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A TWO PARAMETER ITERATIVE METHOD FOR SOLVING ALGEBRAIC SYSTEMS OF DOMAIN DECOMPOSITION TYPE

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Summary. An iterative procedure containing two parameters for linear algebraic systems originating from the domain decompostition technique is proposed. The optimization of the parameters is investigated. A numeric example is given as an illustration.

Keywords: iterative methods, block matrix, domain decomposition

AMS classification: 65N30

The paper generalizes the iterative procedure proposed in [1]. The procedure is appropriate for solving linear algebraic systems of special form originating from the discretization of boundary value problems on composite domains and belongs to the domain decomposition algorithms. Mostly, such algorithms present preconditioners for the CG method, see e.g. [2], [3], [4], [5]. A one parameter method for the use independent of CG method is proposed in [6].

To the contrary of the previous case, we use two parameters and perform their optimization. We thus obtain a reasonable improvement of convergence rate. The method can be easily modified to yield a preconditioner for the CG method. For all details see [7].

1. DESCRIPTION OF THE METHOD

Let a system of linear algebraic equations

$$Mw = d \tag{1}$$

be given, where

$$M = \begin{pmatrix} A & D^T & O \\ D & B & E^T \\ O & E & C \end{pmatrix}, \quad w = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad d = \begin{pmatrix} f \\ g \\ h \end{pmatrix}. \tag{2}$$

We suppose that the matrices A, B, C are square and symmetric, O are null matrices of the corresponding orders.

Let B_1 and B_2 be symmetric matrices such that $B = B_1 + B_2$ and that the matrices

$$A_1 = \begin{pmatrix} A & D^T \\ D & B_1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} B_2 & E^T \\ E & C \end{pmatrix}$$

are positive definite. It is easily seen that the matrix M is positive definite, too.

Let us denote S_U the Schur complement of A in A_1, i.e., $S_U = B_1 - DA^{-1}D^T$ and by S_L the Schur complement of C in A_2, i.e., $S_L = B_2 - E^TC^{-1}E$. Further, we denote $p = DA^{-1}f + E^TC^{-1}h$.

The procedure consists in the construction of a sequence of vectors y_n converging to the y-component of the true solution of (1). It is easily calculated that this exact value fulfils the equation

$$(S_U + S_L)y_\infty = (g - p). \tag{3}$$

Let α, β be real parameters, $0 < \alpha, \beta < 1$. Let an approximation y_n to y_∞ be given. Let $y_{n+1}^{(1)}$ be the y-component of the solution of the system

$$Ax + D^Ty = f, \tag{4_1}$$

$$Dx + B_1y = (1 - \alpha)g + \alpha(DA^{-1}f + S_Uy_n) - (1 - \alpha)(E^TC^{-1}h + S_Ly_n), \tag{4_2}$$

and let $y_{n+1}^{(2)}$ be the y-component of the solution of

$$B_2y + E^Tz = \alpha g - \alpha(DA^{-1}f + S_Uy_n) + (1 - \alpha)(E^TC^{-1}h + S_Ly_n), \tag{5_1}$$

$$Ey + Cz = h. \tag{5_2}$$

The new iteration is defined by

$$y_{n+1} = \beta y_{n+1}^{(1)} + (1 - \beta)y_{n+1}^{(2)}. \tag{6}$$

Remark 1. Let us substitute y_n for y into the system (4), compute x from (4_1) and substitute this value into the left-hand side of (4_2). We obtain

$$Dx + B_1y_n = DA^{-1}f + S_Uy_n.$$

An analogical manipulation with the system (5) gives

$$B_2y_n + E^Tz = E^TC^{-1}h + S_Ly_n.$$

This is the way for computing the right hand sides in the above systems.

After the solution of the systems (4) and (5) for $y_{n+1}^{(1)}$ and $y_{n+1}^{(2)}$ respectively, and the substitution into (6) we have

$$y_{n+1} = \{[\alpha\beta + (1-\alpha)(1-\beta)]I - (1-\alpha)\beta T - \alpha(1-\beta)T^{-1}\}y_n + \\ + [(1-\alpha)\beta S_U^{-1} + \alpha(1-\beta)S_L^{-1}](g-p),$$

where $T = S_U^{-1}S_L$.

The character of the iterations is determined by the spectral radius of the iteration matrix K

$$K = [\alpha\beta + (1-\alpha)(1-\beta)]\ I - (1-\alpha)\beta T - \alpha(1-\beta)T^{-1}.$$

Before studying the optimization of the spectral radius in dependence of the parameters α and β, we show that the process (if convergent) converges to the exact solution. In the case of convergence we have

$$(I-K)\,y_\infty = [(1-\alpha)\beta S_U^{-1} + \alpha(1-\beta)S_L^{-1}](g-p).$$

However, it is easily seen that

$$I - K = P(S_U + S_L),$$

where $P = (1-\alpha)\beta S_U^{-1} + \alpha(1-\beta)S_L^{-1}$. But for a convergent K the matrix P is regular and we obtain (3).

The form of the matrix $I - K$ enables us to rewrite the iterations in the form

$$y_{n+1} = y_n - P\,[(S_U + S_L)\,y_n - (g-p)]\,.$$

The matrix P can be thus considered as a preconditioner for the matrix $S_U + S_L$ and the proposed method as the method of residual iterations with preconditioning. This suggests the use of P as a preconditioner for the method of conjugate gradients for the solution of the equation (3).

2. Optimization of the parameters

The eigenvalues of K depend only on the eigenvalues of T. We will suppose that we know the bounds for the spectrum of this matrix

$$\operatorname{sp}(T) \subset [m, M], \qquad m, M > 0.$$

Introducing new parameters s and t by

$$s = \sqrt{\alpha(1-\alpha)\beta(1-\beta)}, \quad t = \sqrt{\frac{(1-\alpha)\beta}{\alpha(1-\beta)}} \tag{7}$$

we obtain

$$(1-\alpha)\beta = st,$$
$$\alpha(1-\beta) = s/t.$$

Further, we have

$$t \in (0, \infty), \qquad s \in (0, t/(1+t)^2]. \tag{8}$$

Conversely, for every pair (s, t) satisfying (8), the system (7) has at least one solution α, β. In fact, let us denote

$$\psi(q) = \frac{qt}{(q+t)(1+qt)}.$$

We find such a q, $0 < q < 1$, that $s = \psi(q)$. This is always possible because the function ψ is (for a fixed t) positive and increasing on the interval $[0, 1]$ and takes the values between 0 and $t/(1+t)^2$. Setting $\alpha = 1/(1+qt)$ and $\beta = t/(q+t)$, we obtain one solution of (7).

With the new parameters, we have for the matrix K

$$K = \left[1 - s\left(t + \frac{1}{t}\right)\right] I - s\left(tT + \frac{1}{t}T^{-1}\right).$$

For every eigenvalue μ_i of T we have an eigenvalue λ_i of K

$$\lambda_i = 1 - s\left(t + \frac{1}{t}\right) - s\left(t\mu_i + \frac{1}{t\mu_i}\right)$$

or, with the use of the function $\varphi(t) = t + t^{-1} - 2$,

$$\lambda_i = 1 - s\left[4 + \varphi(t) + \varphi(t\mu_i)\right].$$

The spectral radius of K, $\varrho(K) = \max_i |\lambda_i|$ should be minimized in dependence of s and t. Instead of $\max_i |\lambda_i|$, we will consider

$$R(s,t) = \max_{x \in [m,M]} |1 - s[4 + \varphi(t) + \varphi(tx)]|$$

and minimize the function $R(s,t)$ for s, t satisfying (8). We obtain thus an upper bound for the spectral radius $\varrho(K)$.

Theorem. *The function $R(s,t)$ attains its minimum in the region given by* (8) *for*

$$s = \frac{2}{8 + 2\varphi(\sqrt{Mm}) + \varphi\left(\sqrt{\frac{M}{m}}\right)} \quad \text{and} \quad t = \frac{1}{\sqrt{Mm}}$$

and we have

$$\min_{s,t} R(s,t) = \frac{\varphi\left(\sqrt{\frac{M}{m}}\right)}{8 + 2\varphi(\sqrt{Mm}) + \varphi\left(\sqrt{\frac{M}{m}}\right)}. \tag{9}$$

See [7] for the proof.

The value given by (9) is less than one, so the iterative process converge for the optimum values of the parameters. The parameters α and β necessary for the iteration scheme are computed from s and t by the procedure described above.

Remark 2. If the bounds for the spectrum of T are strict, i.e., if $\mu_{\min} = m$ and $\mu_{\max} = M$, and, moreover, if $\mu_k = \sqrt{Mm}$ fore some k the estimate (9) gives the exact value of $\varrho(K)$. In other cases the spectral radius is less than the estimate.

3. Numerical example

As an illustration of the above procedure we will use the following problem. Let O be the L-shaped domain of Fig. 1, consisting of three squares, each with the side of length $\frac{1}{2}$. Let the following boundary value problem be given on O

$$\begin{aligned} \Delta u &= 0 \quad \text{in } O, \\ u &= g \quad \text{on } \partial O, \end{aligned} \tag{12}$$

where g is chosen so that the exact solution be

$$u(x,y) = x^3 - 3xy^2.$$

We introduce a uniform square mesh with the step $\frac{1}{2n}$. Then, we approximate the problem in the usual way with the standard five point scheme. We thus obtain a system of linear algebraic equations. In order to put it into the suitable form (1) with appropriate blocking of M, w and d we gather the equations corresponding to the mesh points lying in O_1 and O_3 in the first block, the equations corresponding to the points on the both interfaces i_1 and i_2 to the second block and the equations corresponding to the mesh points in O_2 to the third block.

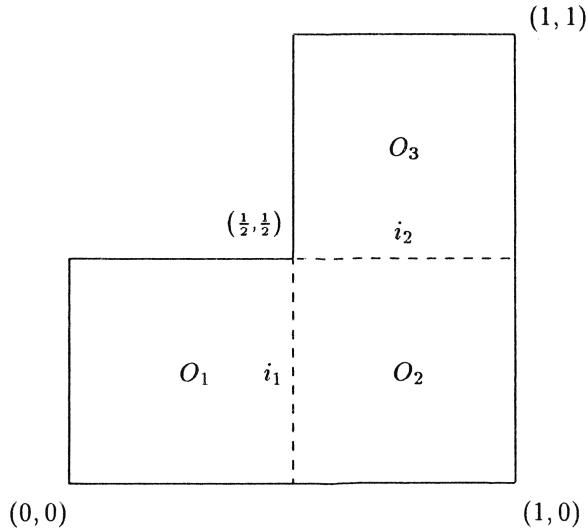

Fig. 1

We use the splitting $B = B_1 + B_2$ with $B_1 = B_2 = B/2$. The application of the method consists in the following steps:

1) We choose the values at the points on i_1 and i_2 arbitrarily.

2) We solve three discrete Dirichlet problems on each square separately. This corresponds to the computing of x from (4_1) and of z from (5_2), see Remark 1.

3) We compute the values of the discrete normal derivatives of these solutions on both interfaces from both sides. We proceed again according to the Remark 1. The splitting of B we made gives just the discrete normal derivatives.

4) We solve the systems (4) and (5), i.e., solve three separate discrete boundary value problems with the Neumann condition on the interfaces. The solution of the discrete boundary value problem on O_2 has, however, to fulfill the Dirichlet condition in the single point $(\frac{1}{2}, \frac{1}{2})$.

5) We compute the new approximation of the values on the interfaces from (6).

For the numerical experiment the null vector was taken for the initial approximation of the values on the interfaces. There were used two choices of the parameters. The first one was the choice of *standard parameters* $\alpha = 0.5, \beta = 0.5$. This choice does not guarantee convergence, in general, but it is reasonable in many cases. It yields convergence in our case. The second choice are the optimal parameters described in §2.

The maximum and minimum numerically computed eigenvalues of the matrix T, i.e., of the generalized eigenvalue problem $S_L u = \mu S_U u$ were taken as the values of M and m, respectively. The values of the optimal parameters for different values of n are shown in Table 1.

n	4	8	16	32	64	128
α	.5454	.5590	.5664	.5699	.5713	.5713
β	.5724	.6186	.6614	.6999	.7337	.7631

Table 1. Optimal parameters

Figures 2 and 3 show the results of the iterative process for the first four iterations. The values shown are the maximum norms of the error on the interfaces. The curves from top to bottom correspond to the values $n = 128$, $n = 64$, $n = 32$, $n = 16$, $n = 8$, $n = 4$.

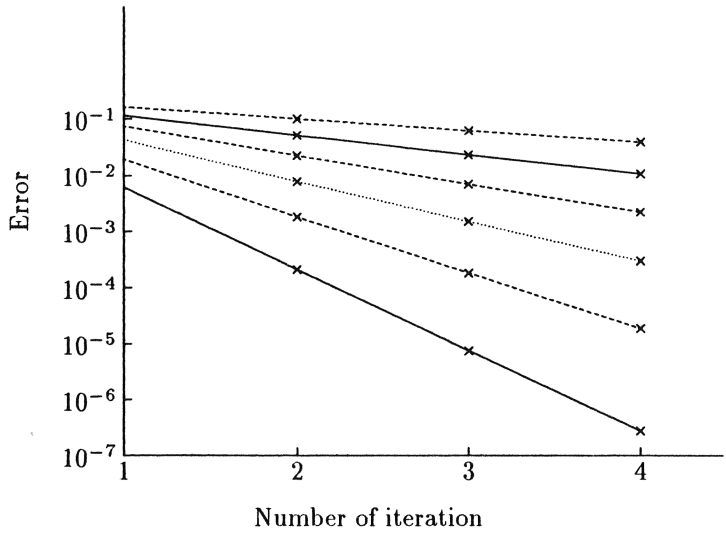

Figure 2. Direct iterations, standard parameters

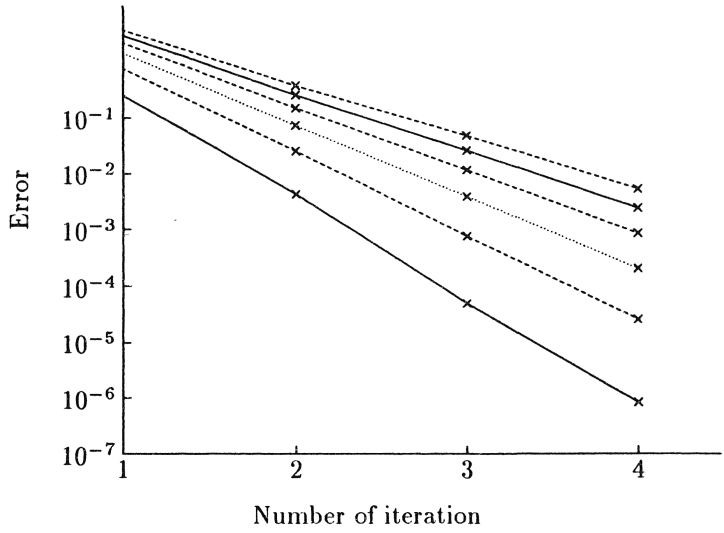

Figure 3. Direct iterations, optimal parameters

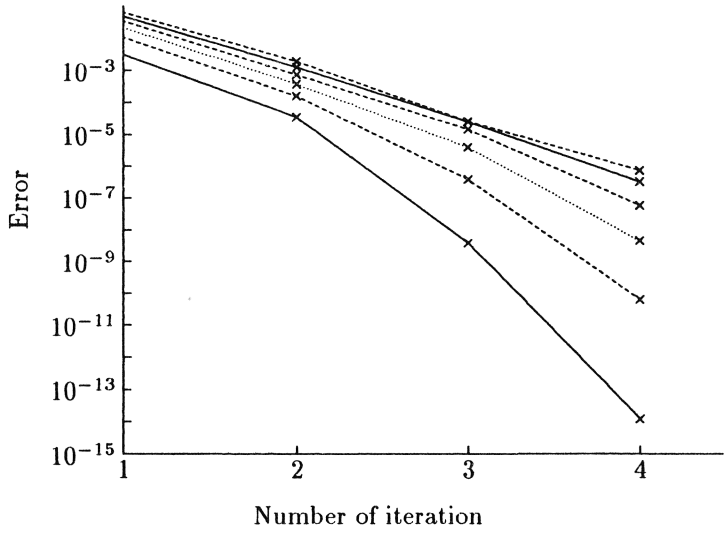

Figure 4. Preconditioned CG method, standard parameters

As it was mentioned in paragraph 1, we can use the matrix P for the preconditioning of the CG method. It follows from [2] that the spectral radius of $P(S_U + S_L)$ is bounded independently on n. The numerical realization of this method is easy and needs only minimal extra computational effort in comparison to the direct method. The results of computations with the preconditioned CG method, again for two choices of iteration parameters, are shown in Figure 4 and 5. The curves from top

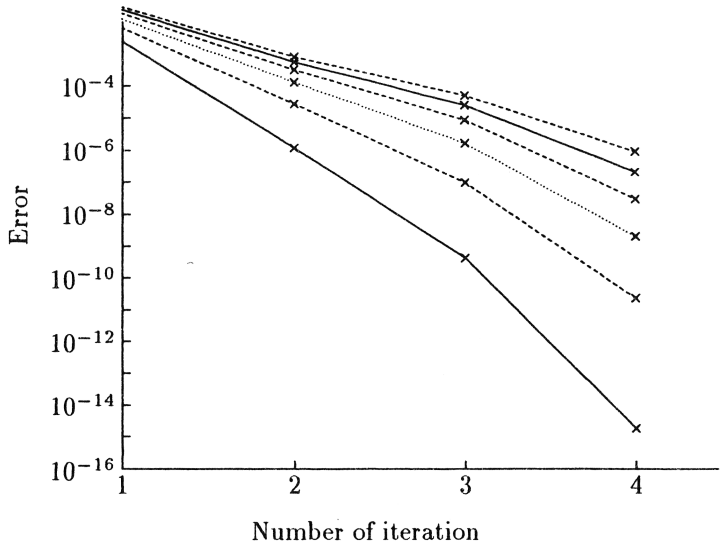

Figure 5. Preconditioned CG method, optimal parameters

to bottom correspond again to the values $n = 128$, $n = 64$, $n = 32$, $n = 16$, $n = 8$, $n = 4$.

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