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GRID ADJUSTMENT BASED ON A POSTERIORI ERROR ESTIMATORS

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Summary. The adjustment of one-dimensional space grid for a parabolic partial differential equation solved by the finite element method of lines is considered in the paper. In particular, the approach based on a posteriori error indicators and error estimators is studied. A statement on the rate of convergence of the approximation of error by estimator to the error in the case of a system of parabolic equations is presented.

Keywords: grid adjustment, principle of equidistribution of monitor, a posteriori error estimate, parabolic equation, finite element method, method of lines

AMS classification: 65M50, 65M15

1. INTRODUCTION

Recently, a variety of techniques for a posteriori error estimation have been theoretically developed and practically applied. A posteriori estimates can serve as a means for the grid adjustment ensuring the optimal number and optimal distribution of grid nodes in the finite element as well as finite difference method.

Obviously, if we want to construct a new, optimal grid we need a solution of the problem on some grid. This approach, therefore, is very suitable e.g. for solving parabolic partial differential equations by the method of lines. The analysis of the approximate solution at a fixed time level yields then a new grid to be used for the time step leading to the next time level.

The subject has been treated by many authors in various ways. Our approach is based on the concepts of error indicator and error estimator introduced and further developed by Babuška and his fellow-workers (see, e.g., [3], [4], [5], [9]).

We formulate a simple parabolic model problem in Section 2 and its discretization by the finite element method and the method of lines in Section 3. Methods for the grid adjustment are surveyed in Section 4. In Section 5 the concepts of the error indicator and error estimator are introduced.

Section 6 is concerned with the approximation of error by an estimator. A statement on the convergence rate of this approximation from [2] is quoted. A model

problem for a system of parabolic differential equations is formulated in Section 7. Moreover, the statement on the convergence rate of the approximation of error by the estimator for a parabolic system is presented there. In conclusion, numerical experience with the grid adjustment is briefly surveyed in Section 8.

2. Model problem

The principal ideas as well as algorithmic procedures connected with the use of an adaptive grid for solving linear parabolic partial differential equations can be demonstrated with the help of the following model problem.

We solve the equation

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial x}\Big(A(x)\frac{\partial}{\partial x}u(x,t)\Big) - Q(x)u(x,t) + f(x,t), \qquad 0 < x < 1, \quad 0 < t \leqslant T, \tag{2.1}$$

where $A(x) \geqslant A > 0$ is a positive smooth function and $Q(x) \geqslant 0$ a nonnegative one, with a fixed $T > 0$ for an unknown function $u(x,t)$. We prescribe the homogeneous Dirichlet boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad 0 \leqslant t \leqslant T, \tag{2.2}$$

and a smooth initial condition

$$u(x,0) = u_0(x), \quad 0 < x < 1. \tag{2.3}$$

Due to many practical applications, the variable x is usually called the *space variable* and the variable t the *time variable*.

We introduce the well-known variational formulation of the model problem which is the starting point for the finite element discretization. Let $H^1 = H^1(0,1)$ be the Sobolev space of functions defined on the interval $(0,1)$ with the norm given by

$$\|w\|_1^2 = \int_0^1 \Big(|w|^2 + \Big|\frac{\partial}{\partial x}w\Big|^2\Big)\,\mathrm{d}x.$$

Further let $H_0^1 = H_0^1(0,1)$ be the subspace of functions $w \in H^1(0,1)$ satisfying the homogeneous Dirichlet boundary conditions (2.2), i.e.

$$w(0) = 0, \quad w(1) = 0.$$

We put

$$(v, w) = \int_0^1 vw \, \mathrm{d}x,$$

$$a(v, w) = \int_0^1 \left(\left(\frac{\partial}{\partial x} v \right) A(x) \frac{\partial}{\partial x} w + vQ(x)w \right) \mathrm{d}x. \tag{2.4}$$

Now we say that a function $u(x, t)$ is the *variational solution* of the problem (2.1), (2.2), (2.3) if it maps, as a function of the variable t, the interval $[0, T]$ into H_0^1, if the identity

$$\left(v, \frac{\partial}{\partial t} u \right) = -a(v, u) + (v, f) \tag{2.5}$$

holds for each $t \in (0, T]$ and all functions $v \in H_0^1$, and if the identity

$$(v, u) = (v, u_0) \tag{2.6}$$

holds for $t = 0$ and all functions $v \in H_0^1$.

Remark 2.1. If the functions f and u_0 are sufficiently smooth and if $u_0(0) = u_0(1) = 0$ (the *consistency condition*) then the model problem (2.1) to (2.3) possesses a unique solution. Moreover, it possesses a unique variational solution, too.

3. Discretization

Finite element solutions of the model problem (2.1) to (2.3) are constructed from the variational formulation (2.5), (2.6). We use finite dimensional subspaces of H_0^1 to this end. We first introduce a partition

$$0 = x_0 < x < \ldots < x_{N-1} < x_N = 1 \tag{3.1}$$

of the interval $(0, 1)$ into N subintervals (x_{j-1}, x_j), $j = 1, \ldots, N$. We further put

$$h_j = x_j - x_{j-1}, \quad j = 1, \ldots, N,$$

and

$$h = \max_{j=1,\ldots,N} h_j.$$

We employ the simplest finite dimensional subspace $S = S^{N,1} \subset H^1$ of piecewise linear functions and the corresponding subspace $S_0 = S_0^{N,1} \subset H_0^1$ of piecewise linear functions satisfying the homogeneous Dirichlet boundary conditions.

We then say that a function $U(x,t)$ is the finite element approximate solution of the variational problem (2.5), (2.6) if it maps, as a function of the variable t, the interval $[0,T]$ into S_0, if the identity

$$\left(V, \frac{\partial}{\partial t} U\right) = -a(V,U) + (V,f) \tag{3.2}$$

holds for each $t \in (0,T]$ and all functions $V \in S_0$, and if the identity

$$(V,U) = (V,u_0) \tag{3.3}$$

holds for $t = 0$ and all functions $V \in S_0$.

Remark 3.1. Choose a basis $\{\varrho_j\}_{j=1}^{N-1}$ for the finite dimensional space $S_0^{N,1}$ formed by the usual piecewise linear basis functions of the finite element method, i.e.

$$\varrho_j(x_j) = 1, \quad \varrho_j(x_r) = 0, \quad r \neq j, \quad \text{for } j = 1, \ldots, N-1.$$

Putting

$$U(x,t) = \sum_{j=1}^{N-1} c_j(t)\varrho_j(x) \tag{3.4}$$

with the coefficients $c_j(t)$ depending on t and employing the test functions

$$V(x) = \varrho_r(x), \quad r = 1, \ldots, N-1,$$

in (3.2), we finally obtain the system

$$\sum_{j=1}^{N-1} \frac{\partial}{\partial t} c_j(t)(\varrho_r, \varrho_j) + \sum_{j=1}^{N-1} c_j(t) a(\varrho_r, \varrho_j) - (\varrho_r, f) = 0, \quad r = 1, \ldots, N-1, \tag{3.5}$$

of $N-1$ equations. This is a system of ordinary differential equations (in the variable t) for the unknown coefficients $c_j(t)$, $j = 1, \ldots, N-1$. The initial conditions for $c_j(0)$ are determined by the identity (3.3) which becomes

$$\sum_{j=1}^{N-1} c_j(0)(\varrho_r, \varrho_j) = (\varrho_r, u_0), \quad r = 1, \ldots, N-1, \tag{3.6}$$

i.e. a system of $N-1$ linear algebraic equations.

The procedure for constructing the approximate solution $U(x,t)$ described above is called the method of lines. In theoretical considerations it is usually assumed that the initial value problem for the ordinary differential system (3.5) is integrated exactly. In practice, the problem is solved by proper numerical software (e.g. LSODI [8] or DDASSL [13]). The error of the time integration can then be controlled by a proper choice of the error tolerance which is an input to the program used.

4. Space grid construction

The programs used to integrate the system (3.2), (3.3) (or (3.5), (3.6)) of ordinary differential equations determine proper time steps to proceed from a time level to the next one. In this way, we obtain the solution of the solved discrete problem (3.2), (3.3) at definite "natural" time levels. We can thus examine the approximate solution of the model problem (2.1) to (2.3) at each such level and we can adjust the space grid properly to proceed to the next time level.

Remark 4.1. The software used is based on backward difference formulae of order up to five. Any grid adjustment thus represents not only an interpolation of the solution from the old grid to the new one but also an initiation of the ordinary system solver which requires much more arithmetic operations than a current time step from the actual time level to the next one. It is thus disadvantageous to change the grid too often.

The concepts of monitor and monitor function are often used to construct the space grid. The *monitor function* is a nonnegative space integrable function $M(x,t)$ defined on $(0,1) \times (0,T)$. The *monitor* for the interval (x_{j-1}, x_j) at the time level t is the integral

$$\int_{x_{j-1}}^{x_j} M(x,t)\,\mathrm{d}x.$$

The *principle of equidistribution* of the monitor then requires the space grid to fulfil the conditions

$$\int_{x_{j-1}}^{x_j} M(x,t)\,\mathrm{d}x = \int_{x_j}^{x_{j+1}} M(x,t)\,\mathrm{d}x, \quad j = 1,\ldots,N-1,$$

for each time t considered.

The individual ways of grid adjustment differ from each other just by the choice of the monitor function. The monitor function should characterize the solution of the problem in some way. It is defined by some relation to the exact solution $u(x,t)$ of the problem but the approximate solution $U(x,t)$ is used in practical computation instead.

There is a vast literature on the subject (see, e.g., [6], [7], [10], [11], [14] and the quotations therein). The most often used approaches are the following.

1. The *arc length*, i.e. the length of the curve which is the graph of the solution $u(x,t)$ in the interval (x_{j-1}, x_j), is chosen for the monitor. The corresponding monitor function is then

$$M(x,t) = \sqrt{1 + \left(\frac{\partial}{\partial x} u(x,t)\right)^2}$$

and the geometric meaning of the monitor

$$\int_{x_{j-1}}^{x_j} \sqrt{1 + \left(\frac{\partial}{\partial x} u(x,t)\right)^2}\, \mathrm{d}x$$

is really the length of the curve $u(x,t)$ in (x_{j-1}, x_j). A longer graph of the solution in a subinterval means that the solution changes more rapidly in this subinterval and the subinterval should be made shorter according to the principle of equidistribution.

2. The magnitude of the *gradient* of the solution

$$M(x,t) = \left|\frac{\partial}{\partial x} u(x,t)\right|,$$

the *curvature* of the graph of the solution

$$M(x,t) = \frac{\left|\frac{\partial^2}{\partial x^2} u(x,t)\right|}{\left(1 + \left(\frac{\partial}{\partial x}(x,t)\right)^2\right)^{3/2}},$$

or a combination of them is taken for the monitor function. The geometric meaning is then similar to the previous case.

3. The *error* of the approximate solution in a suitable norm is chosen for the monitor. For example, the monitor function

$$M(x,t) = |u - U|^2 + \left|\frac{\partial}{\partial x} u - \frac{\partial}{\partial x} U\right|^2$$

defines the monitor

$$\int_{x_{j-1}}^{x_j} \left(|u - U|^2 + \left|\frac{\partial}{\partial x} u - \frac{\partial}{\partial x} U\right|^2\right) \mathrm{d}x = \|u - U\|_{1,j}^2,$$

which is the square of the $H^1(x_{j-1}, x_j)$ norm of the error. The equidistribution principle now leads to shorter intervals where the error of the approximate solution is large.

The knowledge of the approximate solution is sufficient for the use of the first two monitors. The third one, however, requires the information on the error

$$e(x,t) = u(x,t) - U(x,t), \tag{4.1}$$

e.g. an a posteriori error estimate whose construction we describe in the next section.

Remark 4.2. Note that the first two monitors give no way to determine the number N such that the error of the solution (measured in some norm) is smaller than a bound prescribed. Only the third one, connected with the computation of an error estimate, can be used to choose a proper value of N (see, e.g., [3], [5]).

The procedure we have described so far is called the *static regridding* since the grid adjustment, as we have mentioned, is confined to natural time levels. The grid to be used for the calculation of the solution at the next level is constructed from the knowledge of the approximate solution at the actual level.

It is possible to proceed in a more accurate way called the *dynamic regridding*. Let us assume that the nodes of the partition (3.1) depend on time, i.e.

$$x_j = x_j(t), \quad j = 0, 1, \ldots, N, \tag{4.2}$$

but

$$x_0(t) \equiv 0, \quad x_N(t) \equiv 1.$$

Then the basis functions of the finite dimensional space S_0 (depending on the nodes x_j) depend also on time.

Remark 4.3. The equations (3.2), (3.3) for U remain valid for the dynamic regridding but the calculation of the derivative $\frac{\partial}{\partial t}$ of the expression (3.4) should be carried out with regard to (4.2). The system (3.5) is not true in this case.

Moreover, we add the system

(4.3)
$$\frac{\partial}{\partial t}\left(x_j(t) - x_{j-1}(t)\right) = \lambda\left(\frac{1}{N}\int_0^1 M(x,t)\,\mathrm{d}x - \int_{x_{j-1}}^{x_j} M(x,t)\,\mathrm{d}x\right), \quad j = 1, \ldots, N,$$

of ordinary differential equations for the nodes to the system (3.2) and solve all the equations simultaneously in the way we mentioned in Section 3. λ is a positive parameter to control the stability of the grid. The initial value $x_j(0)$ for the solution of the system (4.3) can be any suitable initial partition of the interval $(0, 1)$, e.g. the equidistant one.

The physical meaning of (4.3) is simple. The left-hand part expresses the time change of the length of a subinterval of the partition (i.e. the velocity with which

a pair of nodes move apart or together). On the right-hand part, the first term in the parentheses is an average monitor value over $(0,1)$ while the second term is the actual monitor value in (x_{j-1}, x_j). A positive right-hand part means that the actual monitor value is smaller than average and implies that the velocity of the movement of this pair of nodes is positive, the nodes move apart, and the subinterval is made longer in the next step.

5. A POSTERIORI ERROR ESTIMATOR

To construct an a posteriori error estimate we rewrite (4.1) in the form

$$u(x,t) = e(x,t) + U(x,t)$$

and substitute for $u(x,t)$ into (2.5), (2.6) to arrive at the identity

$$\left(v, \frac{\partial}{\partial t}(e+U)\right) = -a(v, e+U) + (v,f)$$

that holds for each $t \in (0,T]$ and all functions $v \in H_0^1$ and the identity

$$(v, e+U) = (v, u_0)$$

holding for $t = 0$ and all functions $v \in H_0^1$.

We will look for the finite element approximation $E(x,t)$ of the discretization error $e(x,t)$ in the finite dimensional subspace $\hat{S}_0 = \hat{S}_0^{N,2} \subset H_0^1$ such that $V \in \hat{S}_0^{N,2}$ if

$$V(x) = \sum_{j=1}^{N} V_j(x)$$

where $V_j \in \hat{S}_{0,j}^2$ and $\hat{S}_{0,j} = \hat{S}_{0,j}^2$, $j = 1, \ldots, N$, are the spaces of functions equal to quadratic polynomials on $[x_{j-1}, x_j]$, equal to zero elsewhere, and belonging to H_0^1, i.e. satisfying the local homogeneous Dirichlet conditions

$$V_j(x_l) = 0, \quad l = 0, 1, \ldots, N.$$

We then say that $E(x,t)$ is a finite element approximation of the error $e(x,t)$ if it maps, as a function of the variable t, the interval $[0,T]$ into $\hat{S}_0$, if the identity

$$\left(V, \frac{\partial}{\partial t}(E+U)\right) = -a(V, E+U) + (V,f) \tag{5.1}$$

holds for each $t \in (0, T]$ and all functions $V \in \hat{S}_0$, and if the identity

$$(V, E + U) = (V, u_0) \tag{5.2}$$

holds for $t = 0$ and all functions $V \in \hat{S}_0$.

Notice that, according to the definition of the finite dimensional space $\hat{S}_0$, the problem (5.1), (5.2) can be solved as a series of local problems on (x_{j-1}, x_j). In fact, putting

$$\begin{aligned} (v, w)_j &= \int_{x_{j-1}}^{x_j} vw \, \mathrm{d}x, \\ a_j(v, w) &= \int_{x_{j-1}}^{x_j} \left(\left(\frac{\partial}{\partial x} v \right) A(x) \frac{\partial}{\partial x} w + vQ(x)w \right) \mathrm{d}x, \end{aligned} \tag{5.3}$$

we can rewrite (5.1), (5.2) in the form

$$\left(V, \frac{\partial}{\partial t} E \right)_j = -a_j(V, E) - \left(V, \frac{\partial}{\partial t} U \right)_j - a_j(V, U) + (V, f)_j, \tag{5.4}$$

$$(V, E)_j = -(V, U)_j + (V, u_0)_j \tag{5.5}$$

that holds for all functions $V \in \hat{S}_{0,j}$ and $j = 1, \ldots, N$.

Remark 5.1. Choose a basis $\{\sigma_j\}_{j=1}^N$ for the space $\hat{S}_0^{N,2}$ formed by the local quadratic parabolas

$$\sigma_j(x) = \begin{cases} \dfrac{4(x - x_{j-1})(x_j - x)}{(x_j - x_{j-1})^2} & \text{for } x_{j-1} \leqslant x \leqslant x_j, \\ 0 \text{ elsewhere}, & j = 1, \ldots, N. \end{cases}$$

Assuming (3.4), putting

$$E(x, t) = \sum_{j=1}^{N} d_j(t) \sigma_j(x) \tag{5.6}$$

with the coefficients $d_j(t)$ depending on t, and employing the test functions

$$V(x) = \sigma_r(x), \quad r = 1, \ldots, N,$$

in (5.4), we finally obtain ordinary differential equations

$$\begin{aligned} &\frac{\partial}{\partial t} d_r(t)(\sigma_r, \sigma_r)_r + d_r(t) a_r(\sigma_r, \sigma_r) + \sum_{j=r-1}^{r} \frac{\partial}{\partial t} c_j(t)(\sigma_r, \varrho_j)_r \\ &\quad + \sum_{j=r-1}^{r} c_j(t) a_r(\sigma_r, \varrho_j) - (\sigma_r, f)_r = 0, \quad r = 1, \ldots, N, \end{aligned} \tag{5.7}$$

in the variable t for the unknown coefficients $d_r(t)$, $r = 1, \ldots, N$. We put

$$c_0(t) \equiv c_N(t) \equiv 0$$

for the simplicity of notation. Notice that the choice of $\hat{S}_0$ implies that each d_r can be calculated from the rth equation of (5.7) independently of the other equations. The initial condition

$$d_r(0)(\sigma_r, \sigma_r)_r = -\sum_{j=r-1}^{r} c_j(0)(\sigma_r, \varrho_j)_r + (\sigma_r, u_0)_r, \quad r = 1, \ldots, N,$$

for $d_r(0)$ is obtained from (5.5).

In practice the equations (5.7) are added to the system (3.5) and solved simultaneously with it in the way described in Section 3.

Remark 5.2. If the regridding is dynamic we should take into account that the basis functions of the finite dimensional space $\hat{S}_0$ (depending on the nodes x_j) depend also on time and that the calculation of the derivative $\frac{\partial}{\partial t}$ of the expressions (3.4) and (5.6) in both (3.2) and (5.4) has to be carried out with regard to (4.2). The systems (3.5) and (5.7) are not true in this case. We solve the systems (3.2), (5.4), and (4.3) simultaneously.

Remark 5.3. Obviously, the process reverse to the integration by parts, which leads from (2.1), (2.2) to (2.5), gives

$$a_j(V, U) = \int_{x_{j-1}}^{x_j} \Big(- VA(x)\frac{\partial^2}{\partial x^2}U + VQ(x)U \Big)\, \mathrm{d}x$$

and we can thus rewrite (5.4) as

$$\Big(V, \frac{\partial}{\partial t}E\Big)_j = -a_j(V, E) - (V, R)_j$$

where

$$R(x,t) = \frac{\partial}{\partial t}U(x,t) - A(x)\frac{\partial^2}{\partial x^2}U(x,t) + Q(x)U(x,t) - f(x,t)$$

is the residual of (2.1) after substituting U for u.

The restriction of $E(x,t)$ to the interval $[x_{j-1}, x_j]$ approximates the error on this interval. A properly chosen norm (e.g. the H^1 norm) of the restriction can thus be used for the monitor on (x_{j-1}, x_j). The error on each interval of the partition is then characterized by a single number called the *error indicator*. The error on the whole interval $[0, 1]$ can also be characterized by a single number, i.e. by a properly chosen norm of $E(x,t)$. This number is called the *error estimator*.

6. Error analysis

The most important question is now the quality of approximation of the error $e(x,t)$ by the quantity $E(x,t)$ defined by (5.1), (5.2). The answer is given for the parabolic model problem (2.1) to (2.3) and for the approximation by piecewise polynomials of a general degree p by Adjerid et al. [2].

Let p be a positive integer. We introduce the finite dimensional spaces $S^{N,p}$, $S_0^{N,p}$, and $\hat{S}_0^{N,p}$ in the following way. Let $P_p(x_{j-1}, x_j)$ be a class of polynomials of degree p on $[x_{j-1}, x_j]$. Then

$$S^{N,p} = \{W(x) \mid W \in H^1,\ W \in P_p(x_{j-1}, x_j) \text{ for } x \in [x_{j-1}, x_j],\ j = 1, \ldots, N\},$$
$$S_0^{N,p} = \{W(x) \mid W \in H_0^1,\ W \in P_p(x_{j-1}, x_j) \text{ for } x \in [x_{j-1}, x_j],\ j = 1, \ldots, N\},$$

$\hat{S}_0^{N,p+1}$ is such a space that $W \in \hat{S}_0^{N,p+1}$ if

$$W(x) = \sum_{j=1}^{N} W_j(x)$$

where $W_j \in \hat{S}_{0,j}^{p+1}$ and

$$\hat{S}_{0,j}^{p+1} = \{Z(x) \mid Z \in H_0^1,\ Z \in P_{p+1}(x_{j-1}, x_j) \text{ for } x \in [x_{j-1}, x_j],$$
$$Z \equiv 0 \text{ elsewhere}\},\quad j = 1, \ldots, N.$$

We modify the variational formulation of the model problem to present the results of [2]. The equation (2.5) for $u(x,t) \in H_0^1$ remains unchanged, i.e.

$$\left(v, \frac{\partial}{\partial t} u\right) = -a(v,u) + (v,f), \quad t \in (0,T],\ v \in H_0^1, \tag{6.1}$$

but the initial condition is formulated in a way different from (2.6), namely

$$a(v,u) = a(v,u_0), \quad t = 0,\ v \in H_0^1. \tag{6.2}$$

Similarly we have

$$\left(V, \frac{\partial}{\partial t} U\right) = -a(V,U) + (V,f), \quad t \in (0,T],\ V \in S_0^{N,p}, \tag{6.3}$$

$$a(V,U) = a(V,u_0), \quad t = 0,\ V \in S_0^{N,p}, \tag{6.4}$$

instead of (3.2), (3.3) for $U \in S_0^{N,p}$ and

$$\left(V, \frac{\partial}{\partial t}(E+U)\right)_j = -a_j(V, E+U) + (V,f)_j, \quad t \in (0,T],\ V \in \hat{S}_{0,j}^{p+1}, \tag{6.5}$$

$$a_j(V, E+U) = a_j(V,u_0), \quad t = 0,\ V \in \hat{S}_{0,j}^{p+1},\ j = 1, \ldots, N, \tag{6.6}$$

instead of (5.4), (5.5) for $E \in \hat{S}_0^{N,p+1}$.

Let s be a nonnegative integer. We will use the Sobolev space $H^s = H^s(0,1)$ of functions defined on $(0,1)$ with the norm given by

$$\|w\|_s^2 = \sum_{l=0}^{s} \Big(\frac{\partial^l}{\partial x^l} w, \frac{\partial^l}{\partial x^l} w \Big). \tag{6.7}$$

Theorem 6.1. *Let $u \in H_0^1 \cap H^{p+2}$ and $U \in S_0^{N,p}$ and $E \in \hat{S}_0^{N,p+1}$ be solutions of* (6.1), (6.2) *and* (6.3), (6.4) *and* (6.5), (6.6), *respectively. Then there exist positive constants C and δ such that*

$$\|e(.,t)\|_1^2 = \|u(.,t) - U(.,t)\|_1^2 = \|E(.,t)\|_1^2 + \varepsilon, \quad 0 < \delta < t \leqslant T,$$

where

$$|\varepsilon| \leqslant C(u) h^{2p+1}.$$

Proof. The proof is given in [2]. It is based on two lemmas we quote below. □

Lemma 6.1. *Let $W \in S^{N,p}$ interpolate $w \in H^{p+1}$ at x_{j-1}, x_j, and $p-1$ distinct points on (x_{j-1}, x_j), $j = 1, \ldots, N$. Then there exists a positive constant C such that*

$$\|w - W\|_s \leqslant C h^{p+1-s} \|w\|_{p+1}, \quad s = 0, 1, \ldots, p.$$

Proof. See, e.g., Oden and Carey [12]. □

Lemma 6.2. *Let u and U be solutions of* (6.1), (6.2) *and* (6.3), (6.4), *respectively. Further let*

$$a(V, \hat{U}) = a(V, u) \quad \text{for } 0 < t \leqslant T \text{ and all } V \in S_0^{N,p}$$

be the energy projection of u onto $S_0^{N,p}$. If $u_0 \in H_0^1 \cap H^{p+1}$ and u is smooth enough for all terms in (6.8), (6.9), (6.10) *to be bounded then there exist constants $C > 0$*

and $\delta > 0$ such that

$$\|\hat{U} - U\|_1^2 \leqslant Ch^{2(p+1)} \int_0^t \left\| \frac{\partial}{\partial t} u \right\|_{p+1}^2 \mathrm{d}\tau, \quad 0 \leqslant t \leqslant T, \tag{6.8}$$

$$\begin{aligned} \left\| \frac{\partial^n}{\partial t^n} e(.,t) \right\|_0 \leqslant Ch^{p+1} \Bigg(& \|u_0\|_{p+1} + \sum_{l=0}^{n} \left\| \frac{\partial^l}{\partial t^l} u(.,t) \right\|_{p+1} \\ & + \int_{t-\delta}^{t} \left\| \frac{\partial^{n+1}}{\partial t^{n+1}} u(.,\tau) \right\|_{p+1} \mathrm{d}\tau + \int_0^t \left\| \frac{\partial}{\partial t} u(.,\tau) \right\|_2 \mathrm{d}\tau \Bigg), \\ & 0 < \delta < t \leqslant T, \ n \geqslant 0, \end{aligned} \tag{6.9}$$

$$\begin{aligned} \left\| \frac{\partial^n}{\partial t^n} e(.,t) \right\|_1 \leqslant Ch^p \Bigg(& \|u_0\|_{p+1} + \sum_{l=0}^{n} \sup_{t-\delta<\tau<t} \left\| \frac{\partial^l}{\partial t^l} u(.,\tau) \right\|_{p+1} \\ & + \left(\int_{t-\delta}^{t} \left\| \frac{\partial^{n+1}}{\partial t^{n+1}} u(.,\tau) \right\|_p^2 \mathrm{d}\tau \right)^{1/2} \\ & + h \left(\|u_0\|_0 + \int_0^t \|f\|_0 \, \mathrm{d}\tau \right) \Bigg), \quad 0 < \delta < t \leqslant T, \ n \geqslant 0. \end{aligned} \tag{6.10}$$

P r o o f. The proof of (6.8) is given, e.g., in Wait and Mitchell [16] and the proof of (6.9), (6.10) in Thomée [15]. □

The error estimate E defined by (5.4), (5.5) or (6.5), (6.6) is usually called the *parabolic estimate* since it is calculated from the parabolic equation (5.1) or, in particular, from the local parabolic equations (5.4) or (6.5). Some computation may be saved if we assume that the change of the error $e(x,t)$ with time is very slow. We can then put

$$\frac{\partial}{\partial t} E = 0$$

and rewrite (6.5) in the form

$$\left(V, \frac{\partial}{\partial t} U \right)_j = -a_j(V, E+U) + (V,f)_j, \quad t \in (0,T], \ V \in \hat{S}_{0,j}^{p+1}, \ j = 1, \ldots, N. \tag{6.11}$$

The error estimate defined by the equation (6.11) corresponding to an elliptic differential equation for E is called the *elliptic estimate*. Its accuracy does not differ from the accuracy of the parabolic estimate.

Theorem 6.2. *Let $u \in H_0^1 \cap H^{p+2}$ and $U \in S_0^{N,p}$ and $E \in \hat{S}_0^{N,p+1}$ be solutions of (6.1), (6.2) and (6.3), (6.4) and (6.11), (6.6), respectively. Then there exist positive constants C and δ such that*

$$\|e(.,t)\|_1^2 = \|u(.,t) - U(.,t)\|_1^2 = \|E(.,t)\|_1^2 + \varepsilon, \quad 0 < \delta < t \leqslant T,$$

where

$$|\varepsilon| \leqslant C(u)h^{2p+1}.$$

Proof. The proof is analogical to the proof of Theorem 6.1 and is presented in [2]. □

7. Error analysis for a parabolic system

In this section, we will formulate an initial boundary value problem for a system of parabolic partial differential equations. We will further formulate statements analogical to those of Theorems 6.1 and 6.2.

We work with real vector functions and their spaces in this section. Let M be a positive integer. We solve the system of equations

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial x}\Big(A(x)\frac{\partial}{\partial x}u(x,t)\Big) - Q(x)u(x,t) + f(x,t), \qquad 0 < x < 1,\ 0 < t \leqslant T,$$

for an unknown vector function $u = (u_1, \ldots, u_M)^{\mathrm{T}}$, where $A = (A_{ik})$ is a given smooth $M \times M$ symmetric positive definite matrix, $Q = (Q_{ik})$ is a given smooth $M \times M$ symmetric positive semidefinite matrix, and $f = (f_1, \ldots, f_M)^{\mathrm{T}}$ is a given vector. We further impose the homogeneous Dirichlet boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad 0 \leqslant t \leqslant T,$$

and a smooth initial condition

$$u(x,0) = u_0(x), \quad 0 < x < 1,$$

where u_0 is a given M-component vector. The smoothness and consistency conditions are assumed here, too.

We now denote by

$$(v,w) = \int_0^1 v^{\mathrm{T}} w \,\mathrm{d}x \tag{7.1}$$

the inner product of two vector functions $v = (v_1, \ldots, v_M)^{\mathrm{T}}$ and $w = (w_1, \ldots, w_M)^{\mathrm{T}}$. Further we put

$$(v,w)_j = \int_{x_{j-1}}^{x_j} v^{\mathrm{T}} w \,\mathrm{d}x.$$

Let s be a nonnegative integer. Then $H^s = H^s(0,1)$ is the Sobolev space of vector functions defined on $(0,1)$ with the norm given, in the notation (7.1), by (6.7). We, moreover, introduce the subspace $H_0^1 = H_0^1(0,1)$ of vector functions $w \in H^1(0,1)$ satisfying the homogeneous Dirichlet boundary conditions. Finally, we now use the notation

$$a(v,w) = \int_0^1 \left(\left(\frac{\partial}{\partial x} v^{\mathrm{T}} \right) A \frac{\partial}{\partial x} w + v^{\mathrm{T}} Q w \right) \mathrm{d}x,$$

$$a_j(v,w) = \int_{x_{j-1}}^{x_j} \left(\left(\frac{\partial}{\partial x} v^{\mathrm{T}} \right) A \frac{\partial}{\partial x} w + v^{\mathrm{T}} Q w \right) \mathrm{d}x$$

instead of (2.4), (5.3) and introduce the finite dimensional spaces $S_0^{N,p}$ and $\hat{S}_0^{N,p+1}$ of vector functions like in Section 6. We put

$$S_0^{N,p} = \Big\{ W(x) = \big(W_1(x), \ldots, W_M(x)\big)^{\mathrm{T}} \mid W \in H_0^1,\ W_k \in P_p(x_{j-1}, x_j) \\ \text{for } x \in [x_{j-1}, x_j],\ j = 1, \ldots, N,\ k = 1, \ldots, M \Big\}$$

and $\hat{S}_0^{N,p+1}$ is such a space that $W \in \hat{S}_0^{N,p+1}$ if

$$W(x) = \sum_{j=1}^{N} W_j(x)$$

where $W_j \in \hat{S}_{0,j}^{p+1}$ are vector functions and

$$\hat{S}_{0,j}^{p+1} = \Big\{ Z(x) = \big(Z_1(x), \ldots, Z_M(x)\big)^{\mathrm{T}} \mid Z \in H_0^1,\ Z_k \in P_{p+1}(x_{j-1}, x_j) \\ \text{for } x \in [x_{j-1}, x_j],\ Z_k \equiv 0 \text{ elsewhere},\ k = 1, \ldots, M \Big\}, \quad j = 1, \ldots, N.$$

In this notation, the problem for the vector function $u(x,t)$ is formally written in the form (6.1), (6.2). Further, the vector function $U(x,t)$ approximating the exact solution $u(x,t)$ satisfies the equations (6.3), (6.4). Finally, the parabolic error estimate is the vector function $E(x,t)$ fulfilling (6.5), (6.6) and the elliptic error estimate is the vector function fulfilling (6.11), (6.6). The quality of approximation of the vector of error $e(x,t)$ by $E(x,t)$ is stated in the next two theorems.

Theorem 7.1. *Let $u \in H_0^1 \cap H^{p+2}$ and $U \in S_0^{N,p}$ and $E \in \hat{S}_0^{N,p+1}$ be vector solutions of the systems* (6.1), (6.2) *and* (6.3), (6.4) *and* (6.5), (6.6), *respectively. Then there exist positive constants C and δ such that*

$$\|e(.,t)\|_1^2 = \|u(.,t) - U(.,t)\|_1^2 = \|E(.,t)\|_1^2 + \varepsilon, \quad 0 < \delta < t \leqslant T, \tag{7.2}$$

where

$$|\varepsilon| \leqslant C(u)h^{2p+1}.$$

Proof. The proof can be carried out in a way similar to that of [2]. It will be published in detail elsewhere. □

Theorem 7.2. *Let* $u \in H_0^1 \cap H^{p+2}$ *and* $U \in S_0^{N,p}$ *and* $E \in \hat{S}_0^{N,p+1}$ *be vector solutions of the systems* (6.1), (6.2) *and* (6.3), (6.4) *and* (6.11), (6.6), *respectively. Then there exist positive constants* C *and* δ *such that* (7.2) *holds where*

$$|\varepsilon| \leqslant C(u)h^{2p+1}.$$

Proof. The proof is analogical to the proof of Theorem 7.1 and will be published in detail elsewhere. □

8. Numerical experiments

We have performed a large number of numerical experiments with different strategies of the grid adjustment. The basic set of test examples were parabolic equations and parabolic systems taken from [1].

The evaluation of the results shows, in general, that the regridding based on the monitor of the third kind, i.e. on the a posteriori error indicators, gives very reliable and accurate solutions, and rather stable grids. Moreover, this approach provides the possibility to determine N, the number of grid nodes, if some error tolerance is given. The experiments included the dynamic as well as static regridding with both parabolic and elliptic estimates. The test problems successfully solved included also the cases not supported theoretically yet (e.g. some nonlinear problems).

The change to other monitors brings computational work savings since no error estimates are computed. On the other hand, the first and second monitors reflect the behavior of solution much slower and give no hint as far as the choice of N is concerned.

Detailed results of numerical experiments mentioned will be published elsewhere.

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