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NUMERICAL ANALYSIS OF THE NAVIER-STOKES EQUATIONS

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1. INTRODUCTION

This paper discusses some conceptional questions of the numerical simulation of viscous incompressible flow which are related to the presence of boundaries. The governing mathematical model are the well-known Navier-Stokes equations

$$\partial_t u - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \nabla \cdot u = 0, \tag{1.1}$$

in a two- or three-dimensional region Ω, together with appropriate initial and boundary conditions. Despite its rather simple structure this system represents one of the most challenging problems in theoretical and numerical analysis. Its solutions develop a rich variety of structures, e.g., vortex shedding and dynamical layers, which require high quality standards in their reliable numerical simulation. Traditionally the numerical schemes for solving these equations were based on finite difference discretizations on quasi uniform grids. In the last decade the trend has shifted towards discretizations on solution-adapted grids which allow for more flexibility in resolving the fine structure of a flow. One group of such methods is based on the finite element approach which has been developed on the bases of a long experience in structural mechanics. However, the particular difficulties inherent in the Navier-Stokes problem require various modifications of the traditional finite element concepts. These difficulties consist in the combination of the incompressibility constraint $\nabla \cdot u \equiv 0$, and the singular perturbation character for $\nu \ll 1$, with the complicated dynamics of the solutions which require a very fine spatial resolution and the computation over long intervals of time. Through the introduction of the Reynolds-number, $\mathrm{Re} = LU/\nu$, the problem is set in a dimensionless form, where L is the characteristic length, U the characteristic speed, and $\nu = \mu/\varrho$ the dynamic viscosity. This model parameter, Re, is normally considered as a measure for the computational difficulties in modeling the corresponding flow. For $\mathrm{Re} \approx 1$, the problem takes on a parabolic behavior with large dissipative effects, while for $\mathrm{Re} \gg 1$, the transport term dominates giving

the problem more the appearance of a hyperbolic system. In the latter case the phenomenon of *turbulence* (chaotic behavior) threatens rendering any deterministic numerical approach to a heuristic gamble. However, in the intermediate range of moderate Reynolds-numbers, depending of course on the particular problem, *laminar* flow occurs which is amenable to a numerical treatment on save mathematical grounds. Particular difficulties arise when the flow problem has to be solved on a domain with boundaries, either natural rigid boundaries or possibly artificial inlets or outlets. Here, the interaction of the physical boundary conditions and the constraint $\nabla \cdot u = 0$ results in "paradox" phenomena which are not present in the "unbounded" case, e.g., for spatially periodic flows. This leads to interesting and sometimes unexpected mathematical questions the analysis of which, in turn, helps to better understand the computational approaches. In this context the following three topics will be discussed:

1. The treatment of the incompressibility constraint by Chorin's projection method: *The problem of the nonphysical pressure boundary conditions.*
2. The choice of outflow boundary conditions: *The problem of well-posedness.*
3. The modeling of the nonlinearity: *The mechanism behind the success of the nonlinear Galerkin method.*

This selection reflects the current research interests of the author in computational fluid dynamics. Most of the related results have been obtained in collaboration with J. G. Heywood from Vancouver (UBC), [13], [14], and S. Turek, [13], [21], and A. Prohl, [18], both from Heidelberg. The computational results and the illustrative figures are taken from papers of H. Blum [1] and S. Turek [24]. The work on the above questions is still in progress and some of the "answers" presented here are necessarily incomplete and sometimes only conjectured. This is to stimulate further research. Much of the motivation in this study has been raised and supported through fruitful discussions with Ph. Gresho from Lawrence Livermore Nat. Lab., at various occasions, see [8] and [9] for his contributions to the first and to the second question.

Notation

By $L^2(\Omega)$ we denote the Lebesgue space of all square integrable functions over Ω, provided with the usual norm $\|\cdot\| = \|\cdot\|_\Omega$ and inner product $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$. Further, $H^m(\Omega)$ is the mth-order Sobolev space of those $L^2(\Omega)$-functions which possess generalized derivatives of order m in $L^2(\Omega)$. The norm of this (Hilbert-)space is $\|\cdot\|_m = \left(\sum_{|\alpha|\leq m} \|D^\alpha \cdot\|^2\right)^{1/2}$. The completion in $H^1(\Omega)$ of the subspace $D(\Omega)$ of test functions with compact support in Ω is denoted by $H_0^1(\Omega)$. By Poincaré's inequality the Dirichlet form $(\nabla\cdot,\nabla\cdot)_\Omega$ is an inner product on $H_0^1(\Omega)$. These are all

spaces of scalar functions. Spaces of vector functions $u = (u_i)_{i=1,\dots,n}$ (with values in $\mathbb{R}^n$) are written with bold face type, while no distinction is made in the notation of the corresponding norms and inner products, e.g., the norm of the space $\mathbf{H}_0^1(\Omega)$ is $\|\nabla v\| = \left(\sum_{i,j} \|\partial_i v_j\|^2\right)^{1/2}$. All the other notation are self-evident: $\partial_t u = \partial u/\partial t$, $\partial_i u = \partial u/\partial x_i$, $\partial_n u = \nabla u \cdot n$, $\partial_\tau u = \nabla u \cdot \tau$, etc., where n and τ are the normal and tangential unit vector along the boundary $\partial\Omega$.

2. Stable finite element discretization

In setting up a finite element model of the Navier-Stokes problem one starts from the variational formulation of the problem: Find $u = b + v$ and p, with $v \in \mathbf{H}$ and $p \in \mathbf{L}^2(\Omega)$, such that

$$(\partial_t v, \varphi) + \nu(\nabla v, \nabla\varphi) + (v \cdot \nabla v, \varphi) - (p, \nabla \cdot \varphi) = (f, \varphi), \quad \forall \varphi \in \mathbf{H}, \tag{2.1}$$

$$(\chi, \nabla \cdot v) = 0, \quad \forall \chi \in L^2(\Omega). \tag{2.2}$$

The choice of the function space $\mathbf{H} \subset \mathbf{H}^1(\Omega)$ depends on the specific boundary conditions chosen for the problem to be solved. On a finite mesh $\mathbf{T}_h$ (triangulation, etc.) covering the domain Ω, with element width h, one defines spaces $\mathbf{H}_h \times L_h$" $\subset$ "$\mathbf{H} \times L^2(\Omega)$ of piecewise polynomial trail and test functions. The discrete analogues of (2.1), (2.2) then read as follows:

Find $u_h = b_h + v_h$ and p_h, with $v_h \in \mathbf{H}_h$ and $p_h \in L_h$, such that

$$\begin{aligned}(\partial_t v_h, \varphi_h) + \nu(\nabla_h v, \nabla_h \varphi) + (v_h \cdot \nabla v_h, \varphi_h) - (p_h, \nabla_h \cdot \varphi_h) \\ = (f_h \varphi_h), \quad \forall \varphi_h \in \mathbf{H}_h,\end{aligned} \tag{2.3}$$

$$(\chi_h, \nabla_h \cdot v_h) = 0, \quad \forall \chi_h \in L_h. \tag{2.4}$$

Using nodal basis representations for the unknowns $v_h(t)$ and $p_h(t)$, for any fixed h, this may be written as a system of ordinary differential equations with an algebraic constraint, i.e., as a so-called DAE system. In this spatial semi-discretization the spaces $\mathbf{H}_h$ may be "nonconforming", i.e., the discrete velocities v_h are continuous across the interelement boundaries and zero along the rigid boundaries only in an approximate sense; in this case the discrete gradient operator ∇_h occurring in (2.3), (2.4) is to be understood in the "piecewise" sense. In order that (2.3), (2.4) is a numerically stable approximation to (2.1), (2.2), as $h \to 0$, it is crucial that the spaces $\mathbf{H}_h \times L_h$ satisfy a compatibility condition, the so-called Babuška-Brezzi condition (see [7]),

$$\inf_{q_h \in L_h/\mathbb{R}} \sup_{w_h \in \mathbf{H}_h} \left(\frac{(q_h, \nabla_h \cdot w_h)}{\|q_h\| \|\nabla w_h\|} \right) \geqslant \gamma > 0. \tag{2.5}$$

This inequality insures, first, that the problems (2.3), (2.4) possess solutions, which are uniquely determined in $\mathbf{H}_h \times L_h/\mathbf{R}$, and, second, that these discrete solutions converge to the true solution of the variational problem (2.1), (2.2); a rigorous convergence analysis of spatial semi-discretizations of the Navier-Stokes problem has been given in [12; Part 1].

Many stable pairs of finite element spaces $\{\mathbf{H}_h, L_h\}$ have been proposed in the literature (see, e.g., [7], [12; Part 1], and [21]). Below, two particularly simple examples of quadrilateral elements will be described which have satisfactory approximation properties and are applicable in two as well as in three space dimensions. The incompressibility constraint (2.2) is treated here completely implicitly in computing stationary as well as nonstationary flows. Both types of elements may be used in the spatial discretizations underlying the discussions in the following sections.

1) The first example is the natural quadrilateral analogue of the well known triangular finite element of Crouzeix/Raviart (see [7]). This nonconforming element uses piecewise "rotated" bilinear (reference-) shape functions for the velocities, spanned by $\{1, x, y, x^2 - y^2\}$, and piecewise constant pressures. As the nodal values one may take the mean values of the velocity vector over the element edges and the mean values of the pressure over the elements. Though belonging to a classical family of stable "Stokes elements", it has apparently not found much attention in the literature yet. A convergence analysis is given in [21] and very promising computational results are reported in [24]. This element has several important features:

- It is possible to construct a "divergence-free" (local) nodal-basis, which allows the elimination of the pressure from the problem resulting in a positive definite algebraic system for the velocity unknowns alone. Further it admits simple upwind strategies in the case of dominating transport, $\mathrm{Re} \gg 1$.

- The reduced algebraic system can be solved by specially adapted multigrid methods.

All nonstationary 2-D flows shown in this paper have been calculated by a code which uses the divergence-free version of this Stokes element.

2) The second example is even simpler than the first one. It uses continuous bi-linear (or tri-linear) shape functions for both the velocity and the pressure approximations. The nodal values are just the function values of the velocity and the pressure at the vertices of the mesh, making this ansatz particularly attractive in the three dimensional case. This combination of spaces, however, would be unstable, i.e., would violate the condition (2.5), if used together with the variational formulation (2.3), (2.4). In order to get a stable discretization, it was devised by T. J. R. Hughes, et al., [15], to add certain least squares term in the continuity equation (2.4) (pressure

stabilization method),

$$(2.6) \qquad (\chi_h, \nabla \cdot v_h) + \frac{\alpha}{\nu} \sum_{K \in \mathbf{T}_h} h_k^2 (\nabla \chi_h, \nabla p_h)_K = \frac{\alpha}{\nu} \sum_{K \in \mathbf{T}_h} h_k^2 \{(\nabla \chi_h, f)_K + \dots\}.$$

The correction terms on the right hand side have the effect, that this modification is fully consistent, since the additional terms cancel out if the exact solution $\{u, p\}$ of problem (2.1), (2.2) is inserted. It was shown in [15], and later on in a series of mathematical papers (see F. Brezzi and J. Pitkäranta [2], and the literature cited therein) in the context of a more general analysis of such stabilization methods, that this discretization is numerically stable and of optimal order convergent for all practically relevant pairs of spaces $\{\mathbf{H}_h, L_h\}$. This particularly includes the favorable lowest order continuous quadrilateral element of equal degree approximation for velocities and pressures. This element has several important features (see [15] and [10]):

- With the same number of degrees of freedom it is more accurate than tis triangular analogue (and also slightly more accurate than its nonconforming analogue described above) and it also admits simple upwind strategies in the case of dominating transport, $\mathrm{Re} \gg 1$. Further it has a very simple data structure due to the use of the same type of nodal values for velocities and pressures which allows for an efficient vectorization of solution processes.

- Thanks to the stabilization term in the continuity equation (2.4), standard multigrid techniques can be used for solving the algebraic systems with good efficiency.

In the case of higher Reynolds numbers (e.g., $\mathrm{Re} > 1000$ for the 2-D driven cavity, and $\mathrm{Re} > 100$ for the flow around an ellipse) the finite element models (2.3), (2.4) or (2.3), (2.6) become unstable since they essentially use central differences like discretizations of the adjective term. This "instability" most frequently occurs in form of a drastic slow-down or even break-down of the iteration processes for solving the discretized problems; in the extreme case the possibly existing "mathematical" solution contains strongly oscillatory components without any physical meaning. In order to avoid these effects some additional numerical dumping is required. The use of simple first-order artificial viscosity is not advisable since it requires the predetermination of a numerical parameter and generally introduces too much dumping for time depending flows. The better choice is "upwinding" which can be defined also quite naturally for finite element methods (see [10], [24], and the literature cited therein). Here, the upwinding effect is accomplished in the evaluating of the advection term through shifting integration points into the negative flux direction. This modification leads to system matrices which have certain M-matrix properties and

are therefore amenable to efficient multigrid techniques. This is widely exploited in the finite element codes in [10] and [24].

The DAE system (2.3), (2.4) is very stiff, with a ration $O(\nu h^{-2})$ (or even $O(\nu h^{-4})$ in the case of the "divergence-free" element formulation). Therefore, in the choice of time stepping methods for solving this system one is essentially limited to strongly A-stable schemes. Through extensive numerical experience (see [19] and [24]) it has turned out that the second order "fractional step Θ-method" proposed by R. Glowinski, et al. [3], is particularly suitable for simulating nonstationary flows. But, in contrast to [3], where this scheme was combined with a splitting approach separating the incompressibility constraint from the nonlinearity, it is used here as a mere time stepping procedure. The fractional step θ-method is as accurate as the traditional Crank-Nicolson scheme but it possesses much better stability properties with respect to rough initial and boundary data.

THE FIRST PROBLEM: TREATMENT OF THE INCOMPRESSIBILITY CONSTRAINT

Suppose, for simplicity, that problem (1.1) is posed with zero Dirichlet boundary conditions, $u|\partial\Omega = 0$. The classical way of coping with the incompressibility constraint in flow computations is the projection method of A. Chorin [4], which reads as follows. For a given initial value $u^0 \in \mathbf{J}_1(\Omega)$, choose a time step k, and solve for $m \geqslant 1$:

i) $\tilde{u}^m \in \mathbf{H}_0^1(\Omega)$ ("Burgers step"):

$$\frac{1}{k}(\tilde{u}^m - u^{m-1}) - \nu\Delta\tilde{u}^m + \tilde{u}^m \cdot \nabla\tilde{u}^m = f^m, \text{ in } \Omega, \tag{3.1}$$

ii) $u^m = P\tilde{u}^m \in \mathbf{J}_0(\Omega)$ ("Projection step"): $\nabla \cdot u^m = 0$, in Ω,

$$n \cdot u^m|\partial\Omega = 0. \tag{3.2}$$

Here, the function spaces $\mathbf{J}_0(\Omega)$ and $\mathbf{J}_1(\Omega)$ are obtained through the completion of the space $\{\varphi \in \mathbf{D}(\Omega), \nabla \cdot \varphi \equiv 0\}$ of solenoidal test functions with respect to the norm of $\mathbf{L}^2(\Omega)$ and that of $\mathbf{H}_0^1(\Omega)$, respectively. This time stepping scheme can be combined with any spatial discretization method, e.g., the finite element methods mentioned above. The projection step (ii) can equivalently be expressed in the form

$$\text{ii}') \qquad u^m = \tilde{u}^m - k\nabla p^m, \tag{3.3}$$

with some "pressure" $p^m \in H^1(\Omega)$, which in turn is determined through the properties

$$\text{ii}'') \qquad \Delta p^m = \frac{1}{k}\nabla \cdot \tilde{u}^m, \text{ in } \Omega, \quad \partial_n p^m|\partial\Omega = 0. \tag{3.4}$$

This amounts to a Poisson equation for p^m with zero Neumann boundary conditions. It is this nonphysical boundary condition, $\partial_n p^m \big| \partial\Omega = 0$, which has caused a lot of controversial discussion about the principle value of the projection method. Nevertheless, the method has proven to work well for representing the velocity field in many flow problems of physical interest (see, e.g., Ph. Gresho [10]). It is very economical as it requires in each time step only the solution of a (nonlinear) advection-diffusion system for u^m (of Burgers equation type) and a scalar Neumann problem for p^m. Still, it was argued that the pressure p^m were a mere fictious quantity without any physical relevancy. It remained the question: How can such a method work at all? A challenging problem for mathematical analysis!

The first convergence results for the projection method was already given by A. Chorin, but concerned only cases with absent rigid boundaries (all-space or spatially periodic problems). Later on, qualitative convergence was shown even for the pressure, but in a measure theoretical sense, too weak for practical purposes. Only recently, J. Shen [22] was able to prove the following suboptimal convergence estimate:

$$(3.5) \qquad \|u^m - u(t_m)\| + \Big(k \sum_{\mu=0}^{m} \|p^\mu - p(t_\mu)\|^2_{L^2(\Omega)/\mathbb{R}}\Big)^{1/2} = O(\sqrt{k}),\ 0 < t_m \leqslant T,$$

which indicates that the quantities p^m may really be reasonable approximations to the pressure $p(t_m)$. This result has then been sharpened by the author in [20] (see also [18]) to optimal order,

$$(3.6) \qquad \|u^m - u(t_m)\|_{L^2(\Omega)} + \|p^m - p(t_m)\|_{H^{-1}(\Omega)} = O(k),\ 0 < t_m \leqslant T.$$

This finally confirms that Chorin's original method is indeed a first order time stepping scheme for the incompressible Navier-Stokes problem. The key to this new result is the re-interpretation of the projection method in the context of the so-called "pressure stabilization methods". To this end one insert the quantity $u^{m-1} = \tilde{u}^{m-1} - k\nabla p^{m-1}$ into the momentum equation, obtaining

$$(3.7) \qquad \frac{1}{k}(\tilde{u}^m - \tilde{u}^{m-1}) - \nu\Delta\tilde{u}^m + (\tilde{u}^m \cdot \nabla)\tilde{u}^m + \nabla p^{m-1} = f^m, \quad \tilde{u}^m \big| \partial\Omega = 0,$$

$$(3.8) \qquad \nabla \cdot \tilde{u}^m - k\Delta p^m = 0, \partial_n p^m \big| \partial\Omega = 0.$$

This looks like an approximation of the Navier-Stokes equations with a first order (in time) "pressure stabilization" term. Pressure stabilization methods for solving the Navier-Stokes equations have a long history. In these methods the incompressibility constraint is supplemented by terms involving the pressure thereby giving it a similar

appearance as the continuity equation in compressible flow models; this approach is therefore sometimes referred to as "pseudo-compressibility method". The observation is that the projection method can (at least on the continuous level) be viewed as a pressure stabilization method with a global stabilization parameter $\varepsilon = k$, and an explicit treatment of the pressure term. In [20] it is shown how this fact may be used to derive optimal order error estimates for the projection method. One result of this analysis is that the pressure error is of better order in the interior of the domain Ω than at the boundary, i.e., for any strict subdomain $\Omega' \subset\subset \Omega$, there holds

$$\|p^m - p(t_m)\|_{L^2(\Omega')/\mathbb{R}} = O(k), \quad 0 < t_m \leqslant T. \tag{3.9}$$

Moreover, it appears that the pressure error is actually confined to a small boundary strip of width $\delta \approx \sqrt{\nu k}$ and decays exponentially into the interior of Ω (for a discussion of this matter see also Ph. Gresho [8]). In fact, it is conjectured that, setting $d(x) = \operatorname{dist}(x, \partial\Omega)$,

$$|p^m(x) - p(x, t_m)| \leqslant \gamma \exp\Big(-\alpha \frac{d(x)}{\sqrt{\nu k}}\Big)\sqrt{k} + O(k), \quad 0 < t_m \leqslant T. \tag{3.10}$$

This conjecture is supported by numerical experiments for the pressure stabilization method applied to the stationary Stokes problem. In fact, it is even possible to recover the optimal order accuracy of the pressure at the boundary by simply extrapolating the pressure values from the interior of the domain. In Figure 1, the effect of this post-processing is shown for a model Stokes problem solved by using bi-linear finite elements on a uniform rectangular mesh.

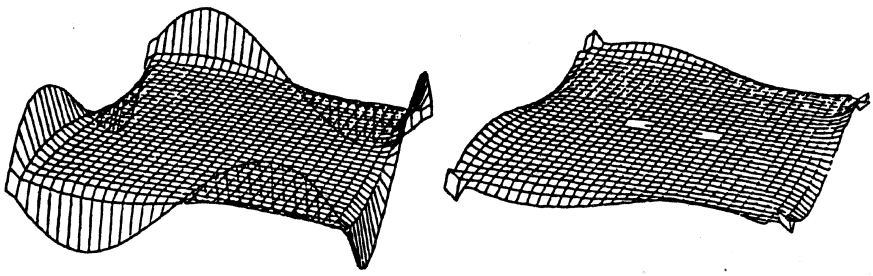

Figure 1. Pressure error plots for a polynomial Stokes solution before and after correction by extrapolation to the boundary (from H. Blum [1]).

Unfortunately, the sharp decay estimate (3.10) has not been proven yet rigorously. In the course of the argument one considers the singularly perturbed Neumann problem

$$\Big(\frac{1}{\nu}\nabla \cdot \Delta_D^{-1}\nabla - \varepsilon\Delta\Big)q = \varepsilon\Delta p, \text{ in } \Omega, \ \partial_n q|\partial\Omega = \partial_n p|\partial\Omega, \tag{3.11}$$

where Δ_D denotes the Laplacian operator corresponding to Dirichlet boundary conditions. Clearly, $\nabla \cdot \Delta_D^{-1} \nabla$ is a zero-order operator mapping $L^2(\Omega)$ into $L^2(\Omega)$. Then, for this problem one would like to know the decay estimate

$$\|q\|_{\Omega_\delta} \leqslant c \exp\Big(-\alpha \frac{\delta}{\sqrt{\nu\varepsilon}}\Big)\sqrt{\varepsilon}\,\|\nabla p\|_\Omega + c\nu\varepsilon\,\|\Delta p\|_\Omega, \tag{3.12}$$

for interior subdomains $\Omega_\delta = \{x \in \Omega, \operatorname{dist}(x, \partial\Omega) > \delta\}$. Such an estimate could be proved in [20] only for the case that the "global" operator $\nabla \cdot \Delta_D^{-1} \nabla$ is replaced by the "local" identity operator. But in the general case the corresponding result is still an open problem.

Let us conclude this discussion by pointing out some possible extensions of the presented analysis to higher order projection methods. One such method which is formally of second order reads: Choose $u^0 \in \mathbf{J}_1(\Omega)$, and $p^0 = 0$, and compute, for $m \geqslant 1$.

i) $\tilde{u}^m \in \mathbf{H}_0^1(\Omega)$:

$$\frac{1}{k}(\tilde{u}^m - u^{m-1}) - \frac{\nu}{2}\Delta(\tilde{u}^m + u^{m-1}) + \frac{1}{2}(\tilde{u}^m \cdot \nabla \tilde{u}^m + u^{m-1} \cdot \nabla u^{m-1}) + \nabla p^{m-1} = f^{m-\frac{1}{2}}, \text{ in } \Omega, \tag{3.13}$$

ii) $p^m \in H^1(\Omega)$: $u^m = \tilde{u}^m - \alpha k \nabla(p^m - p^{m-1})$, for some $\alpha \geqslant \frac{1}{2}$.

For this scheme J. Shen [23] has recently shown the suboptimal error estimate

$$\Big(k \sum_{\mu=0}^{m} \|u^\mu - u(t_\mu)\|^2\Big)^{1/2} + k^{1/2}\Big(k \sum_{\mu=0}^{m} \|\bar{p}^\mu - \bar{p}(t_\mu)\|^2_{L^2(\Omega)/\mathbb{R}}\Big)^{1/2} = O(k^{3/2}), \tag{3.14}$$

where the bar in the pressure term refers to a certain mean value over time. A careful examination of this scheme shows that it may again be interpreted as a certain pressure stabilization method using a stabilization of the form

$$\nabla \cdot u - \alpha k^2 \partial_t \Delta p = 0, \text{ in } \Omega, \ \partial_n p \big| \partial\Omega = 0. \tag{3.15}$$

This observation, in turn, could then lead to improved error estimates of optimal second order. Further, it is conjectured that analogously to (3.10) the following decay estimate holds true:

$$|\bar{p}^{m-1/2}(x) - \bar{p}(t_{m-1/2}, x)| \leqslant c \exp\Big(-\alpha \frac{d(x)}{\sqrt{\nu k}}\Big)\sqrt{k} + O(k^2). \tag{3.16}$$

To prove (or disprove) these statements has to be the subject of future research. In particular, it would be very interesting to see, whether this argument can be carried on to even higher orders of approximation.

4. The second problem: Outflow boundary conditions

Many flow problems are posed in generically unbounded domains, e.g., the flow through a pipe or through a system of pipes, flow around a body, and the flow through a hole in an infinite wall. For computational reasons, such an infinite flow region is usually cut to a bounded domain by introducing artificial inflow and outflow boundaries. While the "inflow" boundary conditions appear to be naturally induced through physical evidence, e.g., parabolic Poisseuille flow profile for the inlet of a pipe, the choice of appropriate "outflow" conditions is a serious problem. "Paradoxes" can arise if one proposes a variational formulation of a problem on the basis of intuition and experience, without carefully checking its full equivalence with recognizable boundary conditions, including flux and pressure conditions. This has serious consequences on the physical relevance of the numerical results obtained on the basis of such a formulation, e.g., by the finite element method. In this context the so-called "do nothing" outflow boundary conditions appears most natural; see [13] for a more detailed discussion of this matter.

For example, in the study of the flow in a channel around an inclined ellipse one may prescribe a Dirichlet condition $u_{|S_1} = b$ for the inflow and the usual no-slip condition along the rigid part of the boundary, Γ. Prescribing "nothing" at the outlet is then expressed by the variational formulation:

Find $u = b + v$ and p, with $v \in \mathbf{H} = \{\varphi \in \mathbf{H}^1(\Omega) : \varphi|\Gamma \cup S_1 = 0\}$ and $p \in L^2(\Omega)$, satisfying $v_{|t=0} = v^0$ and

$$(\partial_t v, \varphi) + \nu(\nabla v, \nabla \varphi) + (v \cdot \nabla v, \varphi) - (p, \nabla \cdot \varphi) = 0, \quad \forall \varphi \in \mathbf{H}, \tag{4.1}$$

$$(\chi, \nabla \cdot v) = 0, \ \forall \chi \in L^2(\Omega). \tag{4.2}$$

This formulation leaving the values for v and p "free" at the outlet S_2 implicitly contains a "natural" condition, $\nu \partial_n v + p = 0$, on S_2, for any sufficiently smooth solution $\{v, p\}$. This "free outflow" boundary condition is mainly motivated by the properties of Poisseuille flow. The numerical method based on (4.1), (4.2) works very well and the results are reasonably independent of the position of the artificial boundary S_2. As an illustration, Figure 2 shows the stream lines of such a flow (relative to the corresponding Stokes flow), obtained by a finite element method using the primitive variables $\{v, p\}$. The computational domain is first taken of length L, and for comparison, is then extended to length 2L. Obviously, this has no visible effect on the pattern of the flow past the obstacle. Now, suppose that one goes further, and formulates the same problem in a bifurcating channel, in exactly same way using the "free" outflow boundary condition. It will be found, see Figure 3,

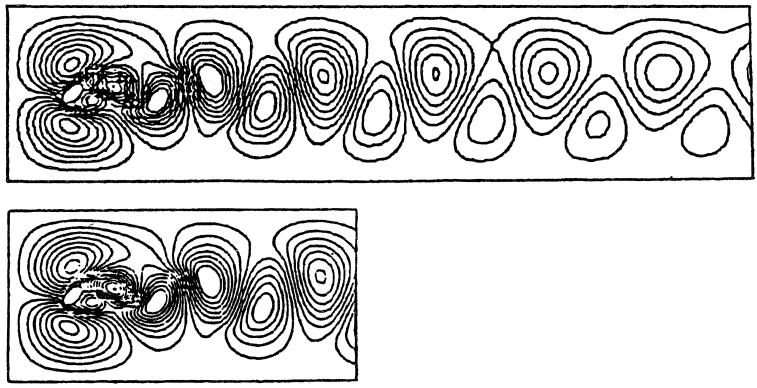

Figure 2. Laminar pipe flow at Re 500 under the "free outflow" boundary condition visualized via "relative" streamlines (from S. Turek [24]).

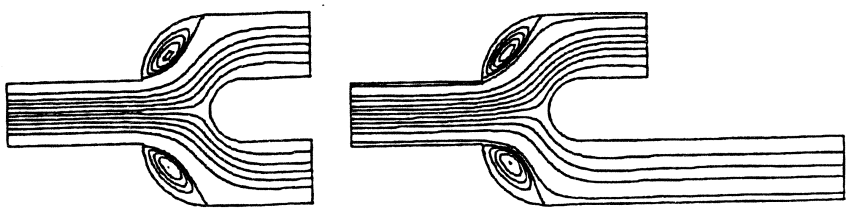

Figure 3. Streamlines of the flow in a bifurcating channel at Re 50 using the "free outflow" boundary condition (from S. Turek [24]).

that the results depend on the relative position of S_2 and S_3, even when they are both placed far downstream. Why?

The solution is that the formulations of both problems are a bit faulty, in that they contain hidden conditions for the pressure which were not discussed in formulating the problem,

$$\nu \partial_n u_n = p, \quad \text{on } \Gamma_{\text{out}}, \Rightarrow |\Gamma_{\text{out}}|^{-1} \int_{\Gamma_{\text{out}}} p \, ds = P = 0. \tag{4.3}$$

Hence, the mean pressure is being set equal to zero at each outlet, which is an essential boundary condition contained in the formulation (4.1), (4.2). This does not affect the velocity if there is only one outlet, but certainly does if there are two. So, variational formulations including "artificial" boundary conditions are a bit tricky. One can propose a formulation, solve it numerically, prove existence and uniqueness theorems for it—all without understanding what the problem is that is being solved. A related "paradox" is inherent in the flow "through a hole" which

has already been observed and analyzed by J. G. Heywood [11]. A thorough analysis of the described phenomenon, and of many others related to pressure drop and flux boundary conditions at artificial boundaries together with various numerical tests, can be found in [13].

The observation of the hidden pressure boundary conditions contained in the "do nothing" approach leads one to consider various non-standard boundary value problems for the Navier-Stokes equations, e.g., involving net flux or pressure drop conditions (see [13], and the literature cited therein). Let Γ be the rigid part of the boundary $\partial\Omega$, and let Γ_i, $i = 1, \ldots, N$, be those segments of $\partial\Omega$ along which inflow or outflow is allowed. Note that in general, it is not a priori determined what is an inflow and what is an outflow segment. Then, the "pressure drop problem" reads as follows:

For prescribed $P_i(t)$, find $u(t)$, such that

$$\partial_t u - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \nabla \cdot u = 0, \text{ in } \Omega, \tag{4.4}$$

$$u_{t=0} = u^0, \quad u\big|\Gamma = b, \quad |\Gamma_i|^{-1} \int_{\Gamma_i} p(t)\,\mathrm{d}s = P_i(t). \tag{4.5}$$

The corresponding natural boundary condition at the "free" segments of $\partial\Omega$ are

$$\nu\partial_n u = (p - P_i)n, \text{ on } \Gamma_i. \tag{4.6}$$

Although the variational problem (4.4), (4.5) looks well set, surprisingly there is a problem with its well posedness. The related Dirichlet problem of the Navier-Stokes equations, stationary as well as nonstationary, is well known to possess weak solutions (not necessarily unique) for any Reynolds-number. The standard argument for this result is based upon the "conservation property" of the nonlinear term in the equation, which is obtained by integration by parts and using the condition $\nabla \cdot u = 0$,

$$(u \cdot \nabla u, u) = 0. \tag{4.7}$$

In the case of "free" boundary this relation is replaced by

$$(u \cdot \nabla u, u) = -\frac{1}{2} \int_{\cup_i \Gamma_i} u_n u^2 \mathrm{d}s, \tag{4.8}$$

which generally does not allow to bound the energy in the system without a priori knowledge of what is an inflow and what is an outflow boundary. As a consequence, in [13] only the following weak result on the solvability of problem (4.5), (4.6) could be obtained:

The "pressure drop problem" possesses a unique solution locally in time. If $P = \sup_{t \geq 0} \sum_i |P_i(t)|$ *and* $\|\nabla u_0\|$ *are sufficiently small, this solution exists for all time and is then also exponentially stable.*

Corresponding results are also known for other types of non-standard boundary conditions including, e.g., prescribed net fluxes across the "free" boundary segments. This leaves the question open, whether one can also expect "global" existence of solutions for "large" data. A positive answer is suggested by numerical tests which do not show any unexpected instability with the discrete analogues of the formulation (4.5), (4.6) in the case of higher Reynolds numbers.

One may suspect that this theoretical difficulty can be avoided simply by changing the variational formulation of the problem, i.e., using other representations of the diffusive and of the advective terms. This, however, is not the case. Replacing in the momentum equation (4.1) the Dirichlet form a) $(\nabla u, \nabla \varphi)$ by b) $(D[u], D[\varphi])$, where $D[u] = \frac{1}{2}(\partial_i u_j + \partial_j u_i)_{ij=1}^n$ is the deformation tensor, results in a nonphysical behavior of the flow (streamlines bent outwards) even for the trivial case of Poisseuille flow through a straight pipe; see the top line in Figure 4. Further, writing the transport term in the form c) $u \cdot \nabla u = u \cdot \nabla u - u \cdot \nabla u^T + \frac{1}{2}\nabla u^2$, or alternatively in the form d) $u \cdot \nabla u = \frac{1}{2}\{u \cdot \nabla u + \nabla \cdot (u^T u)\}$, leads to variational formulations in which the advective terms are conservative (i.e., vanish if v itself is taken as a test function) also for the "free outflow" boundary condition. In the case (c) the corresponding natural outflow boundary condition is $\nu \partial_n u = (p + \frac{1}{2}u^2) \cdot n$, with the so-called "Bernoulli pressure" $\bar{p} = p + \frac{1}{2}u^2$, and in the case (d) it is $\nu \partial_n u = (p + \frac{1}{2}u_n^2) \cdot n$. Both modifications again result in a nonphysical behavior across the outflow boundary (streamlines bent inwards), as is shown on the bottom line in Figure 4.

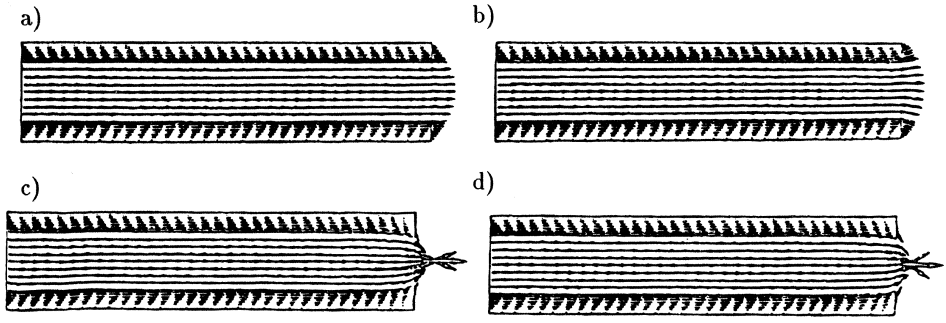

Figure 4. Velocity plots for Poisseuille flow at Re 50 modeled with the forms (a), (b), (c), and (d) described above (from S. Turek [24]).

In the consequence, for physical reasons, it seems to be necessary to deal with the existence question for the original formulation (4.5), (4.6). Only very recently, S. Kracmar and J. Neústupa [16] have obtained an interesting result which may, at least to some extent, solve this problem. They reformulate the problem as a variational inequality, where the amount of possible inflow along Γ_i is restricted by some upper bound M. For this new problem they are then able to prove global existence of weak solutions, which are also solutions of the original problem (4.5), (4.6) if the inforced restriction is satisfied in the strict sense.

5. The third problem: Modeling of the nonlinearity

The traditional ways of dealing with the nonlinearity in the Navier-Stokes equations is either by an explicit coupling or by straightforward linearization, first-order fixed point iteration or second-order Newton iteration. Normally, this approach fails in the range of large Reynolds numbers when the flow turns from "laminar" to "turbulent". In this case turbulence models have to be used which are mostly based on heuristic concepts without a rigorous mathematical justification. Recently an effort has been made by R. Temam and his collaborators (see [6], [5], and literature cited therein) to develop a mathematically based "turbulence model" by applying some new concepts from the theory of infinite dynamical systems to the Navier-Stokes equations. The key words in this context are "*Inertial Manifolds*" and "*Nonlinear Galerkin Method*". Below we will briefly discuss the possible relevancy of these concepts for the numerical solution of the Navier-Stokes equations; for a more detailed account see [14].

An "*inertial manifold*" is roughly spoken an analytical tool for describing the mechanism through which in a "turbulent" flow energy is shifted from low frequency modes into high frequency modes. For the Navier-Stokes equations this is easiest expressed in terms of Fourier expansion with respect to the eigenfunctions $\{a_i\}_{i\in N}$ of the Stokes operator, $\Delta' = P\Delta$. Here, P is the orthogonal projection in $L^2(\Omega)$ onto the subspace $J_0(\Omega)$ of divergence-free vector fields with zero normal components along the boundary. For any fixed $m \in \mathbf{N}$, let $\mathbf{E}_m = \operatorname{span}\{a_i, i = 1, \ldots, m\}$, P_m the L^2-projection onto $\mathbf{E}_m$, and $Q_m = I - P_m$. Further let $u(t) = p(t) + q(t)$ be the orthogonal splitting of the solution of problem (2.1), (2.2), corresponding again to zero Dirichlet boundary conditions, into its "low mode" component $p(t) \in \mathbf{E}_m$ and the corresponding "high mode" component $q(t) \in E_m^{\perp}$. Then, the Navier-Stokes problem can equivalently be written in the form

$$\partial_t p - \nu\Delta' p + P_m N(p+q, p+q) = P_m f, \tag{5.1}$$

$$\partial_t q - \nu\Delta' q + Q_m N(p+q, p+q) = Q_m f, \tag{5.2}$$

where $N(v,w) = v \cdot \nabla w$. Using this notation the "standard" (spectral) Galerkin method based on the eigenfunctions of the Stokes operator has the form

$$\partial_t p_m - \nu \Delta' p_m + P_m N(p_m, p_m) = P_m f. \tag{5.3}$$

For this approximation the error estimate $\|(p_m - p)(t)\| = O(\lambda_m^{5/4-\varepsilon})$, for any $\varepsilon > 0$, is generally best possible due to the incompatibility of the solution along the boundary, $\Delta' u(t)|\partial\Omega \neq 0$. The idea is now to assume the existence of an analytic relation $q = \Phi(p)$, an inertial manifold, such that the system (5.1), (5.2) can equivalently be written as

$$\partial_t p - \nu \Delta' p + P_m N(p + \Phi(p), p + \Phi(p)) = P_m f. \tag{5.4}$$

Although, the existence of such an inertial manifold has not been proven yet for the Navier-Stokes problem in general, this concept has become the basis of the so-called "*nonlinear Galerkin method*". In this approach one tries to model the unknown inertial manifold, if it then exists, by some substitute $\Phi_m(\cdot)$ called "*approximate inertial manifold*". The nonlinear Galerkin method then reads

$$\partial_t \bar{p}_m - \nu \Delta' \bar{p}_m + P_m N(\bar{p}_m + \Phi_m(\bar{p}_m), \bar{p}_m + \Phi_m(\bar{p}_m)) = P_m f. \tag{5.5}$$

As the simplest example of an approximate inertial manifold it is proposed in [5] and [6] to use

$$\bar{q}_m = \Phi_m(\bar{p}_m): \qquad -\nu \Delta' \bar{q}_m = Q_m f - Q_m N(\bar{p}_m, \bar{p}_m), \tag{5.6}$$

which, however, amounts to an *infinite* dimensional problem in the complement space $\mathbf{E}_m^{\perp}$. It has been proven in [5] that this "correction" (at least in two dimensions) actually leads to an improved error behavior of the form $\|(p_m - p)(t)\| = O(\lambda_m^{3/2} |\log \lambda_m|)$. This has widely been taken as evidence of the superiority of the approximate inertial manifold method over the standard Galerkin method. Even more, it is claimed that this approach may ultimately lead to a mathematically rigorous way of turbulence modeling. This point of view now comes into questions through a recent analysis in [14], which shows that the apparent success of the nonlinear Galerkin method has not so much to do with "turbulence modeling" but with the increase of the solution's compatibility at the boundary, which allows for the higher degree of approximation by Fourier modes. This point is easiest explained at a linear model of the Navier-Stokes equations,

$$\partial_t u - \nu \Delta' u + N(b, u) = f, \tag{5.7}$$

where b is a given function. Of course, in this case the question of turbulence modeling is not relevant. However, the mechanism underlying the nonlinear Galerkin method is still active even in this linear situation. Here, the method reads

$$\partial_t \bar{p} - \nu \Delta' \bar{p} + P_m N(b, \bar{p} + \Phi_m(\bar{p})) = P_m f, \tag{5.8}$$

$$\bar{q} = \Phi_m(\bar{p}): \qquad -\nu \Delta' \bar{q} = Q_m f - Q_m N(b, \bar{p}). \tag{5.9}$$

The error analysis for this approximation gives, setting $e = p - \bar{p}$,

$$\|e(t)\|^2 + \nu \int_0^t \|\nabla e(t)\|^2 \mathrm{d}s \leqslant c(t) \max_{[0,t]} \left\{ \|\partial_t q\|^2_{H^{-2}(\Omega)} + \|q\|^2_{H^{-1}(\Omega)} \right\}, \tag{5.10}$$

which, for a sufficiently regular solution u is of the order $O(\lambda_m^{-3})$. A higher (integer) order cannot be realized in general for the spectral Galerkin approximation since this would require the solution u to lie in the domain of definition of the operator $(\Delta')^2$. This amounts to the compatibility condition $\Delta' u = 0$ on $\partial\Omega$, which is generally not satisfied, due to the presence of the pressure term ∇p in the momentum equation. In contrast to (5.10), for the standard Galerkin method the corresponding estimate reads

$$\|e(t)\|^2 + \nu \int_0^t \|\nabla e(t)\|^2 \mathrm{d}s \leqslant c(t) \max_{[0,t]} \left\{ \|q_t\|^2_{H^{-1}(\Omega)} + \|q\|^2 \right\}, \tag{5.11}$$

which is only of the order $O(\lambda_m^{-2})$. The order improvement for the nonlinear Galerkin method obviously becomes possible because of the occurrence of weaker Sobolev norms on the right hand side of (5.10) compared with (5.11). This, in turn, is due to the relation

$$q - \bar{q} \approx \nu \Delta'^{-1} N(b, p - \bar{p}), \tag{5.12}$$

defining the approximate inertial manifold which acts as a regularization process in the scheme. From this simple consideration, the following conclusion may be drawn. The order improvement through the nonlinear Galerkin method possibly occurs for those approximation schemes (like the spectral Galerkin method) which reach optimal order only for those solutions which are sufficiently smooth *and* compatible along the boundary. But this effect is not active for most other "real" discretizations as finite difference, finite volume, or finite element schemes.

Apparently the order restriction of the standard spectral Galerkin method is a problem of the boundary compatibility of the solution u and has nothing to do with "turbulence". In turn, the improvement by the nonlinear Galerkin method is also

directly related to this boundary compatibility. Therefore, the turbulence modeling aspect of the success of the nonlinear Galerkin method appears very questionable. However, there may be some potential for improving on the error constant in a real discretization for a laminar or only slightly turbulent flow.

The nonlinear Galerkin method, as stated above, is not a practicable scheme, since the determining equation (5.6) for the high frequency correction $\bar{q}_m$ is infinite dimensional. To make it practicable, (5.6) should be restricted to a finite dimensional subspace, for example,

$$\bar{q}_m = \Phi_m(\bar{p}_m): \qquad -\nu\Delta'\bar{q}_m = Q_m^{m'} f - Q_m^{m'} N(\bar{p}_m, \bar{p}_m), \tag{5.13}$$

where for some $m' > m$, $Q_m^{m'}$ denotes the projection onto $\operatorname{span}\{a_{m+1}, \ldots, a_{m'}\}$. However, this apparent simplification of the method looses the mentioned order improvement over the standard Galerkin method, unless $m' \geqslant m^{3/2}$. Whether this enormous amount of additional computational work really pays can only be judged on the basis of numerical experiments. An extension of the concept of the nonlinear Galerkin method to finite element discretizations has been proposed by M. Marion and R. Temam [17], where also some qualitative convergence results but no error estimates are given. However, due to the principle differences in the approximation properties of the (global) spectral Galerkin method and those of the (quasi-local) finite element method the application of this idea to the latter method is less obvious. Further research is necessary to explore the potential of this approach for practical methods in computational fluid dynamics. Some quantitative improvement on the computational efficiency in long time simulations may be possible by combining the concept of a reduced system complexity on the high frequency level (but retaining the time derivative!) with that of the multigrid method. This approach, termed "*micro-scale linearization*", reads, again expressed in terms of the spectral Galerkin approximation, as follows:

$$\partial_t \bar{p}_m - \nu\Delta'\bar{p}_m + P_m N(\bar{p}_m + \bar{q}_m, \bar{p}_m + \bar{q}_m) = P_m f. \tag{5.14}$$

$$\partial_t \bar{q}_m - \nu\Delta'\bar{q}_m = Q_m^{2m} f - Q_m^{2m} N(\bar{p}_m, \bar{p}_m). \tag{5.15}$$

Numerical experiments on the basis of this idea applied to finite element discretizations are presently under work. But the corresponding error analysis, qualitative or quantitative, has not been given yet and is presumably a very difficult task.

In applying the concept of microscale linearization to a finite element discretization one first has to decide on the inner product with respect to which the orthogonal splitting $u = u_h + u_h^{\perp}$ is made. Note that the L^2- and the J_1-projections onto finite element spaces are different. A brief examination of the error behavior of finite

element approximations shows that the J_1-inner product is the right choice, if there is then any. Suppose now that J_H and J_h are finite element spaces parameterized by the mesh size $0 < h < H$, and satisfying $J_H \subset J_h \approx J_1(\Omega)$. Note that the inclusion $J_H \subset J_h$ is required here only for notational simplicity. Nearly "divergence-free" approximations can be constructed for instance on the basis of the first example described in Section 2. Let P_h, P_H, and R_h, R_H denote the L^2-projections and the J^1-projections onto J_h and J_H, respectively, and let $P_H^\perp = P_h - P_H$ and $R_H^\perp = R_h - R_H$. Accordingly, let $u_h = u_H + u_H^\perp$, $u_H = R_H u_h$, $u_H^\perp = u_h - R_H u_h$, be the orthogonal splitting of the "fine grid" solution $u_h \in J_h$. Then the Navier-Stokes system (1.1) can equivalently be written in the form

$$\partial_t(u_H + P_H u_H^\perp) + \nu A_h u_H + P_H N(u_H + u_H^\perp, u_H + u_H^\perp) = P_H f, \tag{5.16}$$

$$\partial_t(P_H^\perp u_H + u_H^\perp) + \nu A_h u_H^\perp + P_H^\perp N(u_H + u_H^\perp, u_H + u_H^\perp) = P_H^\perp f, \tag{5.17}$$

where A_h is the discrete analogue of the Stokes operator Δ' in J_h. Neglecting now the time derivatives $\partial_t P_H u_H^\perp$ and $\partial_t P_H^\perp u_H$ (which automatically vanish in the *spectral* Galerkin approximation), and the term $u_H^\perp$ in the nonlinearity of the coarse-grid equation leads to the reduced system

$$\partial_t \bar{u}_H + \nu R_H A_h \bar{u}_H + P_H N(\bar{u}_H + \bar{u}_H^\perp, \bar{u}_H + \bar{u}_H^\perp) = P_H\{f + N(\bar{u}_H^\perp, \bar{u}_H^\perp)\}, \tag{5.18}$$

$$\partial_t \bar{u}_H^\perp + \nu A_h \bar{u}_H^\perp = P_H^\perp\{f - N(\bar{u}_H, \bar{u}_H)\}, \tag{5.19}$$

where the nonlinear coupling between the "small-scale modes" $\bar{u}_H^\perp$ and the "large-scale modes" $\bar{u}_H$ has been reduced. This is what we call "*micro-scale linearization*". It is different from the nonlinear Galerkin method as there the acceleration term $\partial_t \bar{u}_H^\perp$ is neglected in the "small-scale" equation, which appears very questionable if one is not really close to the (unknown) inertial manifold of the continuous model. The "large-scale" equation (5.18) amounts to a standard finite element Galerkin approximation of problem (1.1) in the coarse-grid space J_H. But the evaluation of the "small-scale" equation (5.19) requires the construction of the orthogonal projections $R_H^\perp$ and $P_H^\perp$. This can be done simply by evaluating the identities $R_H^\perp = R_h - R_H$ and $P_H^\perp = P_h - P_H$, which in the case $h \ll H$ essentially amounts to the solution of a fine-grid problem, but with reduced (linear) complexity. For the scheme (5.18), (5.19), based on a *second-order* finite element approximation, e.g., as that described in Section 2, one can derive the error estimate

$$\max_{[0,T]} \|\bar{u}_H - R_H u\|_{L^2(\Omega)} \leqslant c_T(u) H^2, \tag{5.20}$$

which shows that microscale linearization does not reduce the order of the underlying finite element scheme. A proof of this result and others on the basis of the techniques

in [12] will be presented in a forthcoming paper. But the crucial question whether this approach can lead to a quantitative improvement in the accuracy, respectively to a reduction in the computational work, must be left open at present.

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