}_u$ the head comes at first on E_u. Evidently, B_u is also a $[d, f]$-bottleneck square of $[u, \bar{v}_u]$ and the corresponding scheme of B_u and $[u, \bar{v}_u]$ has an odd number of passages.

The number of schemes corresponding to all $[d, f]$-bottleneck squares (of all input pairs) with an odd number of passages is not greater than $D = 2m + 2m^3 + \ldots + 2m^{2[d]-1}$. Now, let be $\operatorname{card} U > D$, then there exist $u_1, u_2 \in U$, $u_1 \neq u_2$ such that the schemes of B_{u_1}, B_{u_2} (and $[u_1, \bar{v}_{u_1}]$, $[u_2, \bar{v}_{u_2}]$, respectively) are the same, we denote them $[e, s_1, \ldots, s_n]$ (r is odd). For shortness, let us denote $v_{(k)} = \bar{v}_{u_k}$ ($k = 1, 2$).

We may write

$$v_{(k)} = v_{(k)}^{(0)} v_{(1)}^{(k)} \ldots v_{(k)}^{(r)}$$

where $v_{(k)}^{(j)}$ ($j = 0, \ldots, r$; $k = 1, 2$) are the words such that under the last letter of input $u_k v_{(k)}^{(0)} \ldots v_{(k)}^{(j-1)}$ ($j = 1, \ldots, r$) the jth passage is performed. Evidently, $v_{(k)}^{(j)} \neq \Lambda$ for $j = 0, \ldots, r$, $k = 1, 2$.

Now, let be $v_k^{\langle k \rangle} = v_{(k)}$, $v_{3-k}^{\langle k \rangle} = v_{(3-k)}^{(0)} v_{(k)}^{(1)} v_{(3-k)}^{(2)} \ldots v_{(k)}^{(r)}$.

It is easily proved that $s(u_1 v_1^{\langle k \rangle}) = s(u_2 v_2^{\langle k \rangle})$ and that for $e = 1$ ($e = -1$) under inputs $u_1 v_1^{\langle k \rangle}$, $u_2 v_2^{\langle k \rangle}$ there holds: the nearest $[f(l(u_k))]$ squares from the end positions of the head (including) to the left (to the right) are in both these cases printed in the same manner*) and on all squares

*) But on the squares B_{u_1}, B_{u_2} different working letters may be.

to the right (to the left) from the end positions there are only blank symbols (noughts). Thus, if $\tilde{u} \in \Sigma^\infty$ and $l(\tilde{u}) \leqq f(l(u_k))$, it is $s(u_1 v_1^{\langle k \rangle} \tilde{u}) = s(u_2 v_2^{\langle k \rangle} \tilde{u})$.

Hence, the important assertion (Lemma 2.6) follows. For its clearer formulation we give first the following definition.

2.4. Definition. For $V \subseteq N_0^\infty$ we define

$\mathscr{D}(V) = \{[v_1, v_2] \mid v_1, v_2 \in N_0^\infty$, there exist an odd number r and words $v_{(k)}^{(j)} \in N_0^\infty$ $(k = 1, 2;\ j = 0, \ldots, r+1)$ such that $v_{(k)}^{(j)} \neq \Lambda$ for $k = 1,2$ and $j = 0, \ldots, r$, $v_{(k)}^{(0)} \ldots v_{(k)}^{(n+1)} \in V$ $(k = 1, 2)$, $v_1 = v_{(1)}^{(0)} \ldots v_{(1)}^{(r)}$, $v_2 = v_{(2)}^{(0)} v_{(1)}^{(1)} v_{(2)}^{(2)} \ldots v_{(1)}^{(r)}\}$,

$\mathscr{D}_0(V) = \{v \mid v \in N_0^\infty$, there exists $v' \in N_0^\infty$ such that $[v, v'] \in \mathscr{D}(V)$ or $[v', v] \in \mathscr{D}(V)\}$.

2.5. There is $\mathscr{D}(V) \subseteq \mathscr{D}_0(V) \times \mathscr{D}_0(V)$ (it is $\mathscr{D}_0(V) = \mathrm{pr}_1 \mathscr{D}(V) \cup \cup\ \mathrm{pr}_2 \mathscr{D}(V)$). Let be $\emptyset \neq V \subseteq N_0^\infty$, $L = \sup_{v \in V} l(v)$ $(\leqq \infty)$, then $\mathscr{D}(V) \neq \emptyset$ if and only if $L \geqq 2$ and it is easily seen that for $[v_1, v_2] \in \mathscr{D}(V)$ there hold inequalities $2 \leqq l(v_1) \leqq L$, $2 \leqq l(v_2) \leqq 2(L-1)$. Moreover, for $L \geqq 2$ $\sup_{v \in \mathscr{D}_0(V)} l(v) = 2(L-1)$.

2.6. Lemma. Let be $U, V \subseteq \Sigma^\infty$, $d \geqq 1$ and let f be a real function on N_0. Let us choose $D = 2(m + m^3 + \ldots + m^{2[d]-1})$. Let card $U > D$ and let for every $u \in U$ there exist $v \in V$ such that $[u, v]$ has a $[d, f]$-bottleneck square. Then there exist $u_1, u_2 \in U$, $u_1 \neq u_2$ and $v_i^{\langle k \rangle}$ $(k, i = 1, 2)$ such that for every $k = 1, 2$ $[v_k^{\langle k \rangle}, v_{3-k}^{\langle k \rangle}] \in \mathscr{D}(V)$ and the words $u_1 v_1^{\langle k \rangle}$, $u_2 v_2^{\langle k \rangle}$ are not $f(l(u_k))$-distinguishable on $\mathbf{T}(\mathfrak{M})$.

2.7. The preceding assertion (and also all assertions based on it) is possible to strengthen (e.g. we may add $l(v_i^{\langle k \rangle}) > f((l(u_k))]$.

2.8. Lemma. *Let T be an event. Let there exist sequences $\{U_n\}$, $\{V_n\}$ of events, a sequence $\{d_n\}$, $d_n \geqq 1$ and a sequence $\{f_n\}$ of real functions on N_0 such that the next conditions are satisfied:*

(a) $\lim_{n \to \infty} (\text{card } U_n)^{\frac{1}{d_n}} = \infty$,

(b) if $n \in N$, $u_1, u_2 \in U_n$, $u_1 \neq u_2$ and if $v_i^{\langle k \rangle}$ $(k, i = 1, 2)$ are such that $[v_k^{\langle k \rangle}, v_{3-k}^{\langle k \rangle}] \in \mathscr{D}(V_n)$ for $k = 1, 2$, then either for $k = 1$ or for $k = 2$ the words $u_1 v_1^{\langle k \rangle}$, $u_2 v_2^{\langle k \rangle}$ are $f_n(l(u_k))$-distinguishable on T,

(c) if $\mathfrak{M}$ is a [1,1]-Turing machine recognizing T, then there exists $C_{\mathfrak{m}} \geqq 1$ such that for almost all n (from N) there holds: for every $u \in U_n$ there exists $v \in V_n$ such that $[u, v]$ has a $[C_{\mathfrak{m}} d_n, f_n]$-bottleneck square (on $\mathfrak{M}$).

Then $T \notin \mathscr{T}_{[1,1]}$.

Proof. Let the suppositions be satisfied and yet there is a [1,1]-Turing machine $\mathfrak{M}$ such that $\mathbf{T}(\mathfrak{M}) = T$ (so, $\mathfrak{B}(T) \subseteq \Sigma_{\mathfrak{m}}$). We shall denote $m = \text{card } S_{\mathfrak{m}}$ and $D_n = 2(m + m^3 + \ldots + m^{2[C_{\mathfrak{m}} d_n]-1})$. There holds

$D_n < (m+1)^{2C_{\mathrm{m}} d_n}$. Let $n_{\mathrm{m}} \in N$ be such that for every $n \geqq n_{\mathrm{m}}$ there holds the assertion from the condition (c). From (a) there follows that there exists $n_0 \geqq n_{\mathrm{m}}$ such that $(\mathrm{card}\ U_{n_0})^{\frac{1}{d_{n_0}}} \geqq (m+1)^{2C_{\mathrm{m}}}$. Thus, $D_{n_0} < (m+1)^{2C_{\mathrm{m}} d_{n_0}} \leqq \mathrm{card}\ U_{n_0}$. According to (c), the suppositions of Lemma 2.6 are satisfied (for $U_{n_0}, V_{n_0}, C_{\mathrm{m}} \cdot d_{n_0}, f_{n_0}$). From this and from condition (b) there follows a contradiction.

2.9. From the proof of 2.8 there follows that instead of condition (a) we may take only condition $\sup_{n \in N} (\mathrm{card}\ U_n)^{\frac{1}{d_n}} = \infty$; of course, this is not an improvement on Lemma 3.5 (if there were satisfied new suppositions, there would be for suitable subsequences also satisfied the former suppositions).

2.10. The preceding Lemma gives sufficient conditions for non-recognition of a set by [1,1]-Turing machines, but these conditions are not suitable for direct application on given events with regard to the character of (c). Suitable sufficient conditions for satisfaction of (c) are established from sect. 3. (Of course, it is possible to prove also more general assertions of type 2.8; the above mentioned Lemma we have quoted with regard to its use in proof of the main theorem.)

3. THE EXISTENCE OF CERTAIN BOTTLENECK SQUARES

3.1. Lemma. *Let $\mathfrak{M}$ be a [1,1]-Turing machine, let $b > 0$, $b' > 0$, $K > 0$, let $d = bK\left(1 + \frac{1}{b'}\right) + 1$. Then for every $u, v \in \Sigma_{\mathrm{m}}^{\infty}$ such that $b' . l(u) \leqq l(v) \leqq b . \lambda_{\mathrm{m}}(uv)$ no more than $\frac{\lambda_{\mathrm{m}}(uv)}{K}$ squares of $t_{\mathrm{m}}(uv)$ are covered more than d times in the computation under uv.*

Proof. Let the suppositions be satisfied for some u, v and let a d^* be such that more than $\frac{\lambda_{\mathrm{m}}(uv)}{K}$ squares of $t_{\mathrm{m}}(uv)$ were covered more than d^* times. Under input uv $l(u) + l(v) + 1$ coverings occur on the whole; because at least $\left[\frac{\lambda_{\mathrm{m}}(uv)}{K} + 1\right]$ squares were covered at least $[d^* + 1]$ times there holds $\left(1 + \frac{1}{b'}\right) l(v) \geqq l(u) + l(v) \geqq \left[\frac{\lambda_{\mathrm{m}}(uv)}{K} + 1\right] \times$ $\times [d^* + 1] - 1 > \frac{\lambda_{\mathrm{m}}(uv)}{K} [d^*] > \frac{l(v)}{Kb} (d^* - 1)$. Therefore, $d^* < Kb \Big(1 +$

$+\frac{1}{b'}\Big) + 1$. Thus, for every $d \geqq Kb\left(1+\frac{1}{b'}\right)+1$ no more than $\frac{\lambda_{\mathrm{m}}(uv)}{K}$ squares of $t_{\mathrm{m}}(uv)$ are covered more than d times.

3.2. Lemma. *Let $b > 0$, $R \geqq 1, f(k) = R(k+2)$. There holds:* if $\mathfrak{M}$ *is a [1,1]-Turing machine and $u, v \in \Sigma_{\mathrm{m}}^{\infty}$ are such that $b(2R+3)\,(l(u)+2) \leqq \leqq l(v) \leqq b\lambda_{\mathrm{m}}(uv)$, then the pair $[u, v]$ has a $[b(2R+3)+2, f]$-bottleneck square.*

Proof. Let the suppositions be satisfied. Let us divide $t_{\mathrm{m}}(uv)$ into five disjoint segments, we denote them (from left to right) as $J_1, \ldots, J_5$, such that J_1 and J_5 have the lengths at least $R\left[\frac{\lambda_{\mathrm{m}}(uv)}{2R+3}\right]$ and J_2, J_3, J_4 have the lengths at least $\left[\frac{\lambda_{\mathrm{m}}(uv)}{2R+3}\right]$. There holds $\lambda_{\mathrm{m}}(u) \leqq l(u)+1 \leqq$

$$\leqq \frac{l(v)}{b(2R+3)} - 1 < \left[\frac{l(v)}{b(2R+3)}\right] \leqq \left[\frac{\lambda_{\mathrm{m}}(uv)}{2R+3}\right] \leqq R\left[\frac{\lambda_{\mathrm{m}}(uv)}{2R+3}\right],$$

hence $t_{\mathrm{m}}(u)$ is coincidental with one of the segments $J_1, \ldots, J_5$ or with two neighbouring ones and in each of this five segments there exists a square which does not belong to $t_{\mathrm{m}}(u)$. Thus, there exists a segment $J \subseteq t_{\mathrm{m}}(uv)$ the length of which is at least $\left[\frac{\lambda_{\mathrm{m}}(uv)}{2R+3}\right]+1$ such that $J \cap (J_1 \cup J_5 \cup t_{\mathrm{m}}(u)) = \emptyset$. Let us choose $K = 2R+3$, $b' = bK$, $d = b\left(1+\frac{1}{b'}\right)K + 1 = b(2R+3)+2$, then (see Lemma 3.1) no more than $\frac{\lambda_{\mathrm{m}}(uv)}{2R+3}$ squares of $t_{\mathrm{m}}(uv)$ are covered more than d times, thus, there exists a square $B \in J \subseteq t_{\mathrm{m}}(uv) - t_{\mathrm{m}}(u)$ which is covered no more than d times. If E is the end of $t_{\mathrm{m}}(uv)$ between which and $t_{\mathrm{m}}(u)$ B lies, then between B and E (including both) are at least $R\left[\frac{\lambda_{\mathrm{m}}(uv)}{2R+3}\right] + 1$ squares, but $R\left[\frac{\lambda_{\mathrm{m}}(uv)}{2R+3}\right] + 1 \geqq R(l(u)+2)+1 > R(l(u)+2) = f(l(u))$. Therefore, B is a $[b(2R+3)+2, f]$-bottleneck square of the pair $[u, v]$.

3.3. Lemma. *Let T be an event and let $\{U_n\}, \{V_n\}$ be sequences of events, let all V_n be nonempty and finite. Let there be satisfied the next conditions:*

(α) if $n \in N$, $u \in U_n$, $v_1, v_2 \in V_n$, $v_1 \neq v_2$, then uv_1, uv_2 are distinguishable on T,

$$(\beta)\ \liminf_{n\to\infty} (\text{card } V_n)^{\frac{1}{\max_{v\in V_n} l(v)}} > 1,^{*)}$$

$$(\gamma)\ \lim_{n\to\infty} \max_{v\in V_n} l(v) = \infty.$$

Then there holds:

if $\mathfrak{M}$ is a [1,1]-Turing machine recognizing T, then there exists a real number $b_{\mathfrak{M}}$ such that for every $n \in N$, $u \in U_n$ there exists $v \in V_n$ such that $l(v) \leqq b_{\mathfrak{M}}\lambda_{\mathfrak{M}}(uv)$.

Proof. Let $\mathfrak{M}$ be a [1,1]-Turing machine with card $S_{\mathfrak{M}} = m$ and card $W_{\mathfrak{M}} = p$ such that $\mathbf{T}(\mathfrak{M}) = T$. Let us denote $\mu_n = \text{card } V_n$ and $k_n(u) = \max_{v\in V_n} \lambda_{\mathfrak{M}}(uv)$ (for $u \in U_n$). According to (α) and to 1.4 there follows that for every $u \in U_n$ the words from the set $\{uv \mid v \in V_n\}$ have mutually distinct codings (i.e. they have exactly μ_n codings) and — because the lengths of work spaces in these codings are not greater than $k_n(u)$ — there holds the inequality $\mu_n \leqq (p+1)^{k_n(u)} . k_n(u) . m$**). So, $\frac{\mu_n}{m} < [2(p+1)]^{k_n(u)}$ and, according to (β) and (γ), there exist $c_{\mathfrak{M}} > 0$ and $n_{\mathfrak{M}} \in N_0$ such that $(2(p+1))^{c_{\mathfrak{M}} \max_{v\in V_n} l(v)} \leqq \frac{\mu_n}{m}$ for $n \geqq n_{\mathfrak{M}}$. Hence, $c_{\mathfrak{M}} \max_{v\in V_n} l(v) < k_n(u)$ for every $n \geqq n_{\mathfrak{M}}$ and $u \in U_n$. For every $1 \leqq n_1 \leqq n_{\mathfrak{M}}$ there are $\max_{v\in V_{n_1}} l(v) \leqq [\max_{1\leqq n\leqq n_{\mathfrak{M}}} \max_{v\in V_n} l(v)]. k_{n_1}(u)$. Thus, for every $n \in N_0$ and $u \in U_n$ there holds the inequality $\max_{v\in V_n} l(v) \leqq b_{\mathfrak{M}}\, k_n(u)$, where

$$b_{\mathfrak{M}} = \max\left(\frac{1}{c_{\mathfrak{M}}},\ \max_{v\in V_1\cup\ldots\cup V_{n_{\mathfrak{M}}}} l(v)\right).$$

Consequently, if $n \in N_0$ and $u \in U_n$, then there exists $v \in V_n$ such that $\lambda_{\mathfrak{M}}(uv) = k_n(u)$, i.e. $l(v) \leqq b_{\mathfrak{M}}\lambda_{\mathfrak{M}}(uv)$ holds for this v.

3.4. Theorem 1. *Let T be an event. Let there exist sequences $\{U_n\}, \{V_n\}$ of nonempty finite events and let there exist a sequence $\{R_n\}$ of real numbers such that $R_n \geqq 1$ for all n and there are satisfying the conditions (α), (β) from Lemma 3.3 and also the condition*

$$(\delta)\ \lim_{n\to\infty} \frac{\min_{v\in V_n} l(v)}{R_n(1 + \max_{n\in V_n} l(u))} = \infty.$$

*) According to the condition (γ) there holds $\max_{v\in V_n} l(v) = 0$ only for finite many n; the condition (β) is equivalent to the condition: there exists $q > 1$ such that the inequality card $V_n \geqq q^{q \max_{v\in V_n} l(v)}$ holds for almost all n.

**) Moreover, a stronger inequality $\mu_n \leqq p^{k_n(u)}\, k_n(u) . m$ is satisfied.

Let $f_n(k) = (k+2)\,R_n$.

Then there holds:

if $\mathfrak{M}$ *is a* [1,1]*-Turing machine recognizing the set* T, *then there exists a real number* $C_{\mathfrak{m}} \geqq 1$ *such that for almost all* n *(from* N*) there holds:*

if $u \in U_n$, *then there exists* $v \in V_n$ *such that the pair* $[u, v]$ *has a* $[C_{\mathfrak{m}} R_n, f_n]$*-bottleneck square.*

Proof. Let $\mathfrak{M}$ be a [1,1]-Turing machine such that $\mathbf{T}(\mathfrak{M}) = T$. According to ($\delta$) there is $\max_{v \in V_n} l(v) \to \infty$. Because ($\alpha$), ($\beta$) are satisfying, there exists b such that for every $n \in N$, $u \in U_n$ there exists $v \in V_n$ such that $l(v) \leqq b\,\lambda_{\mathfrak{m}}(uv)$ (see Lemma 3.3). Moreover, there is $b \geqq 1$: for every $\varepsilon > 0$ there exists (according to (δ)) $n_\varepsilon \in N$ such that $l(v) \geqq \frac{1}{\varepsilon} R_n(l(u) + 1)$ for every $u \in U_{n_\varepsilon}$, $v \in V_{n_\varepsilon}$; so, if $l(v) \leqq b\,\lambda_{\mathfrak{m}}(uv)$, then $l(v) \leqq b(l(u) + l(v) + 1) \leqq b\,l(v)\,(1 + \varepsilon)$, thus, $1 \leqq (1 + \varepsilon)\,b$ for every $\varepsilon > 0$. Now, we choose $C_{\mathfrak{m}} = 7b (\geqq 7)$, $R_n' = \frac{7R_n - 3}{2} - \frac{1}{b}$ (then $1 \leqq R_n \leqq R_n' < \frac{7}{2} R_n$), then $b(2R_n' + 3) + 2 = C_{\mathfrak{m}} \,.\, R_n$. From ($\delta$) easily follows that there exists $n_0 \in N$ such that for every $n \geqq n_0$ the inequation $b(2R_n' + 3)(\max_{u \in U_n} l(u) + 2) \leqq \min_{v \in V_n} l(v)$ is satisfied. Therefore, let $n \geqq n_0$ and let $u \in U_n$; there exists $v \in V_n$ such that $l(v) \leqq b\lambda_{\mathfrak{m}}(uv)$ and with respect to the above $b(2R_n' + 3)\,(l(u) + 2) \leqq l(v)$. Thus, from Lemma 3.2 there follows that the pair $[u, v]$ has a $[b(2R_n' + 3) + 2,\ g_n]$ bottleneck square, where $g_n(k) = R_n'(k + 2) \geqq R_n(k + 2) = f_n(k)$. Therefore, this bottleneck square is also a $[C_{\mathfrak{m}} R_n,\ f_n]$-bottleneck square of $[u, v]$.

4. THE MAIN THEOREM

4.1. Theorem 2. (The main theorem.)

Let T *be an event. Let there exist sequences* $\{U_n\}$, $\{V_n\}$ *consisting of nonempty finite events and let there exist a sequence of real numbers* $\{R_n\}$, $R_n \geqq 1$ *(for all* $n \in N$*) such that there holds:*

$$(1)\ \lim_{n \to \infty} (\text{card } V_n)^{\frac{1}{\max_{v \in V_n} l(v)}} > 1\,,$$

$$(2)\ \lim_{n \to \infty} (\text{card } U_n)^{\frac{1}{R_n}} = \infty\,,$$

$$(3)\ \lim_{n\to\infty} \frac{\min\limits_{v\in V_n} l(v)}{R_n \max\limits_{n\in U_n} l(u)} = \infty,^{*)}$$

(4) if $n \in N_0$, $u \in U_n$, v_1, $v_2 \in V_n$, $v_1 \neq v_2$, then uv_1, uv_2 are distinguishable on T,

(5) if $n \in N$, u_1, $u_2 \in U_n$, $u_1 \neq u_2$ and if $v_i^{\langle k\rangle}(k,\ i = 1,\ 2)$ are such that $[v_k^{\langle k\rangle}, v_{3-k}^{\langle k\rangle}] \in \mathscr{D}(V_n)$ for $k = 1, 2$, then either for $k = 1$ or for $k = 2$ the words $u_1 v_1^{\langle k\rangle}$, $u_2 v_2^{\langle k\rangle}$ are $R_n(l(u_k) + 2)$ — distinguishable on T.

Then $T \notin \mathscr{T}_{[1,1]}$.

Proof. Theorem 2 immediately follows from Theorem 1 and from Lemma 2.8 if we choose $d_n = R_n$, $f_n(k) = R_n(k + 2)$.

4.2. As it is shown in sect. 5 the application of Theorem 2 need not be complicated (though the formulation of Theorem 2 at the first glance could corroborate the contrary). Especially, it is sufficient if instead of the condition (5) the stronger condition

(5′) if $n \in N_0$, u_1, $u_2 \in U_n$, $u_1 \neq u_2$, v_1, $v_2 \in \mathscr{D}_0(V_n)$, then the words u_1v_1, u_2v_2 are $R_n[2 + \min(l(u_1), l(u_2))]$-distinguishable on T

is satisfied. (In fact, its satisfaction is usually proved for v_1, v_2 from a suitable set containing the set $\mathscr{D}_0(V_n)$.)

5. EXAMPLES

The using of the main theorem can be illustrated on many interesting examples. We quote here only three, however, being important also from the theoretical point of view — they give the qualitative comparison of relative strengths of [1,1]- versus [2,1]-, [1,2]- and □-Turing machines (see 1.9). From this view, the first example is the weakest and the third the strongest.

5.1.0. Example "$w\S w_b$"**).

Let $\S \in N$***) and let Σ_0 be a finite subset of N_0, card $\Sigma_0 \geqq 2$, $\S \notin \Sigma_0$, let $\Sigma^{(1)} = \Sigma_0 \cup \{\S\}$. We define

$$T^{(1)} = \{w\S w_b \mid w \in \Sigma^{(1)\infty},\ w_b \text{ is a beginning of } w\}.$$

5.1.1. Let be u_1, u_2, v_1, $v_2 \in \Sigma_0^\infty$, $u_1 \neq u_2$, $l(u_1) = l(u_2)$. Then the

*) Evidently, $\max\limits_{u\in U_n} l(u) = 0$ only for finite many n (see condition (2)).

**) This example is mentioned by P. Strnad [4].

***) Similarly as in the following (at the symbols α, β) we choose the notation according to the references ([1], [3]).

words u_1v_1, u_2v_2 are $(2 + l(u_k))$-distinguishable on $T^{(1)}$ (in fact, they are $(1 + l(u_k))$-distinguishable) for $k=1, 2$: $u_kv_k\S u_k \in T^{(1)}$, $u_{3-k}v_{3-k}\S u_k \in T^{(1)}$ (i.e. they are $[2 + \min(l(u_1), l(u_2))]$-distinguishable on $T^{(1)}$). Now, let u, v_1, $v_2 \in \Sigma_0^\infty$, $v_1 \neq v_2$, $l(v_1) = l(v_2)$. Then the words uv_1, uv_2 are distinguishable on $T^{(1)}$ (e.g. $uv_1\S uv_1 \in T^{(1)}$, $uv_2\S uv_1 \notin T^{(1)}$). Hence, we may take (for $n \in N$) $U_n = \{u \mid u \in \Sigma_0^\infty,\ l(u) = n\}$ and $V_n = \{v \mid v \in \Sigma_0^\infty,\ l(v) = n^2\}$, $R_n = 1$. As $\mathscr{D}_0(V_n) \subseteq \Sigma_0^\infty$ for all n, from the preceding follows that the conditions (4) and (5) (see the main theorem) and also the conditions (1), (2), (3) are satisfied. Thus, according to the main theorem,

$$T^{(1)} \notin \mathscr{T}_{[1,1]}.$$

5.1.2. Evidently, $T^{(1)} \in \mathscr{T}_{[1,2]}$. If card $\Sigma_0 = 1$, then $T^{(1)} \in \mathscr{T}_{[1,1]}$. Always (when $\Sigma_0 \neq \emptyset$) $T^{(1)} \notin \mathscr{T}_0$ (see 1.8; it is possible to choose $U_\infty = \Sigma_0^\infty$).

5.2.0. Rabin's example*).

Let $\alpha, \beta \in N_0$, $\alpha \neq \beta$, let Σ_1, Σ_2 be disjoint finite subsets of N_0 which do not contain α, β, let $\Sigma_1 \neq \emptyset$, card $\Sigma_2 \geqq 2$. We choose $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \{\alpha, \beta\}$ and we define

$$\begin{aligned} T_1^{(2)} &= \{uv\alpha u^{-1} \mid u \in \Sigma_1^\infty,\ v \in \Sigma_2^\infty\},\\ T_2^{(2)} &= \{uv\beta v^{-1} \mid u \in \Sigma_1^\infty, v \in \Sigma_2^\infty\},\\ T^{(2)} &= T_1^{(2)} \cup T_2^{(2)}. \end{aligned}$$

5.2.1. Let be u_1, $u_2 \in \Sigma_1^\infty$, v_1, $v_2 \in \Sigma_2^\infty$, $u_1 \neq u_2$. Then the words u_1v_1, u_2v_2 are $((2 + l(u_k))$-distinguishable on $T^{(2)}$ for $k = 1, 2$: $u_kv_k\alpha u_k^{-1} \in T^{(2)}$, $u_{3-k}v_{3-k}\alpha u_k^{-1} \notin T^{(2)}$. Now, let be $u \in \Sigma_1^\infty$, v_1, $v_2 \in \Sigma_2^\infty$, $v_1 \neq v_2$. Then the words uv_1, uv_2 are distinguishable on $T^{(2)}$ (e.g. $uv_1\beta v_1^{-1} \in T^{(2)}$, $uv_2\beta v_1^{-1} \notin T^{(2)}$). Let us choose $U_n = \{u \mid u \in \Sigma_1^\infty,\ l(u) \leqq n\}$, $V_n = \{v \mid v \in \Sigma_2^\infty,\ l(v) = n^2\}$, $R_n = 1$. As $\mathscr{D}_0(V_n) \subseteq \Sigma_2^\infty$ for all n, from the preceding follows that the conditions (4) and (5) and also the conditions (1), (2), (3) are satisfied. Thus,

$$T^{(2)} \notin \mathscr{T}_{[1,1]}.$$

5.2.2. It is easily to be seen that $T_1^{(2)}$, $T_2^{(2)} \in \mathscr{T}_{[1,1]}$, so $T^{(2)} = T_1^{(2)} \cup T_2^{(2)} \in \mathscr{T}_{[2,1]} \subseteq \mathscr{T}_{[1,2]}$ (see 1.7). Rabin in [1] shows that $T^{(2)} \in \mathscr{T}_{[1,1]}$ for card $\Sigma_1 =$ card $\Sigma_2 = 1$. From 1.8 there follows that $T_1^{(2)} \notin \mathscr{T}_0$ (if $\Sigma_1 \neq \emptyset$), $T_2^{(2)} \notin \mathscr{T}_0$ (if $\Sigma_2 \neq \emptyset$), $T^{(2)} \notin \mathscr{T}_0$ (if $\Sigma_1 \cup \Sigma_2 \neq \emptyset$).

5.3.0. The main example.

For $x, y \in N_0$ we choose $f(x, y) = 2\left[\dfrac{x}{4}\right] + \left[\dfrac{y}{4\left(1 + \left[\dfrac{x}{4}\right]\right)}\right] + 1.$

*) See [1]; here we consider also the case card $\Sigma_1 = 1$, card $\Sigma_2 \geqq 2$ (and we take as u, v also the empty word — but that is not an important difference).

For $x \in N_0$ $4\left(1 + \left[\frac{x}{4}\right]\right) \geqq x + 1$ holds. Thus, $1 \leqq f(x, y) \leqq \frac{x}{2} + \frac{y}{x+1} + 1$ for all $x, y \in N_0$.

Let Σ be the set from 5.2.0 (card $\Sigma_2 \geqq 2$). We define

$$T_1^{(3)} = \{uv\alpha^k u^{-1} \mid u \in \Sigma_1^\infty,\ v \in \Sigma_2^\infty,\ k \geqq f(l(u), l(v))\},$$
$$T_2^{(3)} = \{uv\beta v^{-1} \mid u \in \Sigma_1^\infty,\ v \in \Sigma_2^\infty\},$$
$$T^{(3)} = T_1^{(3)} \cup T_2^{(3)}.$$

5.3.1. For $n \in N$ we choose $U_n = \{u \mid u \in \Sigma_1^\infty,\ n \leqq l(u) \leqq 2n\}$, $V_n = \{v \mid v \in \Sigma_2^\infty,\ l(v) = n^2\}$, $T_n = 5$. Thus, the conditions (1), (2), (3) of the main theorem are satisfied. For $n \in N$, $u \in U_n$, $v_1, v_2 \in V_n$, $v_1 \neq v_2$ the words uv_1, uv_2 are distinguishable on $T^{(3)}$ ($uv_1\beta v_1^{-1} \in T^{(3)}$, $uv_2\beta v_1^{-1} \notin T^{(3)}$), i.e. the condition (4) is satisfied. For $n \in N_0$, $u_1, u_2 \in U_n$, $u_1 \neq u_2$, $v_1, v_2 \in \mathscr{D}_0(V_n)$ there holds $v_1, v_2 \in \Sigma_2^\infty$, $l(v_1), l(v_2) \leqq 2(n^2 - 1)$ (see 2.5). Let us choose $\tilde{u} = \alpha^k u_1^{-1}$, where $k = f(l(u_1), l(v_2))$, then $u_1 v_1 \tilde{u} \in T^{(3)}$, $u_2 v_2 \tilde{u} \notin T^{(3)}$ and $l(\tilde{u}) = l(u_1) + k \leqq l(u_1) + \frac{l(u_1)}{2} + \frac{l(v_1)}{1 + l(u_1)} + 1 \leqq 1 + 3n + \frac{2(n^2 - 1)}{1 + n} = 5n - 1 < R_n[2 + \min(l(u_1), l(u_2))]$; the condition (5) is satisfied. Thus,

$$T^{(3)} \notin \mathscr{T}_{[1,1]}.$$

5.3.2. In 5.1.1 and 5.2.1 it was possible as V_n to choose the set of all words with length n^r (on the corresponding alphabet) for any $r \in N$, $r \geqq 2$. In 5.3.1 this is not possible.

5.3.3. The idea of the main example is to be seen in the proofs 5.3.4, 5.3.5, the choice of the set $T^{(3)}$ was performed such that the set is "similar" to the set $T^{(2)}$ (even $T_2^{(3)} = T_2^{(2)}$) and the function $f(x, y)$ is, as far as possible, simple (also for the price that the $\square$-Turing machine recognizing $T^{(3)}$ would be more complicated).

5.3.4. Lemma. $T^{(3)} \in \mathscr{T}_\square$.

Proof. We describe the idea of the construction of a suitable $\square$-Turing machine*) $\mathfrak{M}^*$ recognizing the set $T^{(3)}$. The square of its tape we denote with pairs of integers**) (analogically as points of the plane — $[x, y]$ design the square lying in the xth column and in the y th line), the initial square is [0,0]. If at first an input word $u \in \Sigma_1^\infty$ comes, $\mathfrak{M}^*$ prints it from

*) A $\square$-Turing machine is defined quite the same as a [1,1]- Turing machine in sect. 1, only to set P we add two further elements (designating the moves up and down).

**) Let us note that in our construction only squares from the set $\{[i, j] \mid i, j \in N_0\}$ will be used.

left to right in squares of the zeroth line such that on every of these squares four subsequent letters of the word u are always printed, of course, with regard to their order). So, after the $(4k)$th tact the head passes from the square $[k-1,0]$ to $[k,0]$, during the other tacts the head stops with the exception of the first and the second tact, when it moves to [0,1] and back to [0,0], at which it signs both these squares by some marker. Thus, during the input u the head passed through the squares [0,0], ..., $\left[\left[\frac{l(u)}{4}\right], 0\right]$. Let after u follow a $v \in \Sigma_2^\infty$. $\mathfrak{M}^*$ again prints the word v four subsequent letters on each square, first on squares in the first line (from the square $\left[\left[\frac{l(u)}{4}\right], 1\right]$ up to $[0, 1]$), then in the second line $\left(\text{from } [0,2] \text{ up to } \left[\left[\frac{l(u)}{4}\right], 2\right]\right)$ etc. Moreover, at the entering on the rth line $(r \geqq 1)$ $\left(\text{i.e. on the square } [\lambda_r, r], \text{ where } \lambda_r = \left[\frac{l(u)}{4}\right]\right.$ $\left.\text{for } r \text{ odd and } \lambda_r = 0 \text{ for } r \text{ even}\right)$ the head uses two tacts for a marking of the square $[\lambda_r, r+1]$ (so that at the printing on the $(r+1)$th line $\mathfrak{M}^*$ may discern the "end" of this line and also of the square $[\lambda_r, r]$ (with regard to the "back moving", see in the following). With fours of letters of the word v $r_0 = \left[\frac{l(v)}{4\left(1 + \left[\frac{l(u)}{4}\right]\right)}\right]$ lines are fully occupied on the whole.

Now we distinguish two cases:

a) After uv there comes $\alpha^k u'$ ($k \in N$, $u' \in \Sigma_1^\infty$). In this case at first the head exactly during $2\left[\frac{l(u)}{4}\right] + 1$ tacts goes to the square $\left[\left[\frac{l(u)}{4}\right], r_0\right]$ (the idea for the construction of $\mathfrak{M}^*$: in the (r_0+1)th line at the first α the head designs its position Q somehow and moves directly to the left end $[0, r_0+1]$ and from there back, at which in reaching again Q it starts to move with half speed to $\left[\left[\frac{l(u)}{4}\right], r_0+1\right]$ and from there after the following tact to $\left[\left[\frac{l(u)}{4}\right], r_0\right]$ — it is to be seen that this is possible to arrange in all cases and after the further r_0 tacts $\left(\text{in the } \left[\frac{l(u)}{4}\right]\text{th column}\right)$ the head is on the square $\left[\left[\frac{l(u)}{4}\right], 0\right]\Big)$. Thus, if there is $k < f(l(u), l(v))$, then the head does not reach the zeroth line and hence $\mathfrak{M}^*$ can reject the word $uv\alpha^k u'$. If there is $k \geqq f(l(u), l(v))$, then the head waits on the square $\left[\left[\frac{l(u)}{4}\right], 0\right]$ till α^k ends and then it compares u' with u^{-1}. Thus, $\mathfrak{M}^*$ decides whether $uv\alpha^k u'$ is in $T^{(3)}$ or not.

b) After uv there comes $\beta v'$ ($v' \in \Sigma_2^\infty$). In this case the head performs the "back moving" — it moves reverse as at the writing of the word v and compares v' with v^{-1}. Thus, $\mathfrak{M}^*$ decides whether $uv\beta v'$ is in $T^{(3)}$ or not.

It is easy to arrange that $\mathfrak{M}^*$ may not accept words of other forms than $uv\alpha^k u'$ and $uv\beta v'$. Thus, this $\mathfrak{M}^*$ recognizes the set $T^{(3)}$.

5.3.5. Lemma. $T_1^{(3)} \in \mathscr{T}_{[1,1]}$, $T_2^{(3)} \in \mathscr{T}_{[1,1]}$.

Proof. Evidently $T_2^{(3)} \in \mathscr{T}_{[1,1]}$. The set $T_1^{(3)}$ is recognized by a suitable [1,1]-Turing machine $\mathfrak{M}$ which simulates the work of the $\square$-Turing machine $\mathfrak{M}^*$ (see proof 5.3.4) such that to square $[i, j]$ of the tape of $\mathfrak{M}^*$ there corresponds the $(i + j)$th square of the tape of $\mathfrak{M}$, but such that the head does not print the letters of word v (under an input uv, $u \in \Sigma_1^\infty$, $v \in \Sigma_2^\infty$), $\mathfrak{M}$ only registers how the fours of its letters pass. (Moreover, here it is necessary to choose several markers.) Of course, $\mathfrak{M}$ rejects all words of other form than $uv\alpha^k u'$.

5.3.6. From Lemma 5.3.5 first of all follows that $T^{(3)} = T_1^{(3)} \cup T_2^{(3)} \in \mathscr{T}_{[2,1]} \subseteq \mathscr{T}_{[1,2]}$ (see 1.7). From the kind of construction of the machine $\mathfrak{M}$ (see proof 5.3.5) it is to be seen that there holds: if card $\Sigma_2 = 1$, then $T^{(3)} \in \mathscr{T}_{[1,1]}$. Again there is (see 1.8) $T_1^{(3)} \notin \mathscr{T}_0$ (if $\Sigma_1 \neq \emptyset$), $T_2^{(3)} \notin \mathscr{T}_0$ (if $\Sigma_2 \neq \emptyset$), $T^{(3)} \notin \mathscr{T}_0$ (if $\Sigma_1 \cup \Sigma_2 \neq \emptyset$).

Remark. With the comparison of the relative strengths of many-dimensional tapes real-time Turing machines the paper [5] deals.

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