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# CONSTRUCTION OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH SOLUTIONS OF PRESCRIBED PROPERTIES

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## I. INTRODUCTION

We shall deal with the second-order linear differential equation

(1) or (1*) $$y'' = q(t)\, y$$

where $q(t)$ or $q(t) < 0$ is a continuous function and such that the dif. equation (1) or (1*) is oscillatory in the whole interval $(-\infty, \infty)$, respectively. By solutions of this equation we understand such functions only that are continuous, their second derivative included, in the whole interval $(-\infty, \infty)$, in which they satisfy the equation (1) or (1*) and are not identically equal to zero. The condition $q(t) < 0$ implies that every solution of (1*) has just one extremant between its every two neighbouring zeros. If we consider the differential equation (1) or (1*) on the interval $j$ only, then we require the function $q$, or $q < 0$, respectively, to be continuous on $j$. If the interval $j$ is unbounded from the left or from the right, then the considered equation is required to be oscillatory from the left or from the right as well.

In what follows, $(a, b)$ denotes an open interval, $[a, b]$ a closed interval. Analogously we denote, e.g., an interval open from the left and closed from the right. Furthermore, by the continuity or derivative of a function at the right end-point of a closed interval we understand the continuity or the derivative of the function at this point from the left. Analogously, for the left end-point of an interval closed on the left. A function belongs to the class $C_I$, when it has continuous derivatives up to the order $n$ (the latter included) on the interval $I$. The derivative of the order zero of a function $f(t)$ is the function $f(t)$. Furthermore, to be brief, denote $C^n_{(-\infty,\infty)}$ by $C^n$.

If we are interested in some special properties of the solution of Eq. (1), e.g., in the position of the zeros or extremants, it is convenient to study the disperisions of Eq. (1) which O. Borůvka, in [2] and [4], defines as follows:

Let $t_0$ be an arbitrary number; $u$, $v$ are solutions of Eq. (1) or (1*) such that $u(t_0) = v'(t_0) = 0$. Then

$\varphi(t_0)$ is the first zero of the solution $u(t)$ of Eq. (1) lying on the right of $t_0$,

$\psi(t_0)$ is the first zero of the function $v'(t)$ of Eq. (1*) lying on the right of $t_0$,

$\chi(t_0)$ is the first zero of the function $u'(t)$ of Eq. (1*) lying on the right of $t_0$,

$\omega(t_0)$ is the first zero of the solution $v(t)$ of Eq. (1*) lying on the right of $t_0$.

The functions $\varphi$, $\psi$, $\chi$, $\omega$ are called basic central dispersions of the 1st, 2nd, 3rd, 4th kind, respectively, with respect to the differential equation (1) or (1*) (further more concisely: b.c. dispersions of the 1st, 2nd, 3rd, 4th kind).

In the same paper it has been proved that for the b.c. dispersion of the 1st kind $\varphi(t)$ of Eq. (1) there holds

$$(\varphi) \qquad \varphi \in C^3,\ \varphi'(t) > 0,\ \lim_{t\to\pm\infty} \varphi(t) = \pm\infty, \quad \varphi(t) > t,$$

for the b.c. dispersion of the 2nd kind $\psi(t)$ of Eq. (1*) there holds

$$(\psi) \qquad \psi \in C^1, \quad \psi'(t) > 0,\ \lim_{t\to\pm\infty} \psi(t) = \pm\infty, \quad \psi(t) > t,$$

for the b.c. dispersion of the 3rd kind $\chi(t)$ of Eq. (1*) there holds

$$(\chi) \qquad \chi \in C^1, \quad \chi'(t) > 0,\ \lim_{t\to\pm\infty} \chi(t) = \pm\infty, \quad \chi(t) > t,$$

for the b.c. dispersion of the 4th kind $\omega(t)$ of Eq. (1*) there holds

$$(\omega) \qquad \omega \in C^1, \quad \omega'(t) > 0,\ \lim_{t\to\pm\infty} \omega(t) = \pm\infty, \quad \omega(t) > t.$$

When dealing with the transformations of the solutions of the differential equation (1) on the solutions of the differential equation $y'' = q_1(t)\, y$ (where the cases $q_1(t) = q(t)$ or $q_1(t) = -1$ are of special interest), it is useful to notice the so called phases, defined by O. Borůvka [4] for the pair $(u, v)$ of independent solutions of the differential equation (1) in the following way:

$$\operatorname{tg} \alpha(t) = \frac{u(t)}{v(t)}, \quad \operatorname{tg} \beta(t) = \frac{u'(t)}{v'(t)},$$

where the continuous solutions $\alpha$, $\beta$, defined by these relations are called the 1st or the 2nd phase of the ordered pair $(u, v)$ of independent solutions of the differential equation (1), respectively. If, in what follows, it is not essential to which pair of solutions of Eq. (1) the 1st or the 2nd phase belongs, then we shall not mention it explicitly. In the same paper, on p. 237, O. Borůvka has derived the relation

$$(2) \qquad -\{\alpha, t\} - \alpha'^2 = -\frac{1}{2}\left(\frac{\alpha''}{\alpha'}\right)' + \frac{1}{4}\left(\frac{\alpha''}{\alpha'}\right)^2 - \alpha'^2 = q(t).$$

Further on, in [2] and [4], he has shown that for any 1st phase $\alpha(t)$ of Eq. (1) and any 2nd phase $\beta(t)$ of Eq. (1*) there holds:

$$(\alpha) \qquad \alpha \in C^3, \quad \alpha' \neq 0, \lim_{t \to \pm\infty} \alpha(t) \,.\, \operatorname{sign} \alpha'(t) = \pm\infty,$$

$$(\beta) \qquad \beta \in C^1, \quad \beta' \neq 0, \lim_{t \to \pm\infty} \beta(t) \,.\, \operatorname{sign} \beta'(t) = \pm\infty,$$

(the condition $q < 0$ being essential).

When we say, in what follows, that a certain function has the property $(\alpha)$ or $(\beta)$, ..., or $(\omega)$, we mean that this function satisfies the conditions mentioned above and denoted by $(\alpha)$ or $(\beta)$, ..., or $(\omega)$. If we require the function to have some of these properties on a certain interval $j$ only, it means that this function permits of an extension on the whole interval $(-\infty, \infty)$ so as to have this property in the whole interval $(-\infty, \infty)$.

In his seminar on differential equations held at the university of Brno, Prof. O. Borůvka has proved a number of relations between the just defined functions.[1] Let me introduce only those I shall need:

"Two functions $\alpha(t)$ or $\beta(t)$ in an interval $j$, $\alpha \in C_j^3$, $\beta \in C_j^1$, $\alpha' \neq 0$ in $j$ (and if $j$ is unbounded from the left or from the right the same goes for $\alpha(t)$) are then, only then, the 1st or the 2nd phase, respectively, of some pair $(u, v)$ of independent solutions of the equation (1), if for every $t \in j$ there holds

$$(3) \qquad \beta(t) = \alpha(t) + \operatorname{Arccotg}\left(\frac{1}{2}\left(\frac{1}{\alpha'}\right)'\right).$$

Arccotg denotes an arbitrary, convenient, branch of this function.

[1] Detailed studies of the differential equation (1) and the new results prof. O. Borůvka has arrived at will form the contents of his book on linear 2nd order differential equations which will be published by the Deutscher Verlag der Wissenschaften, Berlin.

If the relation (3) applies, then the functions

$$u(t) = \sin \alpha(t) \, . \, \exp \left\{ \int_{t_0}^{t} \alpha'(\sigma) \operatorname{cotg} [\beta(\sigma) - \alpha(\sigma)] \, \mathrm{d}\sigma \right\},$$

$$v(t) = \cos \alpha(t) \, . \, \exp \left\{ \int_{t_0}^{t} \alpha'(\sigma) \operatorname{cotg} [\beta(\sigma) - \alpha(\sigma)] \, \mathrm{d}\sigma \right\},$$

where $t_0 \in j$, possess the above mentioned properties and the function $q(t)$ in the equation (1) is determined by the relation (2)."

"Let $\alpha(t)$ be an arbitrary 1st phase of Eq. (1). Then, for the b.c. dispersion of the 1st kind $\varphi(t)$ of Eq. (1) in $(-\infty, \infty)$, there holds

(4) $$\alpha(\varphi(t)) = \alpha(t) + \pi \operatorname{sign} \alpha'."$$

"Let $\beta(t)$ be an arbitrary 2nd phase of Eq. (1*). Then, for the b.c. dispersion of the 2nd kind $\psi(t)$ of Eq. (1*) in $(-\infty, \infty)$, there holds

(5) $$\beta(\psi(t)) = \beta(t) + \pi \operatorname{sign} \beta'."$$

"Let $\alpha(t)$ and $\beta(t)$ be the 1st and the 2nd phase, respectively, of Eq. (1*) corresponding to the same pair $(u, v)$ of linear independent solutions of Eq. (1*). Let $\chi(t)$ and $\omega(t)$ be the b.c. dispersions of the 3rd and the 4th kind respectively with regard to Eq. (1*). Since $q(t) < 0$, there holds $\alpha'(t) \, . \, \beta'(t) > 0$ in the whole interval $(-\infty, \infty)$. Moreover, for

$$0 < \beta(t) - \alpha(t) < \pi, \quad \alpha' > 0, \quad \beta' > 0,$$

there holds

(6) and (7) $$\beta(\chi(t)) = \alpha(t) + \pi \quad \text{and} \quad \alpha(\omega(t)) = \beta(t),$$

for

$$0 < \beta(t) - \alpha(t) < \pi, \quad \alpha' < 0, \quad \beta' < 0,$$

there holds

(8) and (9) $$\beta(\chi(t)) = \alpha(t) \quad \text{and} \quad \alpha(\omega(t)) = \beta(t) - \pi,$$

for

$$0 < \alpha(t) - \beta(t) < \pi, \quad \alpha' > 0, \quad \beta' > 0,$$

there holds

(10) and (11) $$\beta(\chi(t)) = \alpha(t) \quad \text{and} \quad \alpha(\omega(t)) = \beta(t) + \pi,$$

for

$$0 < \alpha(t) - \beta(t) < \pi, \quad \alpha' < 0, \quad \beta' < 0,$$

there holds

(12) and (13) $$\beta(\chi(t)) = \alpha(t) - \pi \quad \text{and} \quad \alpha(\omega(t)) = \beta(t)."$$

"The b. c. dispersions are uniquely characterised by the relations (4) — (13)."

In the second part of the present paper our considerations will be directed to prove the existence and uniqueness of a solution of the non-linear differential equation (3), possessing convenient properties in the whole interval $(-\infty, \infty)$ and in the following third part, we shall use this result together with the relations (3)—(13) to construct differential equations (1) with solutions of certain prescribed properties. The constructions of differential equations (1) whose solutions have certain prescribed properties have already been dealt with by many authors: E. Barvínek [1], J. Chrastina [6], [7], M. Laitoch [9], F. Neuman [10], [11], in the complex domain, e.g., V. Šeda [12]. Some of their results will be arrived at again in the present systematic study of the problem in question.

## II. THE EXISTENCE AND UNIQUENESS THEOREM CONCERNING THE SOLUTION OF THE DIFFERENTIAL EQUATION $\beta(t) = \alpha(t) + \operatorname{arccotg}\left(\frac{1}{2}\left(\frac{1}{\alpha'}\right)'\right)$.

*Let* $\beta(t) \in C^1$, $\beta'(t) > 0$ *for every* $t$ *and suppose, furthermore, there is given* $t_0 \in (-\infty, \infty)$ *and numbers* $\alpha_0 \in (\beta(t_0) + k\pi, \beta(t_0) + \overline{k+1}\pi)$, $\alpha_0' > 0$; $k = 0, \pm 1, \ldots$. *Then the differential equation*

$$\beta(t) = \alpha(t) + \operatorname{arccotg}\left(\frac{1}{2}\left(\frac{1}{\alpha'}\right)'\right) - \overline{k+1}\,\pi, \tag{14}$$

$(0 < \operatorname{arccotg} t < \pi)$, *has just one solution* $\alpha(t) \in C^3$, *satisfying* $\alpha(t_0) = \alpha_0$, $\alpha'(t_0) = \alpha_0'$. *For this solution* $\alpha(t)$ *there applies* $\alpha'(t) > 0$, $\beta(t) + k\pi < \alpha(t) < \beta(t) + \overline{k+1}\pi$ *for every* $t$.

Proof. Suppose that in the theorem above $k = 0$. If we prove that under this additional assumption there exists a solution $\alpha(t)$ of the differential equation (14) satisfying the statement of the theorem, then the function $\alpha(t) + k\pi$ will be the solution of the differential equation (14), complying with the statement of the theorem even for $k \neq 0$.

Every solution $\alpha(t)$ of the differential equation (14) in an interval $I$ satisfies even the differential equation

$$\alpha'' = 2\alpha'^2 \operatorname{cotg}[\alpha(t) - \beta(t)]. \tag{15}$$

At the same time every solution $\alpha(t)$ of the differential equation (15) in $I$, for which $\alpha'(t) > 0$, $\beta(t) < \alpha(t) < \beta(t) + \pi$ in $I$, satisfies the differential equation (14). Thus, if we show that every solution of the differential equation (15), for which $\alpha \in C_I^3$, $\alpha(t_0) = \alpha_0$, $\alpha'(t_0) = \alpha_0'$, necessarily satisfies $\alpha'(t) > 0$ and $\beta(t) < \alpha(t) < \beta(t) + \pi$ in $I$, then such a solution is always the solution of the differential equation (14) in $I$ as well. In this case, with respect to the initial conditions indicated in the theorem, the set of the solutions of the differential equation (14) would be identical with the set of the solutions of the differential equation (15).

Let us, therefore, deal with the differential equation (15). Since $\alpha(t_0) \in (\beta(t_0), \beta(t_0) + \pi)$, there exists just one solution of the differential equation (15) in a certain neighbourhood $(t_0 - \delta, t_0 + \delta)$, $\delta > 0$, of the number $t_0$, complying with the prescribed initial conditions $\alpha(t_0) = \alpha_0$, $\alpha'(t_0) = \alpha_0'$. The function $\beta(t) \in C^1$ and therefore $\alpha(t) \in C^3_{(t_0-\delta,\, t_0+\delta)}$. Because $\alpha'(t_0) > 0$, there exists $\delta^* > 0$, $\delta^* \leqq \delta$ such that, in the interval $(t_0 - \delta^*, t_0 + \delta^*)$, there is $\alpha'(t) > 0$ as well. Thus, it is possible to find such a neighbourhood $I$ of the number $t_0$ and such a solution $\alpha(t)$ of the differential equation (15) meeting the initial conditions in the number $t_0$, that $\alpha(t) \in C_I^3$, $\alpha'(t) > 0$ and also $\alpha(t) \in (\beta(t), \beta(t) + \pi)$ in $I$. Let $(t_0 - \delta, t_0 + \delta)$ be an arbitrary neighbourhood of the point $t_0$ in which the relations are satisfied, $\alpha(t)$ being the corresponding solution of the differential equation (15) in this neighbourhood. The theorem will be proved if we show that this neighbourhood is the interval $(-\infty, \infty)$, or that the definition of the function $\alpha(t)$ may be extended up to the end-points of the neighbourhood $(t_0 - \delta, t_0 + \delta)$ so that the extended function is a solution of the differential equation (15) and meets the mentioned relations even on this closed interval.

One has $\beta(t) \in C^1$ and also $\beta'(t) > 0$. On the closed interval $[t_0 - \delta, t_0 + \delta]$ the function $\beta'(t)$ reaches its minimum $\mu$, which is necessarily positive, and its maximum $M$ as well. Consequently, $0 < \mu \leqq \beta'(t) \leqq M$, for $t \in [t_0 - \delta, t_0 + \delta]$. For the function $\alpha(t)$, extended up to the end-points of the interval $(t_0 - \delta, t_0 + \delta)$, to be of class $C^3_{[t_0-\delta,\, t_0+\delta]}$, we must set

$$\lim_{t \to t_0 \pm \delta \mp} \alpha(t) = \alpha(t_0 \pm \delta), \quad \lim_{t \to t_0 \pm \delta \mp} \alpha'(t) = \alpha'(t_0 \pm \delta).$$

If we sum up the preceding considerations, it is easy to see that for the proof ot the theorem it is necessary and sufficient to show that

**a)** there exists

$$\lim_{t \to t_0 + \delta -} \alpha(t) = \alpha(t_0 + \delta) \quad \text{and} \quad \alpha(t_0 + \delta) \in (\beta(t_0 + \delta), \beta(t_0 + \delta) + \pi),$$

**b)** there exists

$$\lim_{t\to t_0+\delta_-} \alpha'(t) = \alpha'(t_0+\delta) \quad \text{and} \quad \alpha'(t_0+\delta) > 0,$$

**c)** there exists

$$\lim_{t\to t_0-\delta_+} \alpha(t) = \alpha(t_0-\delta) \quad \text{and} \quad \alpha(t_0-\delta) \in (\beta(t_0-\delta), (\beta(t_0-\delta)+\pi),$$

**d)** there exists

$$\lim_{t\to t_0-\delta_+} \alpha'(t) = \alpha'(t_0-\delta) \quad \text{and} \quad \alpha'(t_0-\delta) > 0.$$

Let us now consider the single cases:

**a)** In the interval $[t_0, t_0+\delta)$ there is $\alpha'(t) > 0$, consequently $\alpha(t)$ is an increasing function. Moreover, in this interval there holds $\alpha(t) < \beta(t) + \pi$. Therefore there exists $\lim_{t\to t_0+\delta_-} \alpha(t)$ and we define $\alpha(t_0+\delta) = \lim_{t\to t_0+\delta_-} \alpha(t)$. It remains to show that there cannot occur:

$$\mathbf{a'}) \ \alpha(t_0+\delta) = \beta(t_0+\delta) + \pi, \qquad \mathbf{a''}) \ \alpha(t_0+\delta) = \beta(t_0+\delta).$$

**a')** Let $\alpha(t_0+\delta) = \lim_{t\to t_0+\delta_-} \alpha(t) = \beta(t_0+\delta) + \pi$. Since, in the whole interval $[t_0, t_0+\delta)$, there holds $\beta(t) < \alpha(t) < \beta(t) + \pi$, there exists, for an arbitrary $\varepsilon > 0$, $\bar{\delta}$, $0 < \bar{\delta} < \delta$ such that in the interval $(t_0+\bar{\delta}, t_0+\delta)$ one has $0 < \beta(t) + \pi - \alpha(t) < \varepsilon$. Furthermore, for $x \in (0, \pi/3)$ there holds $2x \geqq \operatorname{tg} x > 0$. If we choose $0 < \varepsilon < \pi/3$ then, in the interval $[t_0+\bar{\delta}, t_0+\delta)$, there holds $\frac{1}{2[\alpha(t) - \beta(t) - \pi]} \geqq \operatorname{cotg}[\alpha(t) - \beta(t)]$. Since $\alpha'(t) > 0$, and even $\beta'(t) > 0$ in $(t_0-\delta, t_0+\delta)$, all the more in $[t_0+\bar{\delta}, t_0+\delta)$, there holds

$$-\frac{\beta'(t)}{2\alpha'(t)[\alpha(t) - \beta(t) - \pi]} > 0.$$

Therefore

$$\frac{1 - \dfrac{\beta'}{\alpha'}}{2(\alpha - \beta - \pi)} > \frac{1}{2(\alpha - \beta - \pi)} \geqq \operatorname{cotg}[\alpha - \beta].$$

For $t \in [t_0+\bar{\delta}, t_0+\delta)$ there accordingly holds

$$\frac{\alpha''(t)}{\alpha'^2(t)} = 2\operatorname{cotg}[\alpha(t) - \beta(t)] < \frac{1 - \dfrac{\beta'(t)}{\alpha'(t)}}{\alpha(t) - \beta(t) - \pi}$$

and because of $\alpha'(t) > 0$ one gets

$$\alpha''/\alpha' = 2\alpha' \operatorname{cotg}[\alpha - \beta] < [\alpha' - \beta']/[\alpha - \beta - \pi].$$

Consequently $\alpha''/\alpha' = [\alpha' - \beta']/[\alpha - \beta - \pi] + f(t)$ where $f(t) < 0$. Therefore $\alpha' = C \,.\, (\beta + \pi - \alpha) \,.\, \exp F(t) \leqq C \,.\, [\beta(t) + \pi - \alpha(t)]$ where $F(t) = \int\limits_{t_0+\bar{\delta}}^{t} f(\sigma)\, \mathrm{d}\sigma$, $C$ is a convenient positive constant; for $F(t)$ decreasing and hence even $\exp F(t)$. At the same time, however, there is $\alpha'(t) > 0$ in the interval $[t_0 + \bar{\delta},\, t_0 + \delta)$. Choose, now, $\delta_0 \geqq \bar{\delta}$ so that, in the interval $[t_0 + \delta_0,\, t_0 + \delta)$, there holds $\beta(t) + \pi - \alpha(t) < \dfrac{\mu}{2C}$. Then, in the interval $[t_0 + \delta_0,\, t_0 + \delta)$, there also holds $0 < \alpha'(t) < \mu/2$. At the same time $\beta'(t) \geqq \mu$ and therefore $\beta'(t) - \alpha'(t) \geqq \mu/2$. Moreover, there is $\beta(t_0 + \delta_0) + \pi > \alpha(t_0 + \delta_0)$ and consequently $\varkappa = \beta(t_0 + \delta_0) + \pi - \alpha(t_0 + \delta_0) > 0$. Let us set $g(t) = \beta(t) + \pi - \alpha(t)$. Evidently $g(t) \in C^1_{[t_0+\delta_0, t_0+\delta)}$, $g(t_0 + \delta_0) = \varkappa > 0$ and $g'(t) \geqq \mu/2$ for $t \in [t_0 + \delta_0,\, t_0 + \delta)$. Hence $g(t) \geqq \varkappa$ and, because $\lim\limits_{t \to t_0+\delta-} g(t)$ exists and is equal to $\beta(t_0 + \delta) + \pi - \alpha(t_0 + \delta) \geqq \varkappa$, one necessarily gets $\beta(t_0 + \delta) + \pi > \alpha(t_0 + \delta)$. But this is a contradiction, as we have assumed $\alpha(t_0 + \delta) = \beta(t_0 + \delta) + \pi$. Consequently the case **a'** is not possible.

**a'')** Let $\alpha(t_0 + \delta) = \beta(t_0 + \delta)$. For $x \in (0, \pi/3)$ there holds $0 < \operatorname{tg} x \leqq 2x$ and to an arbitrary $\varepsilon \in (0, \pi/3)$ there exists $\bar{\delta} \in (0, \delta)$ such that $0 < \alpha(t) - \beta(t) < \varepsilon$ for $t \in [t_0 + \bar{\delta},\, t_0 + \delta)$. Hence, for $t \in [t_0 + \bar{\delta},\, t_0 + \delta)$ one has $\alpha''(t) = 2\alpha'^2(t) \operatorname{cotg}[\alpha(t) - \beta(t)] > 0$; $\alpha'(t)$ is an increasing function in $[t_0 + \bar{\delta},\, t_0 + \delta)$, that is to say, in this case $\alpha'(t_0 + \bar{\delta}) \leqq \alpha'(t)$. Furthermore, in $[t_0 + \bar{\delta},\, t_0 + \delta)$, there holds

$$\left| \frac{\beta'(t)}{\alpha'(t)} - 1 \right| \leqq \frac{\beta'(t)}{\alpha'(t)} + 1 \leqq \frac{M}{\alpha'(t_0 + \bar{\delta})} + 1$$

and therefore

$$\operatorname{cotg}(\alpha - \beta) \geqq \frac{1}{2[\alpha(t) - \beta(t)]} \geqq \frac{\left| \dfrac{\beta'}{\alpha'} - 1 \right|}{2(\alpha - \beta)} \cdot \left[ \frac{M}{\alpha'(t_0 + \bar{\delta})} + 1 \right]^{-1} \geqq$$

$$\geqq \frac{\dfrac{\beta'}{\alpha'} - 1}{2(\alpha - \beta)} \cdot \left[ \frac{M}{\alpha'(t_0 + \bar{\delta})} + 1 \right]^{-1}.$$

Set $k = \left[\frac{M}{\alpha'(t_0+\bar{\delta})} + 1\right]^{-1}$. Evidently $0 < k < 1$. We get cotg $(\alpha - \beta) \geqq \frac{k}{2} \cdot \frac{\frac{\beta'}{\alpha'} - 1}{\alpha - \beta}$. Thus, we can write $\frac{\alpha''}{\alpha'} = k\frac{\beta' - \alpha'}{\alpha - \beta} + f(t)$, $f(t) \geqq 0$ or $\alpha'(t) = C \cdot [\alpha(t) - \beta(t)]^{-k} \cdot \exp F(t)$, $C$ being a convenient constant, $F(t) = \int\limits_{t_0+\bar{\delta}}^{t} f(\sigma)\,d\sigma$. The constant $C$ is positive because $\alpha'(t_0 + \bar{\delta}) > 0$. The function $F(t)$ is non-decreasing and therefore $\alpha'(t) \geqq C \cdot [\alpha(t) - \beta(t)]^{-k} \cdot \exp F(t_0 + \bar{\delta}) = C \cdot [\alpha(t) - \beta(t)]^{-k}$. Choose $m$, $\alpha(t_0 + \bar{\delta}) - \beta(t_0 + \bar{\delta}) > m > 0$ so that $C \cdot m^{-k} > M$. Set $g(t) = \alpha(t) - \beta(t)$ for $t \in [t_0 + \bar{\delta}, t_0 + \delta)$. Evidently $g(t_0 + \bar{\delta}) = \alpha(t_0 + \bar{\delta}) - \beta(t_0 + \bar{\delta}) > m > 0$ and also $g(t) \in C^1_{[t_0+\bar{\delta}, t_0+\delta)}$. Moreover, $\lim\limits_{t\to t_0+\delta_-} g(t) = 0$. As $g(t)$ is continuous on $[t_0 + \bar{\delta}, t_0 + \delta]$, there exists $\bar{t} \in (t_0 + \bar{\delta}, t_0 + \delta)$ such that $g(\bar{t}) = m$. Denote the smallest of these numbers (the existence of the smallest number is guaranteed because these numbers form a non-empty, closed set) by $t_1$. Thus $g(t) > m$ for $t \in [t_0 + \bar{\delta}, t_1)$ and $g(t_1) = m$. There holds $\frac{g(t) - g(t_1)}{t - t_1} < 0$ for $t \in [t_0 + \bar{\delta}, t_1)$ and because there exists $g'(t_1)$, one has $g'(t_1) \leqq 0$. Consequently, $\alpha'(t_1) \leqq \beta'(t_1)$. But, according to what we said before, one has $\alpha'(t_1) \geqq C \cdot [\alpha(t_1) - \beta(t_1)]^{-k} = C \cdot m^{-k} > M \geqq \beta'(t_1)$, hence $\alpha'(t_1) > \beta'(t_1)$, which is a contradiction. For this reason, neither the case **a″** can occur. The only possible case is the case **a**.

**b)** We are to show that $\lim\limits_{t\to t_0+\delta_-} \alpha'(t)$ exists and that this limit is positive.

Let us distinguish the following cases:

**b1)** There exists an interval $[t_0 + \bar{\delta}, t_0 + \delta)$ in which there holds $\beta(t) + \pi > \alpha(t) > \beta(t) + \pi/2$.

**b2)** There exists an interval $[t_0 + \bar{\delta}, t_0 + \delta)$ in which there holds $\beta(t) < \alpha(t) < \beta(t) + \pi/2$.

**b3)** There exists, in the interval $[t_0 + \bar{\delta}, t_0 + \delta)$, an increasing sequence of numbers $\{t_n\}_{n=1}^{\infty}$ converging to the number $t_0 + \delta$ and such that $\alpha(t_n) = \beta(t_n) + \pi/2$.

It is obvious that just one of these three cases occurs. Consider each case separately.

**b1)** In this case there exists an interval $[t_0 + \bar{\delta}, t_0 + \delta)$ in which

$\alpha(t) \in (\beta(t) + \pi/2, \beta(t) + \pi)$. According to (15), $\alpha''(t)$ is—in this interval—negative and therefore $\alpha'(t)$ decreases. And, as one has $\alpha'(t) > 0$ for $t \in [t_0 + \bar{\delta}, t_0 + \delta)$, there exists $\lim\limits_{t \to t_0 + \delta_-} \alpha'(t)$ which is non-negative. It remains to show that this limit is positive. Suppose it is not so and let $\lim\limits_{t \to t_0 + \delta_-} \alpha'(t) = 0$. Then there exists an interval $[t_0 + \bar{\bar{\delta}}, t_0 + \delta), \delta \leqq \bar{\bar{\delta}} < \delta$, in which $0 < \alpha'(t) < \mu \leqq \beta'(t)$ and hence $\beta'(t) - \alpha'(t) > 0$. Furthermore, there holds $\pi/2 < \alpha(t) - \beta(t) < \pi$ for $t \in [t_0 + \bar{\bar{\delta}}, t_0 + \delta)$. Consequently, in the interval $[t_0 + \bar{\bar{\delta}}, t_0 + \delta)$, the function $\operatorname{cotg} [\alpha(t) - \beta(t)] < 0$ is increasing, hence $A = \operatorname{cotg} [\alpha(t_0 + \bar{\bar{\delta}}) - \beta(t_0 + \bar{\bar{\delta}})] \leqq \operatorname{cotg} [\alpha(t) - \beta(t)] < 0$. Write this inequality in the form $\alpha''(t)/\alpha'^2(t) = 2 \operatorname{cotg} [\alpha(t) - \beta(t)] = -f(t)$ where $-2A \geqq f(t) > 0$. We get the relation $\alpha'(t) = [c + \int_{t_0+\bar{\bar{\delta}}}^{t} f(\sigma)\, d\sigma]^{-1}$ for $t \in [t_0 + \bar{\bar{\delta}}, t_0 + \delta)$, where $c$ is a convenient constant; $c > 0$ because $\alpha'(t_0 + \bar{\bar{\delta}}) > 0$. The function $\int_{t_0+\bar{\bar{\delta}}}^{t} f(\sigma)\, d\sigma$ is increasing and therefore there holds

$$\frac{1}{c} \geqq \frac{1}{c + \int\limits_{t_0+\bar{\bar{\delta}}}^{t} f(\sigma)\, d\sigma} \geqq \frac{1}{c - 2A(\delta - \bar{\bar{\delta}})} = B > 0.$$

Hence, in the interval $[t_0 + \bar{\bar{\delta}}, t_0 + \delta)$, there holds $\alpha'(t) \geqq B > 0$, which is a contradiction with regard to the assumption that $\lim\limits_{t \in t_0 + \delta_-} \alpha'(t) = 0$.

**b2)** In this case there is, with respect to the relation (15), $\alpha'' > 0$ in $[t_0 + \bar{\delta}, t_0 + \delta)$ and, consequently, $\alpha'(t)$ is increasing, and $\alpha'(t_0 + \bar{\delta}) > 0$. It is sufficient, both for the existence and the positivity of $\lim\limits_{t \to t_0 + \delta_-} \alpha'(t)$, to show that $\alpha'(t)$ is bounded in some left neighbourhood of the number $t_0 + \delta$.

Since we have extended the function $\alpha(t)$ up to the number $t_0 + \delta$ so as to be continuous from the left, we can set $A = \max\limits_{t \in [t_0 + \bar{\delta}, t_0 + \delta]} [\beta(t) + \pi/2 - \alpha(t)]$ and evidently $A > 0$. Then $B = \operatorname{tg} A \geqq \operatorname{tg} [\beta(t) + \pi/2 - \alpha(t)] = \operatorname{cotg} [\alpha(t) - \beta(t)] > 0$ for $t \in [t_0 + \bar{\delta}, t_0 + \delta)$ and, with respect to the relation (15), $\alpha''/\alpha' \leqq 2B\alpha'$ or $\alpha''/\alpha' = 2B\alpha' - f(t)$, where $f(t) \geqq 0$. Hence one has $\alpha'(t) = \exp \{2B\alpha - \int_{t_0+\bar{\delta}}^{t} f(\sigma)\, d\sigma + c\}$, where $c$ is a convenient constant. As $f(t) \geqq 0$, $\exp\{-\int\limits_{t_0+\bar{\delta}}^{t} f(\sigma)\,d\sigma\}$ is a non-increasing function and consequently $\alpha'(t) \leqq \exp\{2B\alpha(t) + c\}$. In the interval $[t_0 + \bar{\delta}, t_0 + \delta)$ there is $\alpha'(t) > 0$, so that $\alpha(t)$ is increasing, while there exists $\lim\limits_{t\to t_0+\delta_-} \alpha(t) = \alpha(t_0 + \delta)$. For this reason $\alpha'(t) \leqq \exp\{2B\alpha(t_0 + \delta) + c\}$.

**b3)** Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a decreasing zero-sequence. Then there exists an increasing sequence $\{\delta_n\}_{n=1}^{\infty}$, $\delta_n < \delta$ converging to $\delta$ such that, for $t \in [t_0 + \delta_n, t_0 + \delta)$, there holds $|\operatorname{cotg}[\alpha(t) - \beta(t)]| < \varepsilon_n$ and, at the same time, $|\alpha(t) - \alpha(t_0 + \delta)| < 1/4$. Since $\alpha''/\alpha' = 2\alpha' \operatorname{cotg}[\alpha(t) - \beta(t)]$, we can write $\ln \alpha'(t) - \ln \alpha'(t_0 + \delta_n) = 2\int\limits_{t_0+\delta_n}^{t} \alpha'(\sigma) \operatorname{cotg}[\alpha(\sigma) - \beta(\sigma)]\,d\sigma$. According to the mean-value theorem there exists $\bar{\delta}_n \in [\delta_n, \delta)$ such that $\ln \alpha'(t) - \ln \alpha'(t_0 + \delta_n) = 2 \operatorname{cotg}[\alpha(t_0 + \bar{\delta}_n) - \beta(t_0 + \bar{\delta}_n)] \cdot \int\limits_{t_0+\delta_n}^{t} \alpha'(\sigma)\,d\sigma = 2 \operatorname{cotg}[\alpha(t_0 + \bar{\delta}_n) - \beta(t_0 + \bar{\delta}_n] \cdot [\alpha(t) - \alpha(t_0 + \delta_n)]$. Thus $|\ln \alpha'(t) - \ln \alpha'(t_0 + \delta_n)| \leqq \varepsilon_n/2$ for $t \in [t_0 + \delta_n, t_0 + \delta)$, or $\ln \alpha'(t_0 + \delta_n) - \varepsilon_n/2 \leqq \ln \alpha'(t) \leqq \ln \alpha'(t_0 + \delta_n) + \varepsilon_n/2$. Denote $I_n \equiv [\ln \alpha'(t_0 + \delta_n) - \varepsilon_n/2, \ln \alpha'(t_0 + \delta_n) + \varepsilon_n/2)$. The system of closed intervals $I_n$ has the property that the lengths of the intervals converge to zero and every finite subsystem has a non-empty intersection (because this intersection also contains the number $\ln \alpha'(x)$ where $x \in [t_0 + \delta_i, t_0 + \delta)$, $i$ denotes the greatest of the indices of the intervals of the finite subsystem). Therefore the intersection of this system of intervals is one single number, which we denote by $\ln A$, $(A > 0)$. If we now choose, arbitrarily, $\varepsilon \in (0, A)$, then there exists $n$ such that $\varepsilon_n \leqq \min\{|\ln(1 - \varepsilon/A)|, |\ln(1 + \varepsilon/A)|\}$. As $\ln \alpha'(t)$ for $t \in [t_0 + \delta_n, t_0 + \delta)$ as the number $\ln A$ lie in the interval $I_n$; hence, there holds $|\ln \alpha'(t) - \ln A| \leqq \varepsilon_n$. Consequently $\exp\{-\varepsilon_n\} \leqq \alpha'(t)/A \leqq \exp \varepsilon_n$ and, all the more, $1 - \varepsilon/A \leqq \alpha'(t)/A \leqq 1 + \varepsilon/A$ i.e. $|\alpha'(t) - A| \leqq \varepsilon$ in the interval $[t_0 + \delta_n, t_0 + \delta)$. Therefore $\lim\limits_{t\to t_0+\delta_-} \alpha'(t) = A > 0$ exists.

We have shown that any solution of the differential equation (15), defined in the interval $[t_0, t_0 + \delta)$ and satisfying the initial conditions indicated in the theorem, can be extended on the whole interval $[t_0, \infty)$, this extended solution meeting—on $[t_0, \infty)$—the postulates stated in the theorem. Now, we shall show that this solution can be extended even on the interval $(-\infty, \infty)$ the statement of the theorem being complied with.

Consider, therefore, the following equation:

$$\hat{\alpha}'' = 2\hat{\alpha}'^2 \operatorname{cotg}[\hat{\alpha} - \hat{\beta}(t)] \tag{15'}$$

where

$$\hat{\beta}(t) = -\beta(-t) - \pi,$$
$$\hat{a}(-t_0) = -\alpha(t_0) = -\alpha_0,$$
$$\hat{a}'(-t_0) = \alpha'(t_0) = \alpha_0'.$$

Evidently $\hat{a}(-t_0) = -\alpha_0 \in (-\beta(t_0) - \pi, -\beta(t_0)) \equiv (\hat{\beta}(-t_0), \hat{\beta}(-t_0) + \pi)$, $\hat{a}'(-t_0) = \alpha_0' > 0$. Thus, there exists just one solution $\hat{a}(t)$ defined on the interval $[-t_0, \infty)$ for which there holds $\hat{a}(t) \in C^3_{[-t_0, \infty)}$, $\hat{a}(-t_0) = -\alpha_0$, $\hat{a}'(-t_0) = \alpha_0'$. For this solution one then has, on the interval $[-t_0, \infty) : \hat{a}'(t) > 0$, $\hat{a}(t) \in (\hat{\beta}(t), \hat{\beta}(t) + \pi)$. Then the function $\alpha(t) = -\hat{a}(-t)$ is just the only solution of the differential equation (15) on the interval $(-\infty, t_0)$ for which $\alpha(t) \in C^3_{(-\infty, t_0]}$, $\alpha(t_0) = \alpha_0$, $\alpha'(t_0) = \alpha_0'$. For this solution there holds $\alpha'(t) > 0$, $\alpha(t) \in (\beta(t), \beta(t) + \pi)$ for $t \in (-\infty, t_0]$.

If we bind the solution of the differential equation (14) defined on the interval $[t_0, \infty)$ with the solution defined on the interval $(-\infty, t_0]$ and complying with the same initial conditions indicated in the theorem, we get the solution meeting the statement of the theorem on the interval $(-\infty, \infty)$ and the theorem is proved.

Setting, now, $-\bar{a}(t) = \alpha(t)$ and $-\bar{\beta}(t) = \beta(t)$ we can also state the following:

*Let $\beta(t) \in C^1$, $\beta'(t) < 0$ for every $t$. Let, moreover, there be given $t_0 \in (-\infty, \infty)$ and the numbers $\alpha_0 \in (\beta(t_0) + k\pi, \beta(t_0) + \overline{k+1}\,\pi)$, $\alpha_0' < 0$; $k = 0, \pm 1, \pm 2, \ldots$ Then the differential equation* (14) ($0 <$ arccotg $t < \pi$), *has just one solution $\alpha(t) \in C^3$ satisfying $\alpha(t_0) = \alpha_0$, $\alpha'(t_0) = \alpha_0'$. For this solution $\alpha(t)$ there then holds $\alpha'(t) < 0$, $\beta(t) + k\pi < \alpha(t) < \beta(t) + \overline{k+1}\,\pi$ for every $t$.*

## III. APPLICATIONS OF THE PRECEDING RESULTS TO CONSTRUCTIONS OF SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH SOLUTIONS OF PRESCRIBED PROPERTIES

It is interesting to note how far a function may arbitrary be chosen so that there exist a differential equation to which the function in question would be some phase or dispersion, and what number of such differential equations (1) there exists. Or, what properties the solution of a differential equation (1) may be required to have so that such a differential equation exists whose solutions have the required properties, or even, what number of such equations there exists.

From the mentioned results of prof. O. Borůvka it follows that by the choice of an arbitrary function $\alpha(t)$ complying with $(\alpha)$ and required to be first phase of some differential equation (1) the latter is uniquely determined by the relation (2).

E. Barvínek in [1] has shown that if we choose an arbitrary function $\varphi(t)$ complying with $(\varphi)$ and required to be the basic central dispersion of the 1st kind of some differential equation (1), then there exists an infinite number of 1st phases satisfying the relation (4). Namely, except for meeting certain conditions at the ends of the interval $[t_0, \varphi(t_0)]$, one can arbitrarily choose the 1st phase on this interval. After this choice, the 1st phase is uniquely determined and thus, according to (2), even the differential equation (1) is uniquely determined. From the relation (2) it is easy to see that by the 1st phase the equation (1) is determined but the opposite does not hold. There remains the interesting question of how many different differential equations (1) have the same b.c. dispersion of the first kind because the results of E. Barvínek only informs us that there exists an infinite number of different 1st phases. Employing a result of F. Neuman's (see [11]), Prof. O. Borůvka has shown (see [5]) that the cardinal number of the set of all Eqs. (1) having the same b.c. dispersion of the 1st kind is equal to $\aleph$.

J. Chrastina [7] has dealt with the case when one chooses a function $\psi(t)$ meeting $(\psi)$ and required to be the b.c. dispersion of the 2nd kind of some Eq. (1*). He has shown that it is possible, except for certain conditions at the end-points, to choose the 2nd phase arbitrarily even on the interval $[t_0, \psi(t_0)]$.

For completeness it remains to show, what is to be expected in the case we choose a function $\beta(t)$ satisfying $(\beta)$ and required to be *the* 2nd *phase* of the differential equation (1*); or: a function $\chi(t)$ complying with $(\chi)$ and required to be *the b.c. dispersion of the 3rd kind* of Eq. (1*); or: a function $\omega(t)$ meeting $(\omega)$ and required to be *the b.c. dispersion of the 4th kind* of Eq. (1*).

The immediate consequence of the theorems of the second part are, with regard to the relation (3), the following statements solving the given problem for the 2nd phase of Eq. (1*):

*Let there be given an arbitrary function $\beta(t)$ complying with $(\beta)$ and the numbers $t_0$, $\alpha_0 \in (\beta(t_0) + k\pi, \ \beta(t_0) + \overline{k+1}\pi)$ and $\alpha_0' \neq 0$;* $\operatorname{sign} \alpha_0' = \operatorname{sign} \beta'(t)$. *Then there exists just one function $\alpha(t)$ meeting $\alpha(t_0) = \alpha_0$, $\alpha'(t_0) = \alpha_0'$ and such that $\alpha(t)$ and $\beta(t)$ are, respectively, the 1st and the 2nd phase of some pair $(u, v)$ of independent solutions of the differential equation* (1).

Or, with respect to the relation (2), we may formulate:

*Let there be given an arbitrary function $\beta(t)$ satisfying $(\beta)$ and numbers $t_0$, $\alpha_0 \in (\beta(t_0) + k\pi, \ \beta(t_0) + \overline{k+1}\pi)$ and $\alpha_0' \neq 0$;* $\operatorname{sign} \alpha_0' = \operatorname{sign} \beta'(t)$.

*Then there exists just one differential equation* (1) *such that* $\alpha(t)$ *and* $\beta(t)$ *are, respectively, the 1st and the 2nd phase of some pair* $(u, v)$ *of independent solutions of this equation and* $\alpha(t_0) = \alpha_0$, $\alpha'(t_0) = \alpha_0'$. Or, somewhat less restrictive:

*To an arbitrary function* $\beta(t)$ *satisfying* $(\beta)$ *there exists a differential equation* (1) *such that* $\beta(t)$ *is its 2nd phase.*

In what follows, let confine our considerations, for a while, to the case of $\alpha'(t) > 0$ and $\beta'(t) > 0$. Formally we can express this additional postulate so that all the increasing functions meeting $(\alpha)$ or $(\beta)$ have the property $(\alpha^+)$ or $(\beta^+)$, resp. Analogously to the considerations of M. Kuczma [8] and E. Barvínek [1] the following lemma can easily be derived:

Let $\psi(t)$ be a function satisfying $(\psi)$. Then the equation $\beta(\psi(t)) = \beta(t) + d$, $d > 0$, has an infinite number of solutions $\beta(t)$ complying with $(\beta^+)$. These solutions can be got by choosing $\beta(t)$ on the interval $[t_0, \psi(t_0)]$ in such a way that $\beta(t_0) + d = \beta(\psi(t_0))$, $\beta'(t_0) = \beta'(\psi(t_0)) . \psi'(t_0)$, $\beta \in C^1_{[t_0, \psi(t_0)]}$, $\beta'(t) > 0$ for $t \in [t_0, \psi(t_0)]$. Then $\beta(t)$ is uniquely determined on $(-\infty, \infty)$.

Now, if we consider an arbitrary function $\psi(t)$ satisfying $(\psi)$ then, according to the lemma, there exist infinitely many functions $\beta(t)$ satisfying the equation (5) and complying with $(\beta^+)$. If we choose $q(t) < 0$ on the interval $[t_0, \psi(t_0)]$ so that some 2nd phase $\beta(t)$ of Eq. (1*) (considered for $t_0 \leqq t \leqq \psi(t_0)$) meets $\beta(\psi(t_0)) = \beta(t_0) + \pi$, $\beta'(\psi(t_0)) . \psi'(t_0) = \beta'(t_0)$, then $\beta(t)$ is uniquely determined on the interval $(-\infty, \infty)$. Consequently, even the differential equation (1*) is determined on the interval $(-\infty, \infty)$. Thus we have obtained the result of J. Chrastina.

Let us now choose, arbitrarily, $\chi(t)$ on an interval $[t_0, \infty)$ so as to satisfy $(\chi)$. Choose, furthermore, $\beta_0(t)$ on the interval $[t_0, \chi(t_0)]$ in order to meet $(\beta^+)$ and $\beta_0(\chi(t_0)) < \beta_0(t_0) + \pi$. Let $\alpha_0(t)$ for $t \in [t_0, \chi(t_0)]$ be a solution of the differential equation (14) satisfying the conditions $\alpha_0(t_0) = \beta_0(\chi(t_0)) - \pi$, $\alpha_0'(t_0), = \beta_0'(\chi(t_0)) . \chi'(t_0)$ (compare the relation (6)) while setting, in (14), $\beta_0$ instead of $\beta$. That is possible because $\alpha_0(t_0) \in \in (\beta_0(t_0) - \pi, \beta_0(t_0))$ and $\alpha_0'(t) > 0$. Let $\beta_1(t) = \alpha_0(\chi^{-1}(t)) + \pi$ for $t \in [\chi(t_0), \chi^2(t_0)]$. Evidently $\beta_1(\chi(t_0)) = \beta_0(\chi(t_0))$ and $\beta_1'(\chi(t_0) = \alpha_0'(\chi^{-1}(t)) \times \times (\chi^{-1}(t))' |_{\chi(t_0)} = \beta_0'(\chi(t_0))$. At the same time $\beta_1(\chi^2(t_0)) = \alpha_0(\chi(t_0)) + + \pi < \beta_0(\chi(t_0)) + \pi = \beta_1(\chi(t_0)) + \pi$. Denote by $\alpha_1(t)$ for $t \in [\chi(t_0), \chi^2(t_0)]$ the solution of the equation (14) complying with $\alpha_1(\chi(t_0)) = = \beta_1(\chi^2(t_0)) - \pi$, $\alpha_1'(\chi(t_0)) = [\beta_1(\chi(t))]' |_{\chi(t_0)}$, $\beta_1$ standing for $\beta$. In general, we determine $\alpha_n(t)$ on the interval $[\chi^n(t_0), \chi^{n+1}(t_0)]$ as the solution of the equation (14) satisfying $\alpha_n(\chi^n(t_0)) = \alpha_{n-1}(\chi^n(t_0))$, $\alpha_n'(\chi^n(t_0)) =$

$= \alpha'_{n-1}(\chi^n(t_0))$; $\beta_n$ stands for $\beta$ in the equation (14). The function $\beta_n(t)$ on the interval $[\chi^n(t_0), \chi^{n+1}(t_0)]$ is again determined by the relation (6), i.e. $\beta_n(t) = \alpha_{n-1}(\chi^{-1}(t)) + \pi$. If we set $\alpha(t) = \alpha_n(t)$ and $\beta(t) = \beta_n(t)$ for $t \in [\chi^n(t_0), \chi^{n+1}(t_0)]$, $n = 0, 1, 2, \ldots$, then the function $\alpha(t)$ has the property $(\alpha^+)$ and $\beta(t)$ has the property $(\beta^+)$, they satisfy the equation (3) and, together with the function $\chi(t)$, coply with the relation (6), all on the interval $[t_0, \infty)$. Therefore we may summarize the preceding considerations:

*Let $\chi(t)$ be a function defined on an interval $[t_0, \infty)$ on which it satisfies $(\chi)$. Choose $\beta_0(t)$ on $[t_0, \chi(t_0)]$ so that it meets $(\beta^+)$ and $\beta_0(\chi(t_0)) < \beta_0(t_0) + \pi$. Then there exists just one differential equation (1*) on the interval $[t_0, \infty)$ such that $\chi(t)$ is its b.c. dispersion of the 3rd kind and $\beta_0(t)$ coincides, on the interval $[t_0, \chi(t_0)]$, with some 2nd phase of this differential equation.* Or in other words:

*Let $\chi(t)$ be a function defined on $[t_0, \infty)$ and satisfying $(\chi)$. Let $q(t) < 0$ be given on the interval $[t_0, \chi(t_0)]$ so that the 1st phase $\alpha(t)$ and the second phase $\beta(t)$ of some pair of independent solutions of the differential equation (1*) satisfy $\alpha(t_0) = \beta(\chi(t_0)) - \pi < \beta(t_0)$, $\alpha'(t_0) = \beta'(\chi(t_0)) \cdot \chi'(t_0) > 0$. Then the function $q(t)$ can be extended on the interval $[t_0, \infty)$ just in one way so that $q < 0$ and $\chi(t)$ is the b.c. dispersion of the 3rd kind of the differential equation $y'' = q(t)\,y$.*

Choose, in a similar way, an arbitrary function $\omega(t)$ on an interval $(-\infty, t_0]$ and satisfying $(\omega)$. Furthermore, choose $\beta_0(t)$ on $[t_0, \omega(t_0)]$ so that it meets $(\beta^+)$ and $\beta_0(t_0) > \beta_0(\omega(t_0)) - \pi$. Let $\alpha_0(t)$ for $t \in [t_0, \omega(t_0)]$ be a solution of the differential equation (14) satisfying the conditions $\alpha_0(\omega(t_0)) = \beta_0(t_0)$, $\alpha'_0(\omega(t_0)) \cdot \omega'(t_0) = \beta'_0(t_0)$ (compare the relation (7)), having set, in (14), $\beta_0$ instead of $\beta$. It is possible, because $\alpha_0(\omega(t_0)) \in (\beta_0(\omega(t_0)) - \pi, \beta_0(\omega(t_0)))$and $\alpha'_0(\omega(t_0)) > 0$. Let $\beta_{-1}(t) = \alpha_0(\omega(t))$ for $t \in [\omega^{-1}(t_0), t_0]$. Evidently $\beta_{-1}(t_0) = \beta_0(t_0)$ and $\beta'_{-1}(t_0) = \beta'_0(t_0)$. Simultaneously $\beta_{-1}(\omega^{-1}(t_0)) = \alpha_0(t_0) > \beta_0(t_0) - \pi = \beta_{-1}(t_0) - \pi$. Denote by $\alpha_{-1}(t)$, for $t \in [\omega^{-1}(t_0), t_0]$, the solution of the equation (14) satisfying $\alpha_{-1}(t_0) = \alpha_0(t_0)$ and $\alpha'_{-1}(t_0) = \alpha'_0(t_0)$, having set $\beta_{-1}$ for $\beta$. In general, we determine $\alpha_{-n}(t)$ on the interval $[\omega^{-n}(t_0), \omega^{-n+1}(t_0)]$ as the solution of the differential equation (14) satisfying $\alpha_{-n}(\omega^{-n+1}(t_0)) = \alpha_{-n+1}(\omega^{-n+1}(t_0))$ and $\alpha'_{-n}(\omega^{-n+1}(t_0)) = \alpha'_{-n+1}(\omega^{-n+1}(t_0))$, $\beta_{-n}$ standing for $\beta$. The function $\beta_{-n}$ on the interval $[\omega^{-n}(t_0), \omega^{-n+1}(t_0)]$ is again determined by the relation (7), i.e. $\beta_{-n}(t) = \alpha_{-n+1}(\omega(t))$. If we set $\alpha(t) = \alpha_{-n}(t)$ and $\beta(t) = \beta_{-n}(t)$ for $t \in [\omega^{-n}(t_0), \omega^{-n+1}(t_0)]$, $n = 0, 1, 2, \ldots$, then the function $\alpha(t)$ has the property $(\alpha^+)$ and $\beta(t)$ has the property $(\beta^+)$; $\alpha(t)$, $\beta(t)$ satisfy the equation (3) and, together with the function

$\omega(t)$, they comply with the relation $\alpha(t) = \beta(\omega^{-1}(t))$, all on the interval $(-\infty, \omega(t_0)]$. With regard to the relation (7) we may summarize our considerations:

*Let $\omega(t)$ be a function defined on an interval $(-\infty, t_0]$ and meeting $(\omega)$. Let us choose $\beta_0(t)$ on $[t_0, \omega(t_0)]$ so that it satisfies $(\beta^+)$ and $\beta_0(t_0) > \beta_0(\omega(t_0)) - \pi$. Then there exists just one differential equation (1*) on the interval $(-\infty, \omega(t_0)]$ such that $\omega(t)$ is its b.c. dispersion of the 4th kind and $\beta_0(t)$ coincides, on the interval $[t_0, \omega(t_0)]$, with some 2nd phase of this differential equation.*

Or to put it differently:

*Let $\omega(t)$ be a function defined on $(-\infty, t_0]$ and satisfying $(\omega)$. Let $q(t) < 0$ be given on the interval $[t_0, \omega(t_0)]$ in such a way that the first phase $\alpha(t)$ and the second phase $\beta(t)$ of some pair of independent solutions of the differential equation (1*) fulfil $\alpha(\omega(t_0)) = \beta(t_0) > \beta(\omega(t_0)) - \pi$, $\alpha'(\omega(t_0)) \times \omega'(t_0) = \beta'(t_0) > 0$. Then it is possible to extend the function $q(t)$ on the whole interval $(-\infty, \omega(t_0)]$ just in one way so that $q < 0$ and $\omega(t)$ may be the b.c. dispersion of the 4th kind of the differential equation $y'' = q(t)\, y$.*

Now, let us consider the case when $\chi(t)$ is defined on the interval $(-\infty, \infty)$ and meets $(\chi)$. Let us choose an arbitrary number $t_0$ and, furthermore, on the interval $[t_0, \chi(t_0)]$, a function $\beta_0(t)$ complying with $(\beta^+)$ and $\beta_0(\chi(t_0)) < \beta_0(t_0) + \pi$. We want to find $\alpha_{-1}(t)$ as an extension of the first phase $\alpha_0(t)$ on the interval $[\chi^{-1}(t_0), t_0]$ of the studied differential equation (1*). The b.c. dispersion of the 3rd kind $\chi(t)$ of (1*) is given. Then there ought to be valid, with regard to (6), $\alpha_{-1}(t) = \beta_0(\chi(t)) - \pi$. But in general it is not possible to state that $\alpha_{-1}(t)$ meets the condition $(\alpha^+)$ on $[\chi^{-1}(t_0), t_0]$. We only know, for example, that the functions $\beta_0(t)$ and $\chi(t)$ belong to $C^1$. We should be obliged to suppose $\beta_0(t)$ to be chosen in such a way that $\beta_0(\chi(t)) - \pi$ satisfies $(\alpha^+)$ on $[\chi^{-1}(t_0), t_0]$. And, moreover, there would have to exist, in the number $t_0$, a continuous derivative of the 3rd order of the function $\alpha_{-1}(t)$ extended in this manner. Hence, if $\chi(t)$ or $\omega(t)$ are defined on the whole interval $(-\infty, \infty)$ we can only state:

*Let $\chi(t)$ be a function defined on the interval $(-\infty, \infty)$ and complying with $(\chi)$. Let us choose an arbitrary number $t_0$ and a function $q(t) < 0$ on the interval $[t_0, \chi(t_0)]$ so that the 1st phase $\alpha(t)$ and the 2nd phase $\beta(t)$ of some pair of independent solutions of the differential equation (1*) fulfil $\alpha(t_0) = \beta(\chi(t_0)) - \pi < \beta(t_0)$, $\alpha'(t_0) = \beta'(\chi(t_0)) \,.\, \chi'(t_0) > 0$. Then there is just one possible way of extending the function $q(t)$ on the interval $[t_0, \infty)$ so that $q(t) < 0$ and $\chi(t)$ is the b.c. dispersion of the 3rd kind of the differential equation $y'' = q(t)\, y$ on the interval $[t_0, \infty)$.*

Or, less restrictive:

*Let* $\chi(t)$ *be a function defined on* $(-\infty, \infty)$ *and meeting* $(\chi)$, $t_0$ *being an arbitrary number. Then there exists a differential equation* (1*) *such that* $\chi(t)$ *is its b.c. dispersion of the* 3rd *kind on the interval* $[t_0, \infty)$.

An analogous statement can be got for a b.c. dispersion of the 4th kind:

*Let* $\omega(t)$ *be a function defined on the interval* $(-\infty, \infty)$ *and satisfying* $(\omega)$. *Let us choose an arbitrary number* $t_0$ *and a function* $q(t) < 0$ *on the interval* $[t_0, \omega(t_0)]$ *so that the* 1st *phase* $\alpha(t)$ *and the* 2nd *phase* $\beta(t)$ *of some pair of independent solutions of the differential equation* (1*) *meet* $\alpha(\omega(t_0)) = \beta(t_0) > \beta(\omega(t_0)) - \pi$, $\alpha'(\omega(t_0)) . \omega'(t_0) = \beta'(t_0) > 0$. *Then there is just one possible way of extending* $q(t)$ *on the interval* $(-\infty, \omega(t_0)]$ *so that* $q < 0$ *and* $\omega(t)$ *may be the b.c. dispersion of the* 4th *kind of the differential equation* $y'' = q(t)\, y$ *on the interval* $(-\infty, \omega(t_0)]$.

Or, less restrictive:

*Let* $\omega(t)$ *be a function defined on* $(-\infty, \infty)$ *and meeting* $(\omega)$, $t_0$ *being an arbitrary number. Then there exists a differential equation* (1*) *such that* $\omega(t)$ *is its b.c. dispersion of the* 4th *kind on the interval* $(-\infty, t_0]$.

We have restricted our considerations by the postulate $\alpha'(t) > 0$ and $\beta'(t) > 0$. Analogous theorems could be derived even for $\alpha'(t) < 0$ and $\beta'(t) < 0$ because, if $\alpha(t)$ and $\beta(t)$ are the 1st and the 2nd phase, respectively, of some pair of linear independent solutions of a differential equation (1*), then $-\alpha(t)$ and $-\beta(t)$ are again the 1st and the 2nd phase, respectively, of some pair of linear independent solutions of the same differential equation (1*). Then one employs, instead of the functional equations (6) and (7), the relations (12) and (13).

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