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Tables for a statistical quality control test

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TABLES FOR A STATISTICAL QUALITY CONTROL TEST

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Summary. Critical constants for a test of the hypothesis that the mean μ and the standard deviation σ of the normal $N(\mu,\sigma^2)$ population satisfy the constrains $\mu + c\sigma \leqslant M$, $\mu - c\sigma \geqslant m$, are presented. In this setup $m < M$ are prescribed tolerance limits and $c > 0$ is a chosen constant.

Keywords: Two-sided quality control; Normal distribution; Small sample sizes; Hypothesis testing

AMS classification: 62 Q 05, 62 F 03

Let us assume that the statistical population X has a normal $N(\mu,\sigma^2)$ distribution with the parameters from the set $\Theta = \left\{\binom{\mu}{\sigma};\ \mu \in \mathbf{R},\ 0 < \sigma < +\infty\right\}$, i.e. its distribution function has the form

$$F_{\mu,\sigma}(x) = (2\pi\sigma^2)^{-1/2}\int_{-\infty}^{x} \exp[-(x-\mu)^2/2\sigma^2]\,\mathrm{d}x. \tag{1}$$

Let c_Δ be the $\big(1 - (\Delta/2)\big)$ quantile of the $N(0,1)$ distribution determined by the equality

$$\Phi(c_\Delta) = 1 - (\Delta/2), \tag{2}$$

where $\Phi(x) = F_{0,1}(x)$. Let $m < M$ be prescribed tolerance bounds and

$$H_\Delta = \left\{\begin{pmatrix}\mu\\ \sigma\end{pmatrix} \in \Theta;\ \mu + c_\Delta\sigma \leqslant M,\ \mu - c_\Delta\sigma \geqslant m\right\}. \tag{3}$$

Let $x_1,\dots,x_n$ be a random sample from X and

$$\bar{x} = n^{-1}\sum_{j=1}^{n} x_j,\quad s = \left[n^{-1}\sum_{j=1}^{n}(x_j - \bar{x})^2\right]^{1/2}. \tag{4}$$

It has been proposed in [6] to test the hypothesis (3) by means of the distance

$$\hat{\varrho} = \hat{\varrho}\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}, H_\Delta\right) = \inf\left\{\left[\frac{(\bar{x}-\mu)^2 + 2(s-\sigma)^2}{s^2}\right]^{1/2}; \begin{pmatrix}\mu\\ \sigma\end{pmatrix} \in H_\Delta\right\} \tag{5}$$

with the test rule

(6) *reject the null hypothesis* H_Δ *if* $\hat{\varrho} > t$

where the critical constant t is chosen so that

$$\sup\left\{P_\theta\left[\hat{\varrho}\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}, H_\Delta\right) > t\right]; \theta \in H_\Delta\right\} = \alpha. \tag{7}$$

Explicit formulas for the set

$$A_t = \left\{\begin{pmatrix}\bar{x}\\ s\end{pmatrix} \in \Theta; \hat{\varrho}\left(\begin{pmatrix}\bar{x}\\ s\end{pmatrix}, H_\Delta\right) \leqslant t\right\}$$

are in [6] denoted by (1.18) – (1.28), and for the practical use also the direct computation of (5) can be employed. Let $s > 0$. If $\bar{x} \geqslant \frac{M+m}{2}$, then denoting $c = c_\Delta$ we have

(8)

$$\hat{\varrho}^2 = \begin{cases} \dfrac{(\bar{x}-M)^2}{s^2} + 2 & \bar{x} > M + \dfrac{2}{c}s, \\[2ex] \dfrac{(\bar{x}+cs-M)^2}{s^2}\dfrac{2}{2+c^2} & M + \dfrac{2}{c}s \geqslant \bar{x} \geqslant \dfrac{M+m}{2} + \dfrac{2}{c}s - \dfrac{M-m}{c^2}, \\ & \bar{x} + cs > M, \\[2ex] 0 & M + \dfrac{2}{c}s \geqslant \bar{x} \geqslant \dfrac{M+m}{2} + \dfrac{2}{c}s - \dfrac{M-m}{c^2}, \\ & \bar{x} + cs \leqslant M, \\[2ex] \dfrac{\left(\bar{x} - \frac{M+m}{2}\right)^2 + 2\left(s - \frac{M-m}{2c}\right)^2}{s^2} & \dfrac{M+m}{2} + \dfrac{2}{c}s - \dfrac{M-m}{c^2} > \bar{x} \geqslant \dfrac{M+m}{2}, \end{cases}$$

while in the case $\bar{x} \leqslant \frac{M+m}{2}$

(9)

$$\hat{\varrho}^2 = \begin{cases} \dfrac{(\bar{x}-m)^2}{s^2} + 2 & \bar{x} < m - \dfrac{2}{c}s, \\[2ex] \dfrac{(\bar{x}-cs-m)^2}{s^2}\dfrac{2}{2+c^2} & m - \dfrac{2}{c}s \leqslant \bar{x} \leqslant \dfrac{M+m}{2} - \dfrac{2}{c}s + \dfrac{M-m}{c^2}, \\ & \bar{x} - cs < m, \\[2ex] 0 & m - \dfrac{2}{c}s \leqslant \bar{x} \leqslant \dfrac{M+m}{2} - \dfrac{2}{c}s + \dfrac{M-m}{c^2}, \\ & \bar{x} - cs \geqslant m, \\[2ex] \dfrac{\left(\bar{x} - \frac{M+m}{2}\right)^2 + 2\left(s - \frac{M-m}{2c}\right)^2}{s^2} & \dfrac{M+m}{2} - \dfrac{2}{c}s + \dfrac{M-m}{c^2} < \bar{x} \leqslant \dfrac{M+m}{2}. \end{cases}$$

Since according to Theorem 1 in [6] the rejection probability (7) depends on n, c and t only, it can be computed for all $m < M$ simultaneously. Values of the critical constants $t = t(\alpha, n, c_\Delta)$ satisfying (7) are given in the enclosed tables. We remark that for $1 - \Delta = 0.75;\ 0.9;\ 0.95;\ 0.99$ we have $c_\Delta = 1.150349;\ 1.644854;\ 1.959964;\ 2.575829$, respectively. This quantiles of the $N(0,1)$ distribution are taken from the tables [2] .

In the following example we use the data from Example 8.1.7 in [1].

Example. The resistance of 10 samples of the wire of the type B-302 was measured (in ohms) with the following results:

0.129, 0.132, 0.128, 0.120, 0.126, 0.137, 0.124, 0.135, 0.119, 0.123.

Test 95 % concentration of the statistical population in the limits $\langle 0.113; 0.135\rangle$ by testing the hypothesis (3) with $\Delta = 0.05$, $m = 0.113$, $M = 0.135$ at the level of significance $\alpha = 0.05$.

Solution. The sample characteristics (4) are

$$\bar{x} = 0.1273, \quad s = 0.0057628.$$

Since $1 - \Delta = 0.95$, the constant $c_\Delta = 1.959964$ and denoting $c = c_\Delta$ we see that

$$\bar{x} = 0.1273 \geqslant \frac{M+m}{2} = 0.124,$$
$$M + \frac{2}{c}s = 0.1408805 \geqslant \bar{x} \geqslant \frac{M+m}{2} + \frac{2}{c}s - \frac{M-m}{c^2} = 0.1241535,$$
$$\bar{x} + cs = 0.1385949 > M = 0.135.$$

This together with (8) means that

$$\hat{\varrho}^2 = \frac{(\bar{x} + cs - M)^2}{s^2} \frac{2}{2 + c^2} = 0.133234.$$

Hence $\hat{\varrho} = 0.36501 \leqslant t(\alpha, n, c) = t(0.05, 10, 1.959964) = 0.438$ and according to (6) the null hypothesis (3) is not rejected.

We remark that the same conclusion can be achieved by means of the formulas (1.17)–(1.28) from [6]. Since $c = 1.959964$, $t = 0.438 < \sqrt{2}$, the upper bounds (1.24) and (1.18) are

$$B(-1, 1, t) = 0.7391321, \quad B = B(0.113, 0.135, 0.438) = 0.0081305.$$

Since the quantities (1.25) are

$$\delta = 1.7439240, \qquad s^* = 0.6813363$$

and

$$\frac{2}{M - m} s = 0.5238909 < s^*,$$

the limits (1.26) and (1.23) are

$$U\Big(\frac{2}{M - m} s, -1, 1, t\Big) = 0.3653503, \; L(s) = 0.11998, \; U(s) = 0.12802.$$

Hence $s = 0.0057628 < B(m, M, t) = 0.0081305$, $\bar{x} = 0.1273 \in \langle L(s), U(s)\rangle$ which together with (1.21) means that $\binom{\bar{x}}{s}$ belongs to the set (1.17), and according to (6) the null hypothesis (3) is not rejected.

We remark that in (5) the constant 2 was chosen because for this choice of $\hat{\varrho}$ the estimate θ_n^* of the unknown parameter θ from (3), determined by the equality

$$\hat{\varrho}\Big(\binom{\bar{x}}{s}, H_\Delta\Big) = \hat{\varrho}\Big(\binom{\bar{x}}{s}, \theta_n^*\Big)$$

where $\hat{\varrho}(\binom{\bar{x}}{s}, \binom{\mu}{\sigma}) = \big[s^{-2}[(\bar{x} - \mu)^2 + 2(s - \sigma)^2]\big]^{1/2}$, and the maximum likelihood estimate $\hat{\theta}_n^H$ of the unknown parameter from (3), are according to (2.9) in [5] asymptotically equivalent in the sense that the equality $\theta_n^* = \hat{\theta}_n^H + o_P(n^{-1/2})$ holds provided that the null hypothesis is valid. The explicit formulas for $\hat{\theta}_n^H$ can be found in [4] and the asymptotic distribution of the likelihood ratio test statistic for testing (3) is derived in [3].

Validity of (8) and (9) follows from the fact that for $s > 0$ and $\bar{x} \geqslant \frac{M+m}{2}$ we have

(10)

$$\theta_n^*\begin{pmatrix}\bar{x}\\ s\end{pmatrix} = \begin{cases} \begin{pmatrix}M\\ 0\end{pmatrix} & \bar{x} > M + \frac{2}{c}s, \\ \begin{pmatrix}\bar{x}\\ s\end{pmatrix} - \frac{(\bar{x}+cs-M)}{(2+c^2)}\begin{pmatrix}2\\ c\end{pmatrix} & M + \frac{2}{c}s \geqslant \bar{x} \geqslant \frac{M+m}{2} + \frac{2}{c}s - \frac{M-m}{c^2}, \\ & \bar{x} + cs > M, \\ \begin{pmatrix}\bar{x}\\ s\end{pmatrix} & M + \frac{2}{c}s \geqslant \bar{x} \geqslant \frac{M+m}{2} + \frac{2}{c}s - \frac{M-m}{c^2}, \\ & \bar{x} + cs \leqslant M, \\ \begin{pmatrix}\frac{M+m}{2}\\ \frac{M-m}{2c}\end{pmatrix} & \frac{M+m}{2} + \frac{2}{c}s - \frac{M-m}{c^2} > \bar{x} \geqslant \frac{M+m}{2}, \end{cases}$$

while in the case $\bar{x} \leqslant \frac{M+m}{2}$

(11)

$$\theta_n^*\begin{pmatrix}\bar{x}\\ s\end{pmatrix} = \begin{cases} \begin{pmatrix}m\\ 0\end{pmatrix} & \bar{x} < m - \frac{2}{c}s, \\ \begin{pmatrix}\bar{x}\\ s\end{pmatrix} + \frac{(\bar{x}-cs-m)}{(2+c^2)}\begin{pmatrix}-2\\ c\end{pmatrix} & m - \frac{2}{c}s \leqslant \bar{x} \leqslant \frac{M+m}{2} - \frac{2}{c}s + \frac{M-m}{c^2}, \\ & \bar{x} - cs < m, \\ \begin{pmatrix}\bar{x}\\ s\end{pmatrix} & m - \frac{2}{c}s \leqslant \bar{x} \leqslant \frac{M+m}{2} - \frac{2}{c}s + \frac{M-m}{c^2}, \\ & \bar{x} - cs \geqslant m, \\ \begin{pmatrix}\frac{M+m}{2}\\ \frac{M-m}{2c}\end{pmatrix} & \frac{M+m}{2} - \frac{2}{c}s + \frac{M-m}{c^2} < \bar{x} \leqslant \frac{M+m}{2}. \end{cases}$$

The formulas (10) have been proved in [6] for $m = -1$, $M = 1$ as the equality (2.26) by means of the relation (2.24), the proof in the general case is the same. Tables of the critical values $t = t(\alpha, n, c_\Delta)$, satisfying in the notation (3) the equality (7), are given below.

$\Delta = 0.25 \quad c_\Delta = 1.150349$				
n	α			
	0.200	0.100	0.050	0.010
3	0.518	0.704	0.876	1.355
4	0.465	0.615	0.742	1.038
5	0.427	0.559	0.666	0.896
6	0.398	0.518	0.614	0.809
7	0.376	0.487	0.575	0.748
8	0.357	0.461	0.543	0.702
9	0.341	0.440	0.517	0.665
10	0.327	0.421	0.495	0.634
11	0.315	0.406	0.477	0.609
12	0.305	0.392	0.460	0.586
13	0.295	0.379	0.445	0.566
14	0.286	0.368	0.432	0.549
15	0.279	0.357	0.420	0.533
16	0.271	0.348	0.409	0.519
17	0.265	0.340	0.399	0.506
18	0.259	0.332	0.389	0.494
19	0.253	0.324	0.381	0.483
20	0.248	0.318	0.373	0.473
21	0.243	0.311	0.365	0.463
22	0.238	0.305	0.358	0.455
23	0.234	0.300	0.352	0.446
24	0.230	0.294	0.346	0.438
25	0.226	0.289	0.340	0.431
26	0.222	0.285	0.334	0.424

$\Delta = 0.25 \quad c_\Delta = 1.150349$				
n	α			
	0.200	0.100	0.050	0.010
27	0.218	0.280	0.329	0.417
28	0.215	0.276	0.324	0.411
29	0.212	0.272	0.319	0.405
30	0.209	0.268	0.315	0.399
31	0.206	0.264	0.310	0.394
32	0.203	0.261	0.306	0.389
33	0.201	0.257	0.302	0.384
34	0.198	0.254	0.298	0.379
35	0.196	0.251	0.295	0.374
36	0.193	0.248	0.291	0.370
37	0.191	0.245	0.288	0.366
38	0.189	0.242	0.285	0.362
39	0.187	0.239	0.282	0.358
40	0.185	0.237	0.278	0.354
41	0.183	0.234	0.276	0.350
42	0.181	0.232	0.273	0.347
43	0.179	0.230	0.270	0.343
44	0.177	0.227	0.267	0.340
45	0.175	0.225	0.265	0.337
46	0.174	0.223	0.262	0.333
47	0.172	0.221	0.260	0.330
48	0.171	0.219	0.257	0.327
49	0.169	0.217	0.255	0.325
50	0.168	0.215	0.253	0.322

$\Delta = 0.10 \quad c_\Delta = 1.644854$				
n	α			
	0.200	0.100	0.050	0.010
3	0.405	0.575	0.700	0.944
4	0.382	0.526	0.631	0.822
5	0.360	0.489	0.584	0.750
6	0.342	0.460	0.547	0.699
7	0.326	0.436	0.518	0.659
8	0.312	0.416	0.494	0.627
9	0.300	0.399	0.473	0.600
10	0.289	0.384	0.455	0.577
11	0.280	0.371	0.439	0.557
12	0.271	0.359	0.425	0.540
13	0.264	0.348	0.413	0.524
14	0.257	0.339	0.401	0.509
15	0.250	0.330	0.391	0.496
16	0.244	0.322	0.381	0.484
17	0.239	0.314	0.373	0.473
18	0.233	0.307	0.364	0.463
19	0.229	0.301	0.357	0.454
20	0.224	0.295	0.350	0.445
21	0.220	0.289	0.343	0.437
22	0.216	0.284	0.337	0.429
23	0.212	0.279	0.331	0.422
24	0.209	0.274	0.326	0.415
25	0.205	0.270	0.320	0.408
26	0.202	0.266	0.315	0.402

$\Delta = 0.10 \quad c_\Delta = 1.644854$				
n	α			
	0.200	0.100	0.050	0.010
27	0.199	0.262	0.311	0.396
28	0.196	0.258	0.306	0.390
29	0.193	0.254	0.302	0.385
30	0.191	0.251	0.298	0.380
31	0.188	0.247	0.294	0.375
32	0.186	0.244	0.290	0.370
33	0.184	0.241	0.286	0.366
34	0.181	0.238	0.283	0.361
35	0.179	0.235	0.279	0.357
36	0.177	0.233	0.276	0.353
37	0.175	0.230	0.273	0.349
38	0.173	0.227	0.270	0.346
39	0.171	0.225	0.267	0.342
40	0.170	0.223	0.264	0.338
41	0.168	0.220	0.262	0.335
42	0.166	0.218	0.259	0.332
43	0.164	0.216	0.257	0.329
44	0.163	0.214	0.254	0.326
45	0.161	0.212	0.252	0.323
46	0.160	0.210	0.250	0.320
47	0.158	0.208	0.247	0.317
48	0.157	0.206	0.245	0.314
49	0.156	0.204	0.243	0.312
50	0.154	0.203	0.241	0.309

$\Delta = 0.05 \quad c_{\Delta} = 1.959964$				
n	α			
	0.200	0.100	0.050	0.010
3	0.356	0.528	0.646	0.846
4	0.345	0.490	0.593	0.762
5	0.330	0.460	0.553	0.706
6	0.316	0.435	0.521	0.663
7	0.303	0.414	0.495	0.630
8	0.291	0.396	0.473	0.602
9	0.281	0.381	0.454	0.578
10	0.272	0.367	0.438	0.557
11	0.263	0.355	0.423	0.539
12	0.256	0.344	0.410	0.522
13	0.249	0.334	0.399	0.508
14	0.242	0.325	0.388	0.494
15	0.236	0.317	0.378	0.482
16	0.231	0.310	0.369	0.471
17	0.226	0.303	0.361	0.461
18	0.221	0.296	0.353	0.451
19	0.217	0.290	0.346	0.442
20	0.213	0.285	0.339	0.434
21	0.209	0.279	0.333	0.426
22	0.205	0.274	0.327	0.418
23	0.202	0.270	0.322	0.411
24	0.199	0.265	0.316	0.405
25	0.195	0.261	0.311	0.399
26	0.192	0.257	0.307	0.393

$\Delta = 0.05 \quad c_{\Delta} = 1.959964$				
n	α			
	0.200	0.100	0.050	0.010
27	0.190	0.253	0.302	0.387
28	0.187	0.249	0.298	0.382
29	0.184	0.246	0.294	0.377
30	0.182	0.243	0.290	0.372
31	0.180	0.239	0.286	0.367
32	0.177	0.236	0.282	0.362
33	0.175	0.233	0.279	0.358
34	0.173	0.231	0.276	0.354
35	0.171	0.228	0.272	0.350
36	0.169	0.225	0.269	0.346
37	0.167	0.223	0.266	0.342
38	0.165	0.220	0.263	0.339
39	0.164	0.218	0.261	0.335
40	0.162	0.216	0.258	0.332
41	0.160	0.214	0.255	0.329
42	0.159	0.211	0.253	0.325
43	0.157	0.209	0.250	0.322
44	0.156	0.207	0.248	0.319
45	0.154	0.205	0.246	0.316
46	0.153	0.204	0.243	0.314
47	0.152	0.202	0.241	0.311
48	0.150	0.200	0.239	0.308
49	0.149	0.198	0.237	0.306
50	0.148	0.197	0.235	0.303

$\Delta = 0.01 \quad c_\Delta = 2.575829$				
n	α			
	0.200	0.100	0.050	0.010
3	0.289	0.468	0.586	0.762
4	0.294	0.444	0.547	0.704
5	0.288	0.422	0.515	0.660
6	0.279	0.401	0.488	0.625
7	0.270	0.384	0.466	0.596
8	0.261	0.369	0.447	0.572
9	0.253	0.355	0.430	0.551
10	0.246	0.344	0.415	0.532
11	0.239	0.333	0.402	0.516
12	0.233	0.323	0.390	0.501
13	0.227	0.314	0.380	0.488
14	0.221	0.306	0.370	0.475
15	0.216	0.299	0.361	0.464
16	0.212	0.292	0.353	0.454
17	0.207	0.286	0.345	0.444
18	0.203	0.280	0.338	0.435
19	0.200	0.275	0.331	0.427
20	0.196	0.269	0.325	0.419
21	0.193	0.264	0.319	0.412
22	0.189	0.260	0.314	0.405
23	0.186	0.256	0.308	0.398
24	0.183	0.251	0.303	0.392
25	0.181	0.248	0.299	0.386
26	0.178	0.244	0.294	0.381

$\Delta = 0.01 \quad c_\Delta = 2.575829$				
n	α			
	0.200	0.100	0.050	0.010
27	0.176	0.240	0.290	0.375
28	0.173	0.237	0.286	0.370
29	0.171	0.234	0.282	0.365
30	0.169	0.231	0.278	0.361
31	0.167	0.228	0.275	0.356
32	0.165	0.225	0.271	0.352
33	0.163	0.222	0.268	0.348
34	0.161	0.220	0.265	0.344
35	0.159	0.217	0.262	0.340
36	0.157	0.215	0.259	0.336
37	0.155	0.212	0.256	0.333
38	0.154	0.210	0.254	0.329
39	0.152	0.208	0.251	0.326
40	0.151	0.206	0.248	0.323
41	0.149	0.204	0.246	0.320
42	0.148	0.202	0.243	0.316
43	0.146	0.200	0.241	0.314
44	0.145	0.198	0.239	0.311
45	0.144	0.196	0.237	0.308
46	0.143	0.194	0.235	0.305
47	0.141	0.193	0.233	0.303
48	0.140	0.191	0.231	0.300
49	0.139	0.189	0.229	0.298
50	0.138	0.188	0.227	0.295

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Súhrn

TABUĽKY PRE TEST KONTROLY KVALITY

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Práca obsahuje tabuľky kritických hodnôt pre test hypotézy $\mu + c\sigma \leqslant M$, $\mu - c\sigma \geqslant m$ o parametroch normálneho rozdelenia.

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