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ON THE EXISTENCE OF CHAOTIC BEHAVIOUR OF DIFFEOMORPHISMS

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Summary. For several specific mappings we show their chaotic behaviour by detecting the existence of their transversal homoclinic points. Our approach has an analytical feature based on the method of Lyapunov-Schmidt.

Keywords: bifurcations, homoclinic orbits, chaotic behaviour

AMS classification: 58F08, 58F14, 58F30

1. INTRODUCTION

Several papers have appeared recently [1–5] in which Smale's theorem about transversal homoclinic points is proved by analytical methods. Originally S. Smale [6] constructed geometrically a certain "horseshoe" region for a diffeomorphism having a transversal homoclinic point. He showed that the action of some iterate of the diffeomorphism on this compact invariant set is equivalent to the action of the Bernoulli shift. In this way he proved Birkhoff's result that the diffeomorphism has infinitely many periodic points.

The purpose of this paper is to proceed in an analytical approach to show chaotic behaviour of specific examples. K. J. Palmer [3] used the theory of exponential dichotomies which was combined with the method of Melnikov for detecting transversal homoclinic points of ordinary differential equations. The author of this paper developed in [5] a similar method directly for diffeomorphisms. The basic technique was the theory of exponential dichotomies for difference equations. K. J. Palmer [2] used the same technique but the work done in [5] was quite independent of the former. Palmer's ideas from [3] were extended by K. R. Meyer and G. R. Sell [1] to skew product flows. Hence in the largest part of this paper we detect chaotic behaviour of perturbed mappings by showing the existence of their transversal homoclinic points, provided the unperturbed ones have suitable properties.

Now we summarize the contents of the paper. In Section 2, we briefly recall the basic results from [2] and [5]. It is shown that the problem of bifurcations of heteroclinic points of diffeomorphisms can be reduced to a finite dimensional problem by the method of Lyapunov-Schmidt. In Section 3, we study specific examples. In the first part of that section we detect transversal homoclinic points for certain mappings: for ordinary differential equations with impulsive effects; for two singularly perturbed mappings which are discrete versions of singularly perturbed ordinary differential equations. We also show a criterion of transversality of homoclinic points on smooth manifolds. In the second part, we investigate specific mappings possessing invariant compact manifolds which are normally hyperbolic [7], and stable and unstable manifolds of these invariant manifolds have transversal intersections. We show the existence of an invariant compact set of such a mapping which consists of a family of surfaces. This structure of the invariant set is preserved by the action of the mapping and the mapping transfers these surfaces in the same way as the Bernoulli shift acts on some set of infinite bisequences. Similar results have been obtained in [1], [8] and the author of the paper was stimulated by these papers.

2. Difference equations

In this section we briefly recall interesting results from the papers [2], [5]. An analytical approach to the study of bifurcations of homoclinic points is a suitable method as was pointed out in these papers. The main tool is a theory of exponential dichotomy of linear difference equations. Let $\mathbb{Z}$ be the set of integer numbers and $\mathbb{N}$ the set of natural ones.

Definition 2.1 [9]. Let X be a Banach space and $\{T_n\}_{n\in I} \subset \mathscr{L}(X)$. $\{T_n\}_{n\in I}$ has an exponential dichotomy on $I = (\mathbb{Z}, \mathbb{Z}_+ = \mathbb{N} \cup \{0\}, \mathbb{Z}_- = -\mathbb{Z}_+)$, if there are positive constants $M, \theta < 1$ and a family of projections $\{P_n\}_{n\in I} \subset \mathscr{L}(X)$ such that

(1) $T_n P_n = P_{n+1} T_n$ ($n < 0$ for $I = \mathbb{Z}_-$) $\forall n \in I$;

(2) $T_n / \operatorname{Im} P_n$ is an isomorphism from $\operatorname{Im} P_n$ on $\operatorname{Im} P_{n+1}$ ($n < 0$ for $I = \mathbb{Z}_-$) for all $n \in I$;

(3) if $T_{n,m} = T_{n-1} \dots T_{m+1} \cdot T_m$ for $n > m$,

$T_{n,n} = \mathrm{Id}$ (the identity)

then

$$|T_{n,m}(\mathrm{Id} - P_m)x| \leqslant M \cdot \theta^{n-m} \cdot |x| \text{ for } n \geqslant m,$$
$$|T_{n,m} P_m x| \leqslant M \cdot \theta^{m-n} \cdot |x| \text{ for } n < m,$$

where $T_{n,m} P_m x = y \in \operatorname{Im} P_n$ if and only if $P_m x = T_{m,n} y$ for $n < m$.

Remark 2.2. If $\{T_n\}$ is a sequence of isomorphisms, then Definition 2.1 is equivalent to the property that there exists a projection $P \in \mathscr{L}(X)$ satisfying

(1) $|T(m) \cdot P \cdot T^{-1}(s)| \leqslant M \cdot \theta^{m-s}$ for $s \leqslant m$,

(2) $|T(m) \cdot (\mathrm{Id} - P) \cdot T^{-1}(s)| \leqslant M \cdot \theta^{s-m}$ for $m \leqslant s$,

where $T(n) = T_{n-1} \dots T_0$ for $1 \leqslant n$, $T(0) = \mathrm{Id}$ and $T(n) \doteq T_n^{-1} \dots T_{-1}^{-1}$ for $n < 0$. Indeed, we can take $P_n = T(n) \cdot (\mathrm{Id} - P) \cdot T^{-1}(n)$.

We shall recall the basic theorem from [5], which we use later to solve several concrete problems.

Theorem 2.3. *Let $\{A_n\}_{n \in \mathbb{Z}}$ be a sequence of invertible matrices $A_n \in \mathscr{L}(\mathbb{R}^m, \mathbb{R}^m)$ with bounded $|A_n|$, $|A_n^{-1}|$ on $\mathbb{Z}$. Let $\{A_n\}_{n \in I}$ have an exponential dichotomy for $I = \mathbb{Z}_+$ and $I = \mathbb{Z}_-$. Define an operator*

(1) $\quad L\colon X \to X = \left\{ \{a_n\}_{-\infty}^{\infty} \mid \sup_n |a_n| < \infty, a_n \in \mathbb{R}^m \right\}$,

(2) $\quad L(\{a_n\}_{-\infty}^{\infty})_n = a_{n+1} - A_n a_n$.

Then L is a Fredholm operator and $\{f_n\} \in \operatorname{Im} L$ if and only if

$$\sum_{-\infty}^{\infty} c_n^* \cdot f_n = 0$$

for any bounded solution $\{c_n\}_{n \in \mathbb{Z}}$ of the equation

$$c_n = (A_n^*)^{-1} c_{n-1} \tag{2.1}$$

($*$ is the transposition).

Lemma 2.4. *Let $\{A_n\}_{0 \leqslant n}$ have an exponential dichotomy on $\mathbb{Z}_+$, where $A_n \in \mathscr{L}(\mathbb{R}^m)$ are invertible bounded on $\mathbb{Z}_+$ and $|B_n| \to 0$ if $n \to \infty$, $B_n \in \mathscr{L}(\mathbb{R}^m)$. Further we assume that $A_n + B_n$ are invertible. Then $\{A_n + B_n\}_{0 \leqslant n}$ has an exponential dichotomy on $\mathbb{Z}_+$ and, moreover, if P, P' are projections of dichotomies (see Remark 2.2) for $\{A_n\}, \{A_n + B_n\}$, then $\dim \operatorname{Im} P = \dim \operatorname{Im} P'$.*

Proof. See [5, Lemma 3.4]. □

Now consider a C^1-mapping $G\colon U \to \mathbb{R}^m$, U being an open subset of $\mathbb{R}^m$. Assume that G has two fixed points y_1, y_2 which are hyperbolic. Let there exist a sequence $\{x_n\}_{-\infty}^{\infty} \subset U$ such that

$$\lim_{n \to -\infty} x_n = y_1, \quad \lim_{n \to \infty} x_n = y_2, \quad x_{n+1} = G(x_n),$$
$$\det DG(x_n) \neq 0, \quad x_0 \neq y_1, \quad x_0 \neq y_2.$$

We put G into a smooth family $G_e\colon \mathbb{R}^m \to \mathbb{R}^m$, $G_0 = G$. We want to find heteroclinic orbits of G_e for e small near $\{x_n\}_{-\infty}^{\infty}$. We know by [5, p. 361] that using the two above results we can reduce this problem by applying the Lyapunov-Schmidt method to a finite dimensional one. We follow this line in the next section for special cases of G.

Now let us consider a diffeomorphism $f_e\colon \mathbb{R}^m \to \mathbb{R}^m$, $e \in \mathbb{R}$ and f_0 has a one-parametric family $\Gamma = \{\{x_n(c)\}_{-\infty}^{\infty},\ c \in \mathbb{R}\}$ of homoclinic orbits tending to a hyperbolic fixed point. Then using the above notation we put

$$F_e(\{y_n\}_{-\infty}^{\infty})_n = y_{n+1} - f_e(y_n).$$

Since $F_0(\{x_n(c)\}_{-\infty}^{\infty}) = 0$, differentiating we have

$$\operatorname{Ker} DF_0(\{x_n(c)\}_{-\infty}^{\infty}) \supset \operatorname{span}\{x_n'(c)\}_{-\infty}^{\infty}.$$

Let us assume $\dim \operatorname{Ker} DF_0(\{x_n(c)\}) = 1$ for any c small. Then by Theorem 2.3 and the Lyapunov-Schmidt reduction (see [5, p. 358]) we obtain uniformly for e small a bifurcation equation of the equation $F_e(\{y_n\}_{-\infty}^{\infty}) = 0$ of the form

$$Q(c,e) = 0, \qquad Q(.,.) \in \mathbb{R},$$

where c is inherited from Γ. Hence $Q(c,0) = 0$. Generally Q is smooth, thus $Q(c,e) = e \cdot M(c,e)$ and this implies that $M(c,0)$ is the Melnikov function for this problem, i.e., if $\exists\, c_0$, $M(c_0,0) = 0$ and $\frac{\partial}{\partial c}M(c_0,0) \neq 0$, then by the implicit function theorem $Q(c,e) = 0$ has a solution $c = c(e)$, $c(0) = c_0$ for e small. But then $F_e(z) = 0$ has a solution $z = z(e)$ satisfying $z(0) = \{x_n(c_0)\}_{-\infty}^{\infty}$ and thus f_e has a homoclinic orbit near Γ for any e small. If $m = 2$ then a rather tedious computation shows that the adjoint equation

$$c_n = (Df_0(x_n(c))^*)^{-1} c_{n-1}$$

has a bounded solution $c_n = (x_{n+1,2}'(c), -x_{n+1,1}'(c)) \cdot \det Df_0^{-n-1}(x_{n+1}(c))$. Here $z_n = (z_{n,1}, z_{n,2})$. Indeed, by using

$$x_{n+1}'(c) = Df_0(x_n(c))x_n'(c)$$

and the identities

$$(z_{n,2}, -z_{n,1}) = \mathcal{J}.(z_{n,1}, z_{n,2})$$
$$A^* \cdot \mathcal{J} \cdot A = \det A \cdot \mathcal{J},$$

where $\mathscr{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and A is any 2×2 matrix, we obtain

$$\begin{aligned} Df_0(x_n(c))^* c_n &= Df_0(x_n(c))^* . \mathscr{J} . x'_{n+1}(c) \cdot \det Df_0^{-n-1}(x_{n+1}(c)) \\ &= Df_0(x_n(c))^* . \mathscr{J} . Df_0(x_n(c)) x'_n(c) \cdot \det Df_0^{-n-1}(x_{n+1}(c)) \\ &= \det Df_0^{-n-1}(x_{n+1}(c)) \cdot \det Df_0(x_n(c)) \cdot \mathscr{J} . x'_n(c) \\ &= \det Df_0^{-n}(x_n(c)) \cdot \mathscr{J} . x'_n(c) = c_{n-1} \end{aligned}$$

since $Df_0^{-n}(x_n(c)) = Df_0^{-n-1}(f_0(x_n(c))) = Df_0^{-n-1}(x_{n+1}(c)) \cdot Df_0(x_n(c))$.

Applying Theorem 2.3 we see that in this case (see [11])

$$M(c,0) = \sum_{-\infty}^{\infty} \frac{\partial}{\partial e} f_0(x_n(c)) \wedge x'_{n+1}(c) \cdot \det Df_0^{-n-1}(x_{n+1}(c)), \tag{2.2}$$

where $\wedge$ is the "wedge" product.

3. Applications

In this section we will demonstrate chaotic behaviour for several specific problems applying the above results.

Application I. Let M be an m-dimensional smooth manifold and $f\colon M \to M$ a C^∞-diffeomorphism with a hyperbolic fixed point $x \in M$ possessing a homoclinic orbit $\{x_n\}_{-\infty}^{\infty}$, $x_n \to x$ if $n \to \pm\infty$. In this section we apply the above theory to derive a criterion for transversality of the homoclinic orbit $\{x_n\}_{-\infty}^{\infty}$ (see [3, p. 229]). For this purpose we study the set of homoclinic orbits in its neighbourhood. By using local coordinates in a neigbourhood of x_i we reduce this problem as follows: we have diffeomorphisms $f_{-K-1}, \ldots, f_{K+1}$ defined on a neighbourhood of the point $0 \in \mathbb{R}^m$ such that

$$\begin{gathered} f_i(0) = 0 \quad \text{for } -K \leqslant i \leqslant K-1, \\ f_K(0) = z_{K+1}, \qquad f_{-K-1}(z_{-K-1}) = 0, \\ f_{K+1}(z_i) = z_{i+1} \quad \text{for } K+1 \leqslant i \quad \text{or} \quad i \leqslant -K-2, \end{gathered}$$

where the orbit $\{x_n\}_{-\infty}^{\infty}$ is transformed to the sequence

$$z = \{\ldots, z_{-K-1}, 0, 0, \ldots, 0, z_{K+1}, z_{K+2}, \ldots\}.$$

Indeed, for a suitable natural number K and an open neighbourhood U of the point $0 \in \mathbb{R}^m$ we can assume the existence of C^∞-diffeomorphisms $\Phi_{-K-1}, \ldots, \Phi_K$ from U to M such that

$$\begin{gathered}\Phi_{-K-1}(0) = x, \Phi_{-K}(0) = x_{-K}, \ldots, \Phi_K(0) = x_K,\\ \Phi_{-K-1}^{-1}(x_j) \in U \quad \text{for all } j > K \text{ or } j < -K.\end{gathered}$$

Define

$$\begin{aligned}\Phi_{-K-1}^{-1}(x_j) &= z_j \quad \text{for } j < -K \text{ or } j > K,\\ f_j &= \Phi_{j+1}^{-1} \cdot f \cdot \Phi_j \quad \text{for } -K \leqslant j \leqslant K,\\ \Phi_{K+1} &= \Phi_{-K-1}, \quad f_{-K-1} = \Phi_{-K}^{-1} \cdot f \cdot \Phi_{-K-1},\\ f_{K+1} &= \Phi_{-K-1}^{-1} \cdot f \cdot \Phi_{-K-1}.\end{aligned}$$

Further we put

$$\begin{aligned}g_i &= f_i \quad \text{for } -K-1 \leqslant i \leqslant K,\\ g_i &= f_{K+1} \quad \text{for } K < i \text{ or } i < -K-1,\end{aligned}$$

$$G(\{y_n\})_n = y_{n+1} - g_n(y_n), G\colon \tilde{U} \subset X \to X,$$

where $X = \left\{\{y_n\}_{-\infty}^{\infty} \mid y_n \in \mathbb{R}^m, \sup\limits_n |y_n| < \infty\right\}$ and $\tilde{U}$ is a neighbourhood of the point z. From the definition of the sequence z it follows that $G(z) = 0$ and $DG(z)$ is a Fredholm operator with index 0 by Theorem 2.3. Hence if $\operatorname{Ker} DG(z) = \{0\}$, then it is possible to apply the implicit function theorem from which it follows that any small perturbation of the map f has a unique homoclinic orbit near $\{x_n\}_{-\infty}^{\infty}$. Thus $\{x_n\}_{-\infty}^{\infty}$ is transversal. On the other hand, if $\operatorname{Ker} DG(z) \neq \{0\}$ then for a suitable perturbation of the map f we have bifurcation of homoclinic orbits. Then the orbit $\{x_n\}_{-\infty}^{\infty}$ is not transversal. Summing up we obtain the following theorem

Theorem I.1. *A homoclinic orbit $\{x_n\}_{-\infty}^{\infty}$ is transversal if and only if*

$$\sup_n \|Df^n(x_0)v\| < \infty$$

implies $v = 0$ for any $v \in T_{x_0}M$, where $Df^n(x_0)\colon T_{x_0}M \to T_{x_n}M$, T_yM is the tangent space to M at the point $y \in M$ and $\|\cdot\|$ is the norm derived from a Riemann metric on M.

Application II. We present a generalization of the Melnikov method for ordinary differential equations with impulsive effects. Consider the equation

$$x' = f(x) + e \cdot h(x, t) \tag{II.1}$$

with impulsive effects

$$x(i + 0) = x(i - 0) + e^2 \cdot a(e, x(i - 0)), \qquad i = 1, 2, \dots,$$

where $x(i \pm 0) = \lim_{t \to i\pm} x(t)$ and $f : \mathbb{R}^m \to \mathbb{R}^m$ is a smooth map, $a : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ is a smooth map, $h : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ is smooth and 1-periodic in t. Let (II.1) for $e = 0$ have a hyperbolic fixed point $0 \in \mathbb{R}^m$ possessing a homoclinic orbit $\bar{b}(t)$. Let $b(t, x, e)$ be the solution of the equation (II.1) satisfying $b(0, x, e) = x$.

We put

$$F_e : \mathbb{R}^m \to \mathbb{R}^m, \qquad F_e(x) = b(1, x, e),$$
$$G_e : \mathbb{R}^m \to \mathbb{R}^m, \qquad G_e(x) = x + e^2 \cdot a(e, x).$$

We see that G_e is a local diffeomorphism on an arbitrary bounded subset of $\mathbb{R}^m$ for e small. Now we apply the results of Section 2 for the map $H_e = F_e \cdot G_e$. Consider the Banach space X from Section 2 and the mapping

$$R_e : X \to X, \quad R_e(\{y_n\}_{-\infty}^{\infty})_n = y_{n+1} - H_e(y_n).$$

We note that $R_0(\{\bar{b}(t + n)\}_{-\infty}^{\infty}) = 0, t \in \mathbb{R}$. Hence $R_0 = 0$ on the set

$$\mathcal{M} = \big\{\{\bar{b}(t + n)\}_{-\infty}^{\infty} \mid t \in (t_0 - 1, t_0 + 1)\big\} \subset X$$

for $t_0 \in \mathbb{R}$ fixed. The set $\mathcal{M}$ is a 1-dimensional submanifold of X containing the point $\{\bar{b}(t_0 + n)\}_{-\infty}^{\infty}$, at which the tangent space to $\mathcal{M}$ is a vector subspace generated by $\{(\bar{b})'(t_0 + n)\}_{-\infty}^{\infty}$.

Theorem II.1. *Let $\Delta(t)$ be the Melnikov function from* [3, p. 252] *for the ordinary differential equation* (II.1) *without the impulsive effects (i.e. $a = 0$). If there exists a number $t_0 \in \mathbb{R}$ such that $\Delta(t_0) = 0$, $\Delta'(t_0) \neq 0$, then H_e has a transversal homoclinic point for $e \neq 0$ small.*

By the Smale theorem [6] mentioned in Introduction and by Theorem II.1 we obtain that the impulsive equation (II.1) has infinitely many periodic solutions whose periods tend to infinity for $e \neq 0$ small.

Proof. We apply Theorem 4.1 from [3] for the map R_e and the manifold $\mathscr{M}$. The assumptions of this theorem, which ensure bifurcation of homoclinic points near $\bar{b}(t_0)$, involve at most the first derivatives by the parameter e of R_e at the point $e = 0$. Since

$$D_e R_0(\{y_n\}_{-\infty}^{\infty}) = -\{D_e H_0(y_n)\}_{-\infty}^{\infty} = -\{D_e F_0(y_n)\}_{-\infty}^{\infty},$$

we see that these assumptions do not include any assumptions for a. Hence we can assume that $a = 0$. But then we obtain the well-known classical bifurcation problem [3], for which these assumptions have the form of our statement (see [3]). □

Corollary II.2. *Consider the equation*

$$\begin{aligned} x' &= f(x) + e \cdot h(x,t) \quad \text{on } \langle 2i+1, 2i+2\rangle, \qquad i = 1, 2, \dots \\ x' &= e^2 \cdot g(x,t) \quad \text{on } \langle 2i, 2i+1\rangle, \end{aligned}$$

where f, h satisfy the above mentioned assumptions, $g \in C^\infty$ is 1-periodic in t. If there exists $t_0 \in \mathbb{R}$ such that $\Delta(t_0) = 0$, $\Delta'(t_0) \neq 0$, then this equation has an infinite sequence of periodic solutions, whose periods tend to infinity.

Proof. We apply the above theorem, regarding the second equation as an impulsive effect. Thus we derive the mapping a. Let $c(t,x,e)$ be the solution of the equation $x' = e^2 \cdot g(x,t)$ such that $c(0,x,e) = x$. Then $c(t,x,0) = x$ and $D_e c(t,x,0) = 0$ for any $t \in \mathbb{R}, x \in \mathbb{R}^m$ and thus

$$c(1,x,e) = x + e^2 \cdot a(x,e).$$

The assertion of Corollary II.2 we obtain from Theorem II.1. □

For the sake of simplicity, until now we have considered ordinary differential equations with impulsive effects of the second order: $x \to x + e^2 \cdot a(e,x)$. Now using the results of Section 2 for two-dimensional mappings we consider the problem

$$\begin{aligned} &x' = f(x) \\ &x(i+0) = x(i-0) + e \cdot a(e, x(i-0)), \end{aligned} \tag{II.2}$$

where $e \in \mathbb{R}$, a, f are smooth, two-dimensional and f is Hamiltonian possessing a homoclinic orbit $\bar{b}(t)$ to a hyperbolic singular point 0. We use the notation before Theorem II.1 and instead of the mapping H_e we study $\tilde{H}_e = G_e \cdot F_0$. Then $\frac{\partial}{\partial e}\tilde{H}_0(x) = a(0, b(1,x,0))$ and $\tilde{H}_0$ is area preserving possessing the family of homoclinic orbits

$\{x_n(c)\} = \{\bar{b}(c+n)\}$. Finally, using the formula (2.2) for $\tilde{H}_e$ we obtain the Melnikov function for (II.2)

$$c \to \sum_{-\infty}^{\infty} a(0, \bar{b}(c+n)) \wedge f(\bar{b}(c+n)),$$

since $x_n(c) = \bar{b}(c+n)$.

Application III. The next applications deal with two types of degenerate perturbations of diffeomorphisms. The first of them is the case when an unperturbed diffeomorphism has a nonhyperbolic fixed point while the other is a singular perturbation. As a matter of fact, some of these perturbations are Poincaré mappings of ordinary differential equations partly studied in the averaging theory and partly in the theory of singular perturbations.

Consider the mapping $g_e \colon \mathbb{R}^i \times \mathbb{R}^k \to \mathbb{R}^i \times \mathbb{R}^k$,

$$g_e(x,y) = (x + e\cdot Ax + e^2\cdot f(x,y), g(y) + e\cdot h(x,y)), \tag{III.1}$$

where $f, g, h \in C^\infty$, A has only eigenvalues with positive real parts, g has a hyperbolic fixed point 0 with a transversal homoclinic orbit $\{\bar{y}_n\}_{-\infty}^{\infty}$ and $h(0,0) = 0$, $f(0,0) = 0$, $Df(0,0) = 0$.

Theorem III.1. *The mapping* (III.1) *has a transversal homoclinic orbit near to* $\{(0, \bar{y}_n)\}_{-\infty}^{\infty}$ *for any* $e \neq 0$ *small.*

P r o o f. We shall study the case $e > 0$. The map (III.1) has a fixed point $(0,0)$ for any e, and

$$Dg_e(0,0) = \begin{pmatrix} I + e\cdot A & 0 \\ e\cdot D_x h(0,0) & Dg(0) + e\cdot D_y h(0,0) \end{pmatrix}.$$

It is clear that $Dg_e(0,0)$ has no eigenvalue on the unit circle for $e \neq 0$ small, since A has only eigenvalues with positive real parts. We solve the equations

$$\begin{aligned} x_{n+1} &= x_n + e\cdot Ax_n + e^2\cdot f(x_n, y_n), \\ y_{n+1} &= g(y_n) + e\cdot h(x_n, y_n) \end{aligned} \tag{III.2}$$

in $X_1 \times X_2$, where

$$X_1 = \big\{\{x_n\}_{-\infty}^{\infty} \mid x_n \in \mathbb{R}^i, \sup_n |x_n| < \infty\big\},$$
$$X_2 = \big\{\{y_n\}_{-\infty}^{\infty} \mid y_n \in \mathbb{R}^k, \sup_n |y_n| < \infty\big\}.$$

Lemma 1. *The linear mapping $L_e\colon X_1 \to X_1$,*

$$L_e(\{x_n\})_n = x_{n+1} - x_n - e \cdot Ax_n$$

is invertible for $e > 0$ small and $|L_e^{-1}| \leqslant \frac{c}{e}$.

Proof of Lemma 1. Let us solve $L_e x = h$ in X_1, i.e.

$$x_{n+1} = (I + e \cdot A)x_n + h_n. \tag{III.3}$$

According to [10, p. 145] we have

$$1 + e \cdot c \leqslant |I + e \cdot A|.$$

Indeed, by [10] we can assume that $\langle Ax, x\rangle \geqslant c.|x|^2$, $c > 0$. Thus $\langle x + e \cdot Ax, x\rangle \geqslant (1 + e \cdot c) \cdot |x|^2$ and $|I + e \cdot A| \geqslant 1 + e \cdot c$.

Hence (III.3) has a unique bounded solution

$$\begin{aligned} x_n &= -(A_1^{-1}h_n + A_1^{-2}h_{n+1} + \ldots) \quad \text{for } 0 \leqslant n, \\ x_n &= A_1^{-|n|}x_0 - A_1^{-1}h_n - \ldots - A_1^{-|n|}h_{-1} \quad \text{for } n < 0, \end{aligned} \tag{III.4}$$

where $A_1 = I + e \cdot A$. Since $|A_1^{-1}| \leqslant \frac{1}{1+e\cdot c}$, then

$$\begin{aligned} |x_n| &\leqslant (|A_1^{-1}| + |A_1^{-2}| + \ldots) \cdot \sup |h_j|, \quad n \geqslant 0 \\ &\leqslant (\frac{1}{1+e\cdot c} + \frac{1}{(1+e\cdot c)^2} + \ldots) \cdot \sup |h_j| \\ &\leqslant \frac{1}{e \cdot c} \cdot \sup |h_j|. \end{aligned}$$

Analogously we have for $n < 0$, $e > 0$ small

$$|x_n| \leqslant \frac{\tilde{c}}{e} \cdot \sup |h_j|.$$

Thus $|L_e^{-1}| \leqslant \frac{\tilde{c}}{e}$ for $e > 0$ small. □

Now we rewrite the equation (III.2) in the form

$$\begin{aligned} x &= L_e^{-1} \cdot e^2 \cdot F(x, y), \\ 0 &= G(y) + e \cdot H(x, y), \end{aligned} \tag{III.5}$$

where $x = \{x_n\}_{-\infty}^{\infty}, y = \{y_n\}_{-\infty}^{\infty}, F(x,y)_n = f(x_n, y_n), G(y)_n = -y_{n+1} + g(y_n)$, $H(x, y)_n = h(x_n, y_n)$. We know that $|L_e^{-1}| \leqslant \frac{c}{e}$ and hence we can solve the first equation in (III.5) by the implicit function theorem and obtain its solution $x(e, y)$ such that $|x(e, y)| = O(e) \cdot O(|y|)$. Let us solve the second equation

$$0 = G(y) + e \cdot H(x(e, y), y).$$

We have $G(\{\bar{y}_n\}_{-\infty}^{\infty}) = 0$, and the operator $D_y G(\{\bar{y}_n\}_{-\infty}^{\infty})$ is invertible, since $\{\bar{y}_n\}_{-\infty}^{\infty}$ is transversal (see [2, p. 292] or the argument before Theorem I.1). We obtain that the system (III.2) has a unique solution $\{(x_n(e), y_n(e))\}_{-\infty}^{\infty}$ near to $\{(0, \bar{y}_n)\}_{-\infty}^{\infty}$ for $e > 0$ small.

Now we show that $\lim_{n \to \pm\infty} |x_n(e)| + |y_n(e)| = 0$.

Let $\limsup_{n \to +\infty} |x_n(e)| + |y_n(e)| = d > 0$. If n is large then according to (III.4–5) we have for $|x_n(e)| + |y_n(e)|$ small

$$\begin{aligned}
|x_n(e)| &= e^2 \cdot |A_1^{-1} f(x_n(e), y_n(e)) + A_1^{-2} f(x_{n+1}(e), y_{n+1}(e)) + \ldots| \\
&\leqslant (|A_1^{-1}| + |A_1^{-2}| + \ldots) \cdot e^2 \cdot \sup_{n \leqslant j} |f(x_j(e), y_j(e))| \\
&\leqslant \frac{c}{e} \cdot e^2 \cdot \sup_{n \leqslant j} |f(x_j(e), y_j(e))| \\
&= O(e) \cdot \sup_{n \leqslant j} |f(x_j(e), y_j(e))| \\
&= O(e) \cdot \sup_{n \leqslant j} (|x_j(e)| + |y_j(e)|)^2 \\
&= O(e) \cdot 2d.
\end{aligned}$$

Further,

$$\begin{aligned}
&y_{n+1}(e) - Dg(0) y_n(e) = O(|y_n(e)|^2) + O(e) \cdot O(|y_n(e)| + |x_n(e)|), \\
&y_{n+1}(e) - Dg(0) y_n(e) = h_n,
\end{aligned} \tag{III.6}$$

where $\sup_{n \leqslant j} |h_j| = O(d^2) + O(e) \cdot O(d)$.

We show that the equality (III.6) implies

$$\limsup_{n \to \infty} |y_n(e)| = O(\limsup_{n \to \infty} |h_n|).$$

First of all, we assume that $Dg(0)$ has only eigenvalues inside the unit circle. Then we solve (III.6) as in (III.4) and have

$$y_n(e) = Dg(0)^n y_0 + Dg(0)^{n-1} h_0 + \ldots + h_{n-1}.$$

If $\limsup_{n \to \infty} |h_n| = b$, then $|h_n| \leqslant b + \delta$ for $n \geqslant n_0$ large and δ small.

We can assume that $|Dg(0)| < 1$ [12, 3.126 Lemma] and thus

$$\begin{aligned}\limsup_{n\to\infty} |y_n(e)| &= \limsup_{n\to\infty} |Dg(0)^n y_0 + Dg(0)^{n-1} h_0 + \dots \\ &+ Dg(0)^{n-n_0} h_{n_0-1} + Dg(0)^{n-n_0-1} h_{n_0} + \dots + h_{n-1}| \\ &\leqslant (1 + |Dg(0)| + |Dg(0)|^2 + \dots) \cdot (b+\delta) \\ &\leqslant c \cdot (b+\delta)\end{aligned}$$

for $\delta > 0$ arbitrary small. Hence $\limsup_{n\to\infty} |y_n(e)| \leqslant c \cdot b$. Similarly we solve the case when $Dg(0)$ has eigenvalues outside the unit circle.

Summing up we obtain that for $e > 0$, d small

$$d = \limsup_{n\to\infty} |x_n(e)| + |y_n(e)| \leqslant O(e) \cdot d + O(d^2) \leqslant \frac{d}{3} + \frac{d}{3} = \frac{2d}{3}$$

and thus $d = 0$. Similarly we show $\limsup_{n\to -\infty} |x_n(e)| + |y_n(e)| = 0$.

Since $Dg_e(0,0)$ is a hyperbolic matrix for $e > 0$ small and the system of variational equations

$$\begin{aligned}v_{n+1} &= v_n + e \cdot A v_n + e^2 \cdot Df(x_n(e), y_n(e))(v_n, z_n) + f_n, \\ z_{n+1} &= Dg(y_n(e)) z_n + e \cdot Dh(x_n(e), y_n(e))(v_n, z_n) + h_n\end{aligned}$$

has a unique bounded solution for any $\{(f_n, h_n)\}_{-\infty}^{\infty} \in X_1 \times X_2$ (this assertion follows in the same way as in (III.2)), we obtain that $\{(x_n(e), y_n(e))\}_{-\infty}^{\infty}$ is a transversal homoclinic orbit of the mapping g_e for $e > 0$ small (see [2, p. 292] or the argument before Theorem I.1). □

The second degenerate case is the following mapping

$$h_e(x,y) = ((x - g(y))e^{-1} + x, x + e \cdot f(x,y)), e \neq 0,$$

where $x, y \in \mathbb{R}^m$, $f, g \in C^\infty$, $f(0,0) = 0$, $Df(0,0) = 0$.

Theorem III.2. *Let g have a hyperbolic fixed point 0 possessing a transversal homoclinic orbit $\{\bar{y}_n\}_{-\infty}^{\infty}$. Then the mapping h_e has a transversal homoclinic orbit to the fixed point $(0,0)$ near to the orbit $\{(g(\bar{y}_n), \bar{y}_n)\}_{-\infty}^{\infty}$ for $e \neq 0$ small.*

Proof. The linearization of the map h_e at $(0,0)$ has the form

$$Dh_e(0,0) = \begin{pmatrix} e^{-1}I + I & -e^{-1}A \\ I & 0 \end{pmatrix},$$

where $A = Dg(0)$ and I is the identity matrix. Let us compute the eigenvalues d of the matrix $Dh_e(0,0)$.

The equation

$$\det \begin{pmatrix} e^{-1}I + I - d \cdot I & -e^{-1}A \\ I & -d \cdot I \end{pmatrix} = 0$$

yields the equation

$$\det \begin{pmatrix} 0 & E \\ I & -d \cdot I \end{pmatrix} = 0,$$

where $E = -e^{-1}A + (e^{-1}I + I - d \cdot I) \cdot d$, and thus

$$\det E = \det e^{-1}((1 + e \cdot (1-d))d \cdot I - A) = 0.$$

It follows from this equation that $d(1 + e(1-d)) = d_1 \in \sigma(A)$ and

$$d = \begin{cases} \dfrac{1 + e + \sqrt{(1+e)^2 - 4ed_1}}{2e} \to \infty, & \text{if } e \to 0 \\ \dfrac{1 + e - \sqrt{(1+e)^2 - 4ed_1}}{2e} \to d_1, & \text{if } e \to 0. \end{cases}$$

We see that the fixed point $(0,0)$ of the mapping h_e is hyperbolic for $e \neq 0$ small. Let us solve the system

$$\text{(III.7)} \qquad \begin{aligned} x_{n+1} &= e^{-1}(x_n - g(y_n)) + x_n, \\ y_{n+1} &= x_n + e \cdot f(x_n, y_n) \end{aligned}$$

in $X_1 \times X_1$, where $X_1 = \left\{ \{x_n\}_{-\infty}^{\infty} \mid x_n \in \mathbb{R}^m, \sup_n |x_n| < \infty \right\}$. From (III.7) we have

$$\begin{aligned} e \cdot (x_{n+1} - x_n) &= x_n - g(y_n), \\ y_{n+1} &= x_n + e \cdot f(x_n, y_n) \end{aligned}$$

and

$$\begin{aligned} e \cdot (x_{n+1} - x_n) &= y_{n+1} - g(y_n) - e \cdot f(x_n, y_n), \\ y_{n+1} &= x_n + e \cdot f(x_n, y_n). \end{aligned}$$

Define the map $F_e \colon X_1 \times X_1 \to X_1 \times X_1$,

$$\begin{aligned} &F_e(\{x_n\}_{-\infty}^{\infty}, \{y_n\}_{-\infty}^{\infty})_n \\ &\quad = (-e(x_{n+1} - x_n) + y_{n+1} - g(y_n) - ef(x_n, y_n), y_{n+1} - x_n - ef(x_n, y_n)). \end{aligned}$$

Then

$$F_0(\{g(\bar{y}_n)\}_{-\infty}^{\infty}, \{\bar{y}_n\}_{-\infty}^{\infty}) = 0,$$
$$DF_0(\{g(\bar{y}_n)\}_{-\infty}^{\infty}, \{\bar{y}_n\}_{-\infty}^{\infty})(\{u_n\}, \{v_n\}) = \{u_{n+1} - Dg(\bar{y}_n)u_n, v_{n+1} - u_n\}.$$

Since $\{\bar{y}_n\}_{-\infty}^{\infty}$ is a transversal homoclinic orbit of the map g we obtain that the linear map

$$DF_0(\{g(\bar{y}_n)\}_{-\infty}^{\infty}, \{\bar{y}_n\}_{-\infty}^{\infty})$$

is invertible (see [2, p. 292] or the argument before Theorem I.1). Using the implicit function theorem we obtain that (III.7) has a bounded orbit

$$\Gamma_e = \{(x_n(e), y_n(e))\}_{-\infty}^{\infty}$$

near to the orbit $\{(g(\bar{y}_n), \bar{y}_n)\}_{-\infty}^{\infty}$. Further, we have

$$\begin{aligned} y_{n+1}(e) - Dg(0)y_n(e) &= O(|y_n(e)|^2) + e(x_{n+1}(e) - x_n(e)) \\ &\quad + e \cdot f(x_n(e), y_n(e)), \\ x_n(e) &= y_{n+1}(e) - e \cdot f(x_n(e), y_n(e)). \end{aligned}$$

Applying the same procedure as in the proof of Theorem III.1 we obtain

$$\limsup_{n \to \pm\infty} |x_n(e)| + |y_n(e)| = 0.$$

Finally, the variational equation along the orbit Γ_e has the form

$$\begin{aligned} u_{n+1} &= (u_n - Dg(y_n(e))v_n) \cdot e^{-1} + u_n + h_n, \\ v_{n+1} &= u_n + e \cdot Df(x_n(e), y_n(e))(u_n, v_n) + g_n. \end{aligned}$$

This implies

$$\begin{aligned} e \cdot u_{n+1} &= v_{n+1} - Dg(y_n(e))v_n + e \cdot h_n \\ &\qquad - e \cdot Df(x_n(e), y_n(e))(u_n, v_n) - g_n + e \cdot u_n, \\ v_{n+1} &= u_n + e \cdot Df(x_n(e), y_n(e))(u_n, v_n) + g_n. \end{aligned}$$

It is clear that this equation has a unique bounded solution for any $\{(h_n, g_n)\}_{-\infty}^{\infty} \in X_1 \times X_1$. Hence Γ_e is a transversal homoclinic orbit of the mapping h_e to the point $(0, 0)$ for $e \neq 0$ small (see [2, p. 292] or the argument before Theorem I.1). □

Application IV. Finally, we shall investigate problems which were stimulated by the papers due to G. R. Sell and K. R. Meyer [1] and S. Wiggins [8]. Let us consider

$$(*)\qquad \begin{aligned} &g_e\colon \mathbb{R}^m \times M \to \mathbb{R}^m \times M,\\ &g_e(x,y) = (f(x) + e \cdot h(x,y), r(y)), \end{aligned}$$

where M is a compact C^∞-manifold, $h \in C^\infty$ and $r\colon M \to M$ is a diffeomorphism. Assume that $0 \in \mathbb{R}^m$ is a hyperbolic fixed point of a diffeomorphism f, possessing a transversal homoclinic orbit $\{x_n\}_{-\infty}^{\infty}$, i.e., $x_n \to 0$ if $n \to \pm\infty$. We see that g_0 has an invariant set

$$\bigcup_n \{x_n\} \times M \cup \{0\} \times M.$$

We shall show that the perturbed mapping g_e for e small has a similar property. For this purpose we introduce the Banach space

$$\overline{X} = \{z\colon M \to \mathbb{R}^m \mid z \text{ is continuous}\}$$

with the norm $\|z\| = \sup_M |z(\cdot)|$. Define a mapping $F_e\colon \overline{X} \to \overline{X}$,

$$F_e(z(\cdot))(y) = f(z(r^{-1}(y))) + e \cdot h(z(r^{-1}(y)), r^{-1}(y)).$$

The map F_e is well defined and

$$F_0(z)(y) = f(z(r^{-1}(y))).$$

Theorem IV.1. *F_e has a unique small fixed point z_e.*

Proof. $F_0(0) = 0$ and F_e is C^1-smooth and thus we have

$$DF_0(0)v = Df(0)v(r^{-1}), \quad v \in \overline{X}.$$

By the assumption the matrix $Df(0)$ is hyperbolic, i.e., $Df(0)$ has no eigenvalues on the unit circle. We can assume $Df(0) = (A, B)$ with $|A| < 1$, $|B^{-1}| < 1$. Then $DF_0(0)v = (Au(r^{-1}), Bw(r^{-1}))$, where $v = (u, w)$ is the natural decomposition of v. We have $\|Au(r^{-1})\| \leqslant |A| \cdot \|u\|$. Further, let us solve $H(w) = Bw(r^{-1}) = \overline{w}$. Then $w(r^{-1}) = B^{-1}\overline{w}$ and $w = B^{-1}\overline{w}(r)$. Hence $\|H^{-1}(\overline{w})\| = \|w\| \leqslant |B^{-1}| \cdot \|\overline{w}\|$. We see that $DF_0(0)$ is a hyperbolic linear operator on $\overline{X}$, and in particular, $I - DF_0(0)$ is invertible. Thus the assertion of the theorem follows from the implicit function theorem. □

Remark IV.2. (1) The fixed point z of the map F_e satisfies the equality

$$z = f(z(r^{-1})) + e \cdot h(z(r^{-1}), r^{-1}).$$

Hence the graph M_e of the map z_e is an invariant set of the diffeomorphism g_e.

(2) Theorem IV.1 is also a consequence of the well-known results of the theory of invariant sets of diffeomorphisms [7].

By Remark IV.2 and Theorem IV.1, g_e has an invariant set M_e near to $\{0\} \times M$. We see that F_0 has an orbit $\{a_n\}_{-\infty}^{\infty}$, where $a_n(y) = x_n$ for any $y \in M$, and $a_{n+1} = F_0(a_n)$ and $a_n \to 0$ if $n \to \pm\infty$. Hence $\{a_n\}_{-\infty}^{\infty}$ is a homoclinic orbit of F_0. Further we shall investigate the mapping F_e in the same way as for ordinary diffeomorphisms [2]. Let

$$X = \{\{z_n\}_{-\infty}^{\infty} \mid z_n \in \overline{X}\}$$

be a Banach space with a norm

$$|\{z_n\}_{-\infty}^{\infty}| = \sup_n \|z_n\|.$$

The map F_e generates a map $H_e \colon X \to X$ defined by

$$H_e(\{z_n\}_{-\infty}^{\infty})_n = z_{n+1} - F_e(z_n).$$

Hence we obtain

$$H_0(\{a_n\}_{-\infty}^{\infty}) = 0$$

and

$$DH_0(\{a_n\}_{-\infty}^{\infty})\{v_n\}_{-\infty}^{\infty} = \{v_{n+1} - DF_0(a_n) \cdot v_n\}_{-\infty}^{\infty},$$
$$v_{n+1} - DF_0(a_n)v_n = v_{n+1} - Df(x_n)v_n(r^{-1}).$$

By the proof of Theorem IV.1 the set $\{A_n\}_{-\infty}^{\infty}$, $A_n = DF_0(0)$ has an exponential dichotomy on $\mathbb{Z}$, since $DF_0(0)$ is a hyperbolic linear operator on $\overline{X}$. Since $x_n \to 0$ if $n \to \pm\infty$, thus by Lemma 2.4 we have an exponential dichotomy of the sequences $\{DF_0(a_n)\}_{-\infty}^{0}$, $\{DF_0(a_n)\}_0^{\infty}$. On the other hand, the sequence $\{Df(x_n)\}_{-\infty}^{\infty}$ has an exponential dichotomy on $\mathbb{Z}$, since $\{x_n\}_{-\infty}^{\infty}$ is a transversal orbit of f (see [2, p. 292] or the argument before Theorem I.1). Hence the equation $v_{n+1} = DF_0(a_n)v_n$, which is equivalent to the equation

$$v_{n+1} = Df(x_n)v_n(r^{-1}),$$

has only the trivial, zero, bounded solution on $\mathbb{Z}$. This implies that $\{DF_0(a_n)\}_{-\infty}^{\infty}$ has an exponential dichotomy and thus $DH_0(\{a_n\}_{-\infty}^{\infty})$ is invertible. Using the implicit function theorem we obtain the following theorem.

Theorem IV.3. *Let g_e have the form $(*)$ and let a_n, $n \in \mathbb{Z}$ be maps defined by $a_n(y) = x_n \, \forall y \in M$. Then for any $n \in \mathbb{Z}$, e small, there exists a C^0-mapping $z_{n,e}: M \to \mathbb{R}^m$ such that $g_e(M_e^n) = M_e^{n+1}$, $M_0^n = \operatorname{graph} a_n$ and $M_e^n \to M_e$ if $n \to \pm\infty$, where M_e^n is the graph of the map $z_{n,e}$, and M_e is the invariant set from Theorem IV.1.*

Remark IV.4. The property $M_e^n \to M_e$ if $n \to \pm\infty$ is proved in the same way as for ordinary diffeomorphisms [2, pp. 295–297], since

$$z_{n+1,e} = F_e(z_{n,e}), \qquad z_e = F_e(z_e),$$

and thus

$$z_{n+1,e} - z_e = F_e(z_{n,e}) - F_e(z_e).$$

Let $w_n = z_{n,e} - z_e$, then

$$\begin{aligned} w_{n+1} &= DF_e(z_e)w_n + F_e(z_e + w_n) - F_e(z_e) - DF_e(z_e)w_n, \\ w_{n+1} &= DF_e(z_e)w_n + O(|w_n|^2). \end{aligned}$$

Using an exponential dichotomy of $\{DF_0(0)\}_{-\infty}^{\infty}$ and the fact that z_e is near to 0 we have an exponential dichotomy of $\{DF_e(z_e)\}_{-\infty}^{\infty}$. Now we proceed as in Theorem III.1 (see the proof of the assertion for (III.6)).

From Theorem IV.3 and from the openness property of transversality it follows that for e sufficiently small the homoclinic orbit $\{z_{n,e}\}_{-\infty}^{\infty}$ is transversal for F_e and thus the set $\{z_{n,e}\}_{-\infty}^{\infty} \cup \{z_e\}$ is hyperbolic for F_e [2, p. 279]. Now we can repeat the proof of the shadowing lemma (see [2, p. 282]) to obtain

Theorem IV.5. *The assertion of the shadowing lemma* [2, p. 282] *holds for F_e on the hyperbolic set $\{z_{n,e}\}_{-\infty}^{\infty} \cup \{z_e\}$.*

Using Theorem IV.5 we can repeat the proof of Theorem 5 of [1] obtaining

Theorem IV.6. *There exist $e_0 > 0$ and a natural number $n_0 > 0$ such that for any e, $|e| < e_0$ and all $n_0 \leqslant n$ there exists a compact invariant set $\Omega_n \subset \overline{X}$ of F_e such that the restriction F_e/Ω_n is equivalent to the Bernoulli shift on $B_n(K)$ (see* [1, p. 76]*).*

Remark IV.7. Proofs of Theorems IV.5–6 are straightforward extensions of the proofs of [1, Theorem 5] and [2, Theorem 4.8] and thus we have not presented them. Also the space $B_n(K)$ and the Bernoulli shift is described in [1, p. 76].

Theorem IV.6 has a consequence for the mapping g_e which we formulate in the next theorem.

Theorem IV.8. *For any e small and $n_0 \leqslant n$ (n_0 from Theorem IV.6) the mapping g_e has a compact invariant set A_n in the form*

$$D(B_n(K) \times M),$$

and

(1) $D(a, b) = (C(a, b), b)$ *for any* $a \in B_n(K)$, $b \in M$,
(2) $C\colon B_n(K) \times M \to \mathbb{R}^m$ *is continuous,*
(3) $D(\{a_1\} \times M) \neq D(\{a_2\} \times M)$ *if* $a_1 \neq a_2$,
(4) *for any* $x \in B_n(K)$ *there exists a unique* $y = y(x) \in B_n(K)$ *such that* $g_e(D(\{x\} \times M)) = D(\{y\} \times M)$.

In this way we define a map $B_n(K) \to B_n(K)$, $x \to y(x)$, *which is precisely the Bernoulli shift. Hence* g_e *has a chaotic dynamics on* A_n.

Remark IV.9. We note that these results hold if M is only a compact topological space, r is a homeomorphism and h is continuous and continuously differentiable in x. If moreover M is a C^∞-manifold and h, r are smooth, and

$$\text{(a)} \qquad \sup_M |Dr(\cdot)| \cdot \|B^{-1}\| < 1, \quad \sup_M |Dr^{-1}(\cdot)| \cdot \|A\| < 1,$$

where $Df(0) = (A, B)$ and A, B have eigenvalues inside and outside the unit circle, respectively, then the maps $z_e, z_{n,e}$ and $C(a, \cdot)$ are C^1-smooth. Hence the set A_n consists of C^1-smooth manifolds. Indeed, introducing $\overline{X} = \{z\colon M \to \mathbb{R}^m \mid z \text{ is } C^1\text{-smooth}\}$ we are able to repeat the proofs of the above theorems. We see that the condition (a) expresses the 1-normal hyperbolicity of the map g_0 on $\{0\} \times M$ in the sense of [7].

Similar results we obtain also for the general skew product mapping studied in [1, 8]. Let us consider a diffeomorphism

$$\text{(IV.2)} \qquad g\colon \mathbb{R}^m \times M \to \mathbb{R}^m \times M,$$

where $g(x, y) = (f(x, y), r(y))$, $f, r \in C^\infty$. Note that $r\colon M \to M$ is a diffeomorphism. Let $f(0, \cdot) = 0$, i.e. g has an invariant manifold $\{0\} \times M$. On the space $\overline{X}$ with the norm $\|z\| = \sup_M |z(\cdot)|$ we define the map $\overline{F}\colon \overline{X} \to \overline{X}$,

$$\overline{F}(z)(\cdot) = f(z \cdot r^{-1}(\cdot), r^{-1}(\cdot)).$$

We see that $\overline{F}(0) = 0$ and $D\overline{F}(0)v = D_x f(0, r^{-1}(\cdot))v(r^{-1}(\cdot))$.

Further, for $\overline{F}$ we define $F\colon X \to X$,

$$F(\{z_n\}_{-\infty}^{\infty})_n = z_{n+1} - \overline{F}(z_n).$$

Definition IV.10 ([7, p. 5]). The mapping g is 0-normally hyperbolic on $\{0\} \times M$ if there exist a smooth mapping $P\colon M \to \mathscr{L}(\mathbb{R}^m)$ and a number b, $0 < b < 1$ such that

(1) $P(y) \cdot P(y) = P(y)$, $D_x f(0, y) \operatorname{Im} P(y) = \operatorname{Im} P(r(y))$
$D_x f(0, y) \operatorname{Im}(I - P(y)) = \operatorname{Im}(I - P(r(y)))$ for all $y \in M$,

(2) $|D_x f(0, y) \cdot P(y)| \leqslant b$, $|D_x f^{-1}(0, y) \cdot (I - P(r(y)))| \leqslant b$ for all $y \in M$,

(3) $\dim \operatorname{Im} P(y)$ does not depend on $y \in M$

($\cdot$ is the composition of matrices).

Let g be 0-normally hyperbolic on $\{0\} \times M$, then by using the projections $P(\cdot)$ we obtain that the set $\{\operatorname{Im} P(y), \operatorname{Im}(I - P(y))\}_{y \in M}$ gives an invariant hyperbolic splitting of $\mathbb{R}^m \times M$ for $D_x f(0, y)$. Since

$$D\overline{F}(0)v = D_x f(0, r^{-1}(\cdot))v(r^{-1}(\cdot)),$$

we put

$$X_1 = \{v \in \overline{X} \mid v(y) \in \operatorname{Im} P(y)\},$$
$$X_2 = \{v \in \overline{X} \mid v(y) \in \operatorname{Im}(I - P(y))\}.$$

Then clearly (X_1, X_2) is an invariant hyperbolic splitting of $\overline{X}$ for the map $D\overline{F}(0)$. Thus $\{D_n\}_{-\infty}^{\infty}$, $D_n = D\overline{F}(0)$ has an exponential dichotomy on $\mathbb{Z}$. Hence the existence of local stable and unstable manifolds $\overline{W}^s_{\text{loc}}$, $\overline{W}^u_{\text{loc}}$ of the point can be shown in the standard way as is used for instance in [2, p. 295]. Let us assume that the sequence $\{u_n\}_{-\infty}^{\infty} \subset \overline{X}$ is such that $\overline{F}(u_n) = u_{n+1}, u_n \to 0$ if $n \to \pm\infty$ and the sequence $\{D\overline{F}(u_n)\}_{-\infty}^{\infty}$ has an exponential dichotomy on $\mathbb{Z}$. Now we proceed as in the proof of Theorem IV.6 to prove the existence of an invariant set A_n of the map g for n natural large, which has the properties from Theorem IV.8.

Further we investigate the case when there exist smooth mappings $v_i\colon M \to \mathbb{R}^m$, $i = 1, \ldots, m$ such that

$$\operatorname{span}\{v_1, \ldots, v_s\}(y) = \operatorname{Im} P(y),$$
$$\operatorname{span}\{v_{s+1}, \ldots, v_m\}(y) = \operatorname{Im}(I - P(y)).$$

Then we can assume that the decomposition of $\mathbb{R}^m \times M$ given by the projection $P(\cdot)$ is constant, i.e. we have

$$D_x f(0,y) = (A(y), B(y)), \tag{IV.3}$$

where $|A(\cdot)| < b, |B^{-1}(\cdot)| < b < 1, A(\cdot) \in \mathscr{L}(\mathbb{R}^s), B(\cdot) \in \mathscr{L}(\mathbb{R}^{m-s})$. Further we assume that g is normally hyperbolic on $\{0\} \times M$, i.e. ([7, p. 5])

$$|A(\cdot)| \cdot |Dr^{-1}(\cdot)| < 1 \quad |B^{-1}(\cdot)| \cdot |Dr(\cdot)| < 1.$$

Then according to [7] g has local stable and unstable C^1 -manifolds for $\{0\} \times M$, which can be expressed as graphs of functions

$$(x_1, y) \to (x_1, G_1(x_1, y), y),$$
$$(x_2, y) \to (G_2(x_2, y), x_2, y),$$

respectively, where $x_1 \in U^s$, $x_2 \in U^{m-s}$, $G_1 : U^s \times M \to \mathbb{R}^{m-s}$, $G_2 \colon U^{m-s} \times M \to \mathbb{R}^s$, U^s, U^{m-s} are open neighbourhoods of the points $0 \in \mathbb{R}^s$ and $0 \in \mathbb{R}^{m-s}$. Since g is a diffeomorphism it is possible to define global stable and unstable manifolds W^s, W^u for $\{0\} \times M$ as the sets of all graphs of $z \in \overline{X}$ satisfying

$$g^n(G) \to \{0\} \times M \quad \text{if } n \to \pm\infty, \quad G = \operatorname{graph} z.$$

Theorem IV.11. *The sequence $\{D\overline{F}(u_n)\}_{-\infty}^{\infty}$ has an exponential dichotomy provided that the manifolds W^s and W^u have a transversal intersection in the graph of a mapping u_0.*

Proof. Since g is a diffeomorphism, $\overline{F}$ is a diffeomorphism as well. Then $\overline{F}$ has global stable and unstable manifolds $\overline{W}^s$, $\overline{W}^u$, respectively. Since g has a special form, the sequence $\{D\overline{F}(u_n)\}_{-\infty}^{\infty}$ has an exponential dichotomy provided $\overline{W}^s, \overline{W}^u$ have a transversal intersection in u_0 (see [2, p. 287 and 300]), i.e.

$$T_{u_0}\overline{W}^s \oplus T_{u_0}\overline{W}^u = \overline{X}.$$

On the other hand, the set $\overline{W}^s$ consists of such $z \in \overline{X}$ for which $\operatorname{graph} z \subset W^s$. The same holds for $\overline{W}^u$. Hence

$$T_{u_0}\overline{W}^s \cong \{z \in \overline{X} \mid z(y) \in T_{u_0(y)}W^s/M(y)\},$$
$$T_{u_0}\overline{W}^u \cong \{z \in \overline{X} \mid z(y) \in T_{u_0(y)}W^u/M(y)\},$$

where $M(y) = T_{u_0(y)}(\text{graph}\, u_0)$ and by "/" the factor space is denoted. This implies

$$T_{u_0}\overline{W}^s \oplus T_{u_0}\overline{W}^u$$
$$\cong \{z \in \overline{X} \mid z(y) \in T_{u_0(y)}W^s/M(y) \oplus T_{u_0(y)}W^u/M(y)\}.$$

According to the assumptions of the theorem, W^s and W^u have a transversal intersection in the graph u_0, i.e. we have

$$T_{u_0(y)}W^s/M(y) \oplus T_{u_0(y)}W^u/M(y) \cong \mathbb{R}^m$$

for $y \in M$. The proof is complete. □

As a consequence of Theorem IV.11 we obtain

Theorem IV. 12. *Let the invariant set* $\{0\} \times M$ *of the map* g *be both trivially normally hyperbolic (i.e.* g *has the property* (IV.3)*) and normally hyperbolic* ([7])*. If the global stable and unstable manifolds of this set have a transversal intersection in* S*, where* $S = \text{graph}\, z$ *for a* C^1*-mapping* $z \colon M \to \mathbb{R}^m$*. Then* g *has an invariant set* A_n *for any large natural number* n *possessing the properties from Theorem* IV.8.

We note that the trivial hyperbolicity of $\{0\} \times M$ is not essential. We have considered it only for the sake of simplicity. By [7] the usual normal hyperbolicity ensures the existence of the local stable and unstable manifolds of $\{0\} \times M$.

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