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A CHARACTERIZATION OF THE GAMMA DISTRIBUTION IN TERMS OF CONDITIONAL MOMENT

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Summary. A characterization of the Gamma distribution in terms of the k-th conditional moment presented in this paper extends the result of Shunji Osaki and Xin-xiang Li (1988).

Keywords: characterization of distribution, Gamma distribution, conditional moment

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1. INTRODUCTION

The problem of characterization of statistical distributions is of great interest because of its applications in reliability theory and statistics.

The books by Kagan, Linnik, Rao [3] and by Galambos and Kotz [2] were the basis of an increasing interest for research in this area. Most of the results deal with the exponential distribution (see e.g. [1], [2], [3]).

As the Gamma distribution is often used to model a wide variety of positive valued random unities in reliability theory and queueing system analysis, the characterization of this distribution is an appealing and interesting problem with a great number of applications. There are several characterizations of the Gamma distributions (see e.g. [3], [4], [5] and [6]). In this paper, we give a new one based on a relationship between the k-th conditional moment and the failure rate (Section 2).

2. Characterization of the Gamma distribution

Let X be a non-negative, absolutely continuous r.v. with distribution function (df) $F(x)$. Then $\overline{F}(x) = 1 - F(x)$, $f(x) = F'(x)$ and $r(x) = f(x)/\overline{F}(x)$ are respectively the survival function, the density function (pdf) and the failure rate of X. In the following, it is said that the r.v. X has the Gamma distribution with parameters (α, β) with $\alpha > 0$ and $\beta > 0$ (denoted by $X \sim \Gamma(\alpha, \beta)$) if $f(x) = \beta^{\alpha}\left[\Gamma(\alpha)\right]^{-1} \cdot \mathrm{e}^{-\beta x} x^{\alpha - 1} \cdot I\{x > 0\}$.

Without loss of generality the characterization below is obtained for the Gamma distribution with parameters $(\lambda, 1)$. Let us denote, for any positive integer k and for any number $a \neq 0$, by $a^{(k)}$ the product $a^{(k)} = a(a-1)\ldots(a-k+1)$ and put $a^{(0)} = 1$.

Theorem. *Let X be a non-negative r.v. with* df $F(x)$, pdf $f(x)$ *such that $f'(x)$ is continuous on $(0, \infty)$ and with failure rate $r(x)$. Then $X \sim \Gamma(\lambda, 1)$ if and only if for an integer $k \geqslant 1$, the representation*

$$E\{X^k \mid X \geqslant y\} = \left[\sum_{i=0}^{k-1} (\lambda + k - 1)^{(i)} y^{k-i}\right] r(y) + (\lambda + k - 1)^{(k)} \tag{1}$$

holds for all $y \geqslant 0$.

Proof. (*Necessity*): Suppose $X \sim \Gamma(\lambda, 1)$, i.e. $f(x) = x^{\lambda-1}\mathrm{e}^{-x}/\Gamma(\lambda)$. For any given integer j, it is easy to obtain that the following recurrence relation holds:

$$\int_y^{\infty} t^j f(t)\,\mathrm{d}t = y^j f(y) + (\lambda + j - 1)\int_y^{\infty} t^{j-1} f(t)\,\mathrm{d}t \quad \text{for all } y \geqslant 0. \tag{2}$$

In the case of $y = 0$, (2) is the well-known relation between the moments $\mathrm{E}X^j$ and $\mathrm{E}X^{j-1}$ of a Gamma distributed random variable X, namely $\mathrm{E}X^j = (\lambda + j - 1)\mathrm{E}X^{j-1}$.

Dividing (2) by y^j and denoting $I_j = \int_y^{\infty}\left(\frac{t}{y}\right)^j f(t)\,\mathrm{d}t$ we rewrite (2) as a recurrence relations between the terms of the sequence $\{I_j\}$, i.e. $I_j = f(y) + \frac{(\lambda+j-1)}{y} I_{j-1}$. Therefore, noting that $I_0 = \overline{F}(y)$, we obtain the representation

$$I_j = \sum_{i=0}^{j-1} \frac{(\lambda + j - 1)^{(i)}}{y^i} f(y) + \frac{(\lambda + j - 1)^{(j)}}{y^j} \overline{F}(y).$$

Consequently, for any given integer $j \geqslant 1$ we have

$$\int_y^{\infty} t^j f(t)\,\mathrm{d}t = \sum_{i=0}^{j-1} (\lambda + j - 1)^{(i)} y^{j-i} f(y) + (\lambda + j - 1)^{(j)} \overline{F}(y).$$

Now, setting $j = k$, dividing by $\overline{F}(y)$ and taking into account that $\mathrm{E}\{X^k \mid X \geqslant y\} = \frac{1}{\overline{F}(y)} \int_y^\infty t^k f(t)\, dt$ we complete the proof of necessity.

(*Sufficiency*): Let us assume that (1) is true, then we have to prove that $X \sim \Gamma(\lambda, 1)$. We write (1) in the form of an equation with an unknown function

$$\int_y^\infty t^k f(t)\,\mathrm{d}t = \sum_{i=0}^{k-1} (\lambda + k - 1)^{(i)} y^{k-i} f(y) + (\lambda + k - 1)^{(k)} \int_y^\infty f(t)\,\mathrm{d}t.$$

Differentiating both sides of the last equation with respect to y gives

$$-y^k f(y) = \sum_{i=0}^{k-1} (\lambda + k - 1)^{(i)} \left[y^{k-i} f'(y) + (k-i) k^{k-i-1} f(y)\right] - (\lambda + k - 1)^{(k)} f,$$

which is a differential equation of order one with respect to the unknown function $f(y)$. The last equation can be written in the form

$$\text{(3)} \qquad \left\{ y \sum_{i=0}^{k-1} (\lambda + k - 1)^{(i)} y^{k-i-1} \right\} f'(y)$$
$$= \left\{ (\lambda + k - 1)^{(k)} - y^k - \sum_{i=0}^{k-1} (\lambda + k - 1)^{(i)} (k-i) y^{k-i-1} \right\} f(y).$$

Now applying the auxiliary identity $(\lambda - 1 + k - i)(\lambda + k - 1)^{(i)} = (\lambda + k - 1)^{(i+1)}$ it can easily be verified that

$$(\lambda + k - 1)^{(k)} - y^k - \sum_{i=0}^{k-1} (\lambda + k - 1)^{(i)} (k-i) y^{k-i-1} = (\lambda - 1 - y) \sum_{i=0}^{k-1} (\lambda + k - 1)^{(i)} y^{k-i-1}.$$

To prove this relation we proceed as follows: We write

$$(*) = (\lambda - 1 - y) \sum_{i=0}^{k-1} (\lambda + k - 1)^{(i)} y^{k-i-1} + \sum_{i=0}^{k-1} (\lambda + k - 1)^{(i)} (k-i) y^{k-i-1}$$
$$= \sum_{i=0}^{k-1} (\lambda + k - 1)^{(i+1)} y^{k-i-1} - \sum_{i=0}^{k-1} (\lambda + k - 1)^{(i)} y^{k-i}.$$

The change of the summation index i by $j = i + 1$ in the first term yields to

$$(*) = \sum_{j=1}^{k} (\lambda + k - 1)^{(j)} y^{k-j} - \sum_{i=0}^{k-1} (\lambda + k - 1)^{(i)} y^{k-i}$$
$$= (\lambda + k - 1)^{(k)} - y^k,$$

which we wanted to verify.

Then the equation (3) is equivalent to the equation

$$\left\{\sum_{i=0}^{k-1}(\lambda+k-1)^{(i)}y^{k-i-1}\right\}yf'(y)=\left\{\sum_{i=0}^{k-1}(\lambda+k-1)^{(i)}y^{k-i-1}\right\}(\lambda-1-y)f(y).$$

Hence for $y \geqslant 0$ we derive

$$yf'(y)=(\lambda-1-y)f(y).$$

Consequently, $f(y)=A\mathrm{e}^{-y}y^{\lambda-1}$ for some constant $A>0$. Therefore, the solution of (3) is $f(y)=A\mathrm{e}^{-y}y^{\lambda-1}$. The constant A can be determined from $\int_0^\infty f(y)\,\mathrm{d}y=1$. Then we have $f(y)=y^{\lambda-1}\mathrm{e}^{-y}/\Gamma(\lambda)$, which is the gamma pdf with parameters $(\lambda,1)$. $\square$

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