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OPTIMAL OSCILLATORY TIME FOR A CLASS OF SECOND ORDER NONLINEAR DISSIPATIVE ODE

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Summary. The oscillatory properties of the equation

$$\ddot{u} + g(t, \dot{u}) + f(t, u) = 0$$

are investigated. The result is applicable to some second order in time evolution eqations.

Keywords: oscillatory time, second order nonlinear ODE

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1. INTRODUCTION

Throughout the paper standard notation is used. In particular, $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$ for any $u \in \mathbf{R}$, and similarly for functions. If J is an interval in $\mathbf{R}$ with end points t_1, t_2 then $|J|$ denotes its length $t_2 - t_1$. The dot $\cdot$ stands for the derivative $\mathrm{d}/\mathrm{d}t$.

In Sec. 2 we introduce the set $\mathscr{O}$ of couples $(q, p) \in \mathbf{R}^2$ for which, roughly speaking, solutions of the inequality

$$\ddot{u} + 2p\,\dot{u}^+ + qu \leqslant 0, \qquad t \in \mathbf{R},$$

and/or

$$\ddot{u} + 2p|\dot{u}| + qu \leqslant 0, \qquad t \in \mathbf{R}$$

admit positive local maxima (the "maximum principle" is not valid on sufficiently large intervals). Then the so-called summit function ϑ is introduced which to any $(q, p) \in \mathscr{O}$ assigns the first positive point of maximum of a solution of the equation

$\ddot{u} + 2p\,\dot{u}^+ + q\,u = 0$ satisfying $u(0) = 0$, $\dot{u}(0) = c > 0$. This correspondence is independent of c. For $c = 1$ such a solution considered for $t \in [0, \vartheta(q,p)]$ is found explicitly as the restriction of the function A, an auxiliary function recalled in Sec. 3, which stems from the linearly damped oscillation theory, as expected. In Sec. 4, this solution suitably extended beyond the end point $\vartheta(q,p)$ (as the solution of the equation $\ddot{u} + 2n\,\dot{u}^- + q\,u = 0$) yields the universal comparison function C. It is this function that makes it possible to establish, in Sec. 6, the non-existence on large intervals of positive (respectively, negative) solutions of a class of nonlinear inequalities of the type

$$\ddot{u} + g(t,\dot{u}) + f(t,u) \leqslant 0 \quad (\geqslant 0),$$

with appropriate assumptions on f and g surveyed in Secs. 5 and 6. Optimal estimates of the length of such intervals are given (in terms of the function ϑ). In Sec. 7 the results are applied to the corresponding equation

$$\ddot{u} + g(t,\dot{u}) + f(t,u) = 0.$$

As a consequence we obtain in Sec. 8 a criterion for this equation to be oscillatory at $+\infty$ together with the optimal (in a sense to be specified) oscillatory time. As a special case we get results of the paper Zuazua (1990) which inspired the present investigation.

2. The summit function ϑ

Let us denote

$$\mathcal{O} = \left\{(q,p) \in \mathbf{R}^2 \mid q > 0, p > -\sqrt{q}\right\}.$$

On the region $\mathcal{O}$ we define the summit function

$$\vartheta(q,p) = \begin{cases} \dfrac{\pi}{\sqrt{q-p^2}} + \dfrac{1}{\sqrt{q-p^2}} \arctan \dfrac{\sqrt{q-p^2}}{p}, & -\sqrt{q} < p < 0, \\ \dfrac{\pi}{2\sqrt{q}}, & p = 0, \\ \dfrac{1}{\sqrt{q-p^2}} \arctan \dfrac{\sqrt{q-p^2}}{p}, & p > 0,\ p \neq \sqrt{q}, \\ \dfrac{1}{\sqrt{q}}, & p = \sqrt{q}. \end{cases}$$

Owing to the relation

$$\operatorname{arctanh} z = -\mathrm{i} \arctan \mathrm{i}z, \qquad z \in \mathbf{R},$$

where i is the imaginary unit, an equivalent expression for ϑ if $p > \sqrt{q}$ is

$$\vartheta(q,p) = \frac{1}{\sqrt{p^2 - q}} \operatorname{arctanh} \frac{\sqrt{p^2 - q}}{p}.$$

Let us mention some properties of the summit function.

- ϑ is a real positive continuous function on $\mathcal{O}$;
- $\lim_{(q,p) \to \partial \mathcal{O}} \vartheta(q,p) = +\infty$;
- for any $q > 0$, $\vartheta(q,\cdot)$ is decreasing on $(-\sqrt{q}, +\infty)$ and $\lim_{p \to +\infty} \vartheta(q,p) = 0$;
- for any $p \in \mathbf{R}$, $\vartheta(\cdot,p)$ is decreasing on $((p^-)^2, +\infty)$ and $\lim_{q \to +\infty} \vartheta(q,p) = 0$.

We shall frequently use the notation

$$\vartheta(q,p) = \vartheta_p^q.$$

3. An auxiliary function A

For $(t,q,p) \in \mathbf{R} \times \mathcal{O}$ we define

$$A(t,q,p) = \begin{cases} \dfrac{1}{\sqrt{q - p^2}} \exp(-pt) \sin(\sqrt{q - p^2}\, t), & p > -\sqrt{q},\ p \neq \sqrt{q}, \\ t \exp(-\sqrt{q}\, t), & p = \sqrt{q}. \end{cases}$$

We may alternately define

$$A(t,q,p) = \frac{1}{\sqrt{p^2 - q}} \exp(-pt) \sinh(\sqrt{p^2 - q}\, t), \quad p > \sqrt{q},$$

which is due to the well-known relation

$$\sinh z = -\mathrm{i} \sin \mathrm{i}z, \qquad z \in \mathbf{R}.$$

The function A is a real continous function on $\mathbf{R} \times \mathcal{O}$. For any $(q,p) \in \mathcal{O}$ the function

$$a(t) = A(t,q,p)$$

is the unique solution of the initial-value problem

$$\ddot{a} + 2p\dot{a} + qa = 0, \qquad t \in \mathbf{R},$$
$$a(0) = 0, \quad \dot{a}(0) = 1.$$

Moreover,

a) $a(t) > 0, \qquad t \in (0, \vartheta_p^q],$

b) $\dot{a}(t) > 0, \qquad t \in [0, \vartheta_p^q), \qquad \dot{a}(\vartheta_p^q) = 0,$

c) $a \in C^\infty(\mathbf{R}).$

4. The universal comparison function C

Let $(q,p) \in \mathcal{O}$ and $(q,n) \in \mathcal{O}$. We define

$$C(t,q,p,n) = \begin{cases} A(t,q,p), & t \in [0, \vartheta_p^q], \\ \exp(-p\vartheta_p^q + n\vartheta_n^q) A(\vartheta_p^q + \vartheta_n^q - t, q, n), & t \in (\vartheta_p^q, \vartheta_p^q + \vartheta_n^q]. \end{cases}$$

For q, p, n fixed we set

$$c(t) = C(t,q,p,n). \tag{4.1}$$

Then

$$c \in C^2[0, \vartheta_p^q + \vartheta_n^q], \tag{4.2}$$

$$\ddot{c} + 2(p\dot{c}^+ + n\dot{c}^-) + qc = 0, \qquad t \in [0, \vartheta_p^q + \vartheta_n^q], \tag{4.3}$$

$$c(0) = c(\vartheta_p^q + \vartheta_n^q) = 0, \qquad c(t) > 0, \quad t \in (0, \vartheta_p^q + \vartheta_n^q), \tag{4.4}$$

$$\begin{aligned} &\dot{c}(0) = 1, \quad \dot{c}(t) > 0, \quad t \in [0, \vartheta_p^q), \qquad \dot{c}(\vartheta_p^q) = 0, \\ &\dot{c}(t) < 0, \quad t \in (\vartheta_p^q, \vartheta_p^q + \vartheta_n^q], \quad \dot{c}(\vartheta_p^q + \vartheta_n^q) = -\exp(-p\vartheta_p^q + n\vartheta_n^q). \end{aligned} \tag{4.5}$$

Three particular cases of special interest. If $n = p$, $n = 0$, $n = -p$, then the function c provides a solution of the equation

$$\ddot{c} + 2p\,|\dot{c}| + qc = 0, \tag{4.6}$$

$$\ddot{c} + 2p\dot{c}^+ + qc = 0, \tag{4.7}$$

$$\ddot{c} + 2p\dot{c} + qc = 0, \tag{4.8}$$

respectively.

The function c is symmetric with respect to $t = \vartheta_p^q$ if and only if $n = p$. A very special case $p = n = 0$ yields

$$c(t) = \frac{1}{\sqrt{q}} \sin(\sqrt{q}\, t), \qquad t \in \left[0, \frac{\pi}{\sqrt{q}}\right],$$

a solution of the equation $\ddot{c} + qc = 0$. In general, c is not concave on $[0, \vartheta_p^q + \vartheta_n^q]$ unless simultaneously $p \geqslant 0$ and $n \geqslant 0$.

Due to the fact that Eq. (4.3) is autonomous any shift of the function c,

$$(\mathscr{S}_h\, c)(t) = c(t - h), \qquad h \in \mathbf{R},$$

satisfies the same equation.

5. Auxiliary results on nonlinear equations

Let us recall some notions and results concerning locally absolutely continuous ($W^1_{1,loc}$) solutions of systems of first order nonlinear equations

$$\dot{U} = F(t, U) \tag{5.1}$$

with the initial condition

$$U(t_0) = U^0. \tag{5.2}$$

We apply them to the equation

$$\ddot{u} + g(t, \dot{u}) + f(t, u) = 0 \tag{5.3}$$

with the initial conditions

$$u(t_0) = u_0, \quad \dot{u}(t_0) = u_1. \tag{5.4}$$

The latter equation (together with the corresponding inequalities) will be the subject of our further investigation.

Let $\tau_0 \subset \mathbf{R}$ be an open interval and $\Omega \subset \mathbf{R}^n$ a region. A function $F\colon \tau_0 \times \Omega \longrightarrow \mathbf{R}$ is said to satisfy the Carathéodory conditions if

- $F(t,\cdot)\colon \Omega \longrightarrow \mathbf{R}$ is continuous for (almost) every $t \in \tau_0$;
- $F(\cdot,U)\colon \tau_0 \longrightarrow \mathbf{R}$ is measurable for every $U \in \Omega$;
- for each compact set $G \subset \Omega$ there exists a function $M \in L_{1,loc}(\tau_0)$ such that
$$\|F(t,U)\| \leqslant M(t), \qquad U \in G,\ t \in \tau_0.$$

A function $F\colon \tau_0 \times \Omega \longrightarrow \mathbf{R}$ is said to satisfy the local Lipschitz condition with respect to U if

- for each compact set $G \subset \Omega$ there exists a function $\lambda \in L_{1,loc}(\tau_0)$ such that
$$\|F(t,U^1) - F(t,U^2)\| \leqslant \lambda(t)\|U^1 - U^2\|, \qquad U^1, U^2 \in G,\ t \in \tau_0.$$

Let $F\colon \tau_0 \times \Omega \longrightarrow \mathbf{R}^n$ satisfy the Carathéodory conditions and the local Lipschitz condition with respect to U (or have any other "uniqueness property" guaranteeing the uniqueness of the solution of the initial-value problem (5.1), (5.2)). Then for any $(t_0, U^0) \in \tau_0 \times \Omega$ there exists a unique solution $U : \tilde{\tau}_0 \longrightarrow \Omega$, $U \in W^1_{1,loc}(\tilde{\tau}_0; \Omega)$ defined for a maximal time interval $\tilde{\tau}_0 = \tilde{\tau}_0(t_0, U^0)$. This (maximal existence) interval is open and U is called the maximal solution of (5.1), (5.2). The solution is global, which means that $\tilde{\tau}_0 = \tau_0$ if, for example,

- $\Omega = \mathbf{R}^n$ and there exist functions M, $N \in L_{1,loc}(\tau_0)$ such that
$$\|F(t,U)\| \leqslant M(t)\,\|U\| + N(t), \qquad U \in \mathbf{R}^n,\ t \in \tau_0.$$

Now, let f and g be two functions

$$f\colon J_0 \times I_0 \longrightarrow \mathbf{R},$$
$$g\colon J_0 \times I_1 \longrightarrow \mathbf{R},$$

where J_0, I_0 and I_1 are open intervals in $\mathbf{R}$. Let

(5.5) f and g satisfy the Carathéodory conditions,

(5.6) f and g satisfy the local Lipschitz condition with respect to the second variable.

Before applying the above mentioned results we introduce the following notation for convenience in writing. If $J \subset \mathbf{R}$ is a compact interval we denote

$$\mathscr{A}(J) = \{u \mid u \in W^1_1(J; I_0),\ \dot{u} \in W^1_1(J; I_1)\}.$$

For any interval $J_0 \subset \mathbf{R}$ we define

$$\mathscr{A}(J_0) = \bigcap_{\substack{J \subset J_0 \\ J \text{ compact}}} \mathscr{A}(J).$$

For any $t_0 \in J_0$, $u_0 \in I_0$, $u_1 \in I_1$ there exists a unique solution $u \in \mathscr{A}(\tilde{J}_0)$ of the initial-value problem (5.3), (5.4) defined for a maximal time duration in J_0.

6. Conjugacy of inequalities

Let f and g be two functions satisfying hypotheses (5.5), (5.6) (this will be assumed tacitly throughout the rest of the paper).

We shall assume that $u \in \mathscr{A}(J)$ satisfies

$$\ddot{u} + g(t, \dot{u}) + f(t, u) \leqslant 0 \quad \text{on } J \quad (\subset J_0) \tag{6.1}$$

and

$$f(t, u(t)) \geqslant q\, u(t)^+, \qquad t \in J, \tag{6.2}$$

$$g(t, \dot{u}(t)) \geqslant 2\left(p\dot{u}(t)^+ + n\dot{u}(t)^-\right), \qquad t \in J \tag{6.3}$$

for some $q \geqslant 0$ and $p, n \in \mathbf{R}$.

To verify these assumptions in practice we introduce a convenient notation. For any couple $(p, n) \in \mathbf{R}^2$ we set

$$V_{p,n} = \left\{(x, y) \in \mathbf{R}^2 \mid y \geqslant p\,x^+ + n\,x^-\right\},$$

$$V_{p,-\infty} = \bigcup_{n \in \mathbf{R}} V_{p,n}, \qquad V_{-\infty,n} = \bigcup_{p \in \mathbf{R}} V_{p,n}.$$

The assumptions (6.2), (6.3) are fulfilled if the following uniform inclusions of graphs of functions f and g are valid:

$$\mathscr{G}(f(t, \cdot)) \subset V_{q,-\infty}, \qquad t \in J, \tag{6.4}$$

$$\mathscr{G}(g(t, \cdot)) \subset V_{2p,2n}, \qquad t \in J. \tag{6.5}$$

In other terms,

$$q = \operatorname{essinf}\left\{\frac{f(t,u)}{u} \mid t \in J,\, u \in I_0 \cap \{u \geqslant 0\}\right\} \geqslant 0,$$

$$2p = \operatorname{essinf}\left\{\frac{g(t,v)}{v} \mid t \in J,\, v \in I_1 \cap \{v \geqslant 0\}\right\} \in \mathbf{R},$$

$$2n = -\operatorname{esssup}\left\{\frac{g(t,v)}{v} \mid t \in J,\, v \in I_1 \cap \{v \leqslant 0\}\right\} \in \mathbf{R}.$$

Lemma 6.1. *Let $t_0 \in J$. If $u \geqslant 0$ on J and $u(t_0) = \dot{u}(t_0) = 0$ then $u \equiv 0$ on J.*

Proof. Denote

$$\mathscr{M} = \{t' \in J \mid u(t) = \dot{u}(t) = 0,\ t \in [t_0, t']\}.$$

The set $\mathscr{M}$ is not empty and let $\tilde{t} = \sup \mathscr{M}$. Assume that $\tilde{t}$ is less than the right end point of the interval J. By (6.2) and (6.3), $f(\tilde{t}, u(\tilde{t})) \geqslant 0$, $g(\tilde{t}, \dot{u}(\tilde{t})) \geqslant 0$ and by (6.1), $\ddot{u}(\tilde{t}) \leqslant 0$. Hence there exists a neighbourhood of $\tilde{t}$ such that the graph of u lies below or on the tangent at $\tilde{t}$. By assumption, $u \geqslant 0$, hence $u \equiv 0$ in this neighbourhood and this is a contradiction with the definition of $\tilde{t}$. A similar reasoning yields that $\inf\{t' \in J \mid u(t) = \dot{u}(t) = 0,\ t \in [t', t_0]\}$ equals to the left end point of J and the assertion follows. □

Lemma 6.2. *Let $t', t'' \in J$.*
a) If $u \geqslant 0$ on J and $\dot{u}(t') \geqslant 0$ then $\dot{u} \geqslant 0$ on $J \cap \{t \leqslant t'\}$.
b) If $u \geqslant 0$ on J and $\dot{u}(t'') \leqslant 0$ then $\dot{u} \leqslant 0$ on $J \cap \{t \geqslant t''\}$.

Proof. In view of (6.2), (6.3) we have by (6.1)

$$\ddot{u} + g(t, \dot{u}) \leqslant 0 \qquad \text{on } J. \tag{6.6}$$

Multiplying (6.6) by $\dot{u}^-$ (an absolutely continuous function) and using (6.3) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|\dot{u}^-|^2 \geqslant 4n\,|\dot{u}^-|^2 \qquad \text{on } J.$$

Thus,

$$|\dot{u}^-(t)|^2 \leqslant |\dot{u}^-(t')|^2 \exp(-4n(t'-t)), \qquad t \in J \cap \{t \leqslant t'\}.$$

Since $\dot{u}^-(t') = 0$ by assumption we get $\dot{u}^- \equiv 0$ on $J \cap \{t \leqslant t'\}$ and the proof of a) is complete.

Multiplying (6.6) by $\dot{u}^+$ we obtain in a similar way

$$\frac{\mathrm{d}}{\mathrm{d}t}|\dot{u}^+|^2 \leqslant 4p\,|\dot{u}^+|^2 \qquad \text{on } J$$

and

$$|\dot{u}^+(t)|^2 \leqslant |\dot{u}^+(t'')|^2 \exp(-4p(t-t'')), \qquad t \in J \cap \{t \geqslant t''\}.$$

Hence, $\dot{u}^+ \equiv 0$ on $J \cap \{t \geqslant t''\}$ and the proof of b) follows. □

Remark 6.1. The function u cannot attain a non-negative minimum at an interior point of the interval J unless $u \equiv 0$ (or $u \equiv M$, M an arbitrary non-negative constant, if $q = 0$).

Now, let us assume more specifically

$$(q,p) \in \mathcal{O}, \quad (q,n) \in \mathcal{O}.$$

Lemma 6.3. a) *If for some $t_0 \in J$, $J^+ = [t_0 - \vartheta_p^q, t_0] \subset J$ and $u(t_0) > 0$, $\dot{u}(t_0) \geqslant 0$ then there exists $t^* \in [t_0 - \vartheta_p^q, t_0)$ such that*

$$u(t^*) = 0, \quad \dot{u}(t^*) > 0.$$

b) If for some $t_0 \in J$, $J^- = [t_0, t_0 + \vartheta_n^q] \subset J$ and $u(t_0) > 0$, $\dot{u}(t_0) \leqslant 0$ then there exists $t^{**} \in (t_0, t_0 + \vartheta_n^q]$ such that

$$u(t^{**}) = 0, \quad \dot{u}(t^{**}) < 0.$$

Proof of a). Let us denote $\mathcal{M} = \{t \in J^+ \mid u(t) \leqslant 0\}$. We prove, by contradiction, that the set $\mathcal{M}$ is not empty. So, let $u > 0$ on J^+. By Lemma 6.2 we know that $\dot{u} \geqslant 0$ on J^+. Now, we shall define the comparison function γ as a suitable shift of the universal comparison function c given by (4.1), namely,

$$\gamma = \mathcal{S}_{t_0 - \vartheta_p^q}\, c.$$

For our purposes it is enough to consider this function only for $t \in J^+$. The function γ satisfies Eq. (4.3) and has the following properties:

$$\gamma(t_0 - \vartheta_p^q) = 0, \qquad \gamma(t) > 0, \quad t \in (t_0 - \vartheta_p^q, t_0],$$
$$\dot{\gamma}(t) > 0, \quad t \in [t_0 - \vartheta_p^q, t_0), \qquad \dot{\gamma}(t_0) = 0.$$

Thus, taking into account (6.2), (6.3) we arrive at the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot\gamma u - \gamma\dot u) \geqslant -2p(\dot\gamma u - \gamma \dot u) \qquad \text{on } J^+.$$

Consequently,

$$(\dot\gamma u - \gamma\dot u)(t) \leqslant (\dot\gamma u - \gamma\dot u)(t_0)\exp(2p(t_0 - t)) \leqslant 0, \qquad t \in J^+.$$

In particular, for $t = t_0 - \vartheta_p^q$ we get

$$\dot\gamma(t_0 - \vartheta_p^q)\, u(t_0 - \vartheta_p^q) \leqslant 0,$$

a contradiction. Let us define $t^* = \sup \mathscr{M}$. Clearly, $u(t^*) = 0$ and $\dot u(t^*) \geqslant 0$. The case $\dot u(t^*) = 0$ is excluded by Lemma 6.1. The proof of a) is complete.

To prove b) we use again the properties of the function γ considered now on the interval J^-:

$$\gamma(t) > 0, \quad t \in [t_0, t_0 + \vartheta_n^q), \qquad \gamma(t_0 + \vartheta_n^q) = 0,$$
$$\dot\gamma(t_0) = 0, \qquad \dot\gamma(t) < 0, \quad t \in (t_0, t_0 + \vartheta_n^q].$$

We arrive analogously at the conclusion $t^{**} = \inf\{t \mid t \in J^-,\ u(t) \leqslant 0\}$. □

Theorem 6.1. *Let*

- $(q, p) \in \mathscr{O}$, $(q, n) \in \mathscr{O}$,
  $\mathscr{G}(f(t,\cdot)) \subset V_{q,-\infty}$, $\mathscr{G}(g(t,\cdot)) \subset V_{2p,2n}$, $\quad t \in J$,
- $u \in \mathscr{A}(J)$, $\ddot u + g(t, \dot u) + f(t, u) \leqslant 0$ *on* J,
- $u \geqslant 0$ *on* J.

Then

$$\textit{either} \qquad |J| \leqslant \vartheta_p^q + \vartheta_n^q$$
$$\textit{or} \qquad u \equiv 0 \quad \textit{on } J.$$

Proof. Let $|J| > \vartheta_p^q + \vartheta_n^q$. Let $J \supset [t_1, t_2]$, $t_2 - t_1 > \vartheta_p^q + \vartheta_n^q$ and

$$\mathscr{M} = \{t' \in [t_1, t_2] \mid \dot u(t) \leqslant 0,\ t \in [t', t_2]\}.$$

By Lemmas 6.3 a) and 6.2 the set $\mathscr{M}$ is not empty and $\inf \mathscr{M} \leqslant t_1 + \vartheta_p^q$. Due to the fact that $u \geqslant 0$ we have $u(t_1 + \vartheta_p^q) = 0$ and consequently $\dot u(t_1 + \vartheta_p^q) = 0$. By Lemma 6.1, $u \equiv 0$ on J. □

Remark 6.2. The inequality (6.1) is said to be conditionally conjugate in an interval J if for every $u \in \mathscr{A}(J)$ satisfying (6.1) and $u \geqslant 0$ there exist t^*, $t^{**} \in J$, $t^* \neq t^{**}$ such that $u(t^*) = u(t^{**}) = 0$. Under the assumptions of Theorem 6.1 the inequality (6.1) is conditionally conjugate on every interval ($\subset J_0$) the length of which is greater than $\vartheta_p^q + \vartheta_n^q$. This number is optimal for the considered class of inequalities in the sense that on any interval of length less than or equal to $\vartheta_p^q + \vartheta_n^q$ we can always find functions f and g obeying the assumptions of the theorem and a non-trivial solution of the corresponding inequality which is non-negative on this interval. In fact, the functions f and g can be chosen in the form

$$f(t,u) = qu, \qquad g(t,v) = 2(pv^+ + nv^-), \tag{6.7}$$

as the following theorem shows.

Theorem 6.2. *Let* $(q,p) \in \mathscr{O}$, $(q,n) \in \mathscr{O}$, $u \in \mathscr{A}(J)$. *Then the statement*

$$u \geqslant 0 \text{ on } J,\ \ddot{u} + 2(p\dot{u}^+ + n\dot{u}^-) + qu \leqslant 0 \text{ on } J \Longrightarrow u \equiv 0 \text{ on } J$$

holds true if and only if

$$|J| > \vartheta_p^q + \vartheta_n^q.$$

Proof. In view of Theorem 6.1 it suffices to show that the implication is not valid if $|J| \leqslant \vartheta_p^q + \vartheta_n^q$. To this end, we take an appropriate shift of the universal comparison function (4.1) which represents a non-trivial non-negative solution on the interval $[0, \vartheta_p^q + \vartheta_n^q]$. □

Analogous lemmas and theorems can be proved for the reversed inequality and its non-positive solutions. For example, putting $z = -u$, $G(t,w) = -g(t,-w)$, $F(t,z) = -f(t,-z)$ we can use the results of Theorem 6.1 to obtain

Theorem 6.3. *Let*

- $(-Q,-P) \in \mathscr{O}$, $(-Q,-N) \in \mathscr{O}$,
  $\mathscr{G}(-f(t,\cdot)) \subset V_{-\infty,-Q}$, $\quad \mathscr{G}(-g(t,\cdot)) \subset V_{-2P,-2N}$, $\qquad t \in J$,
- $u \in \mathscr{A}(J)$, $\quad \ddot{u} + g(t,\dot{u}) + f(t,u) \geqslant 0$ *on* J,
- $u \leqslant 0$ *on* J.

Then

$$\text{either} \quad |J| \leqslant \vartheta(-Q,-P) + \vartheta(-Q,-N)$$
$$\text{or} \quad u \equiv 0 \quad \text{on } J.$$

Proof. Apply Theorem 6.1 to the inequality

$$\ddot{z} + G(t, \dot{z}) + F(t, z) \leqslant 0$$

with $p = -N,\ n = -P,\ q = -Q$. □

7. Conjugacy of equations

Combining Theorems 6.1 and 6.3 we get results on the equation

$$\ddot{u} + g(t, \dot{u}) + f(t, u) = 0. \tag{7.1}$$

Theorem 7.1. *Let us assume*

$$(q, p), (q, n), (-Q, -P), (-Q, -N) \in \mathcal{O}, \tag{7.2}$$

$$\begin{aligned}
&(7.3_1) \quad & qu &\leqslant f(t, u), & & t \in J,\ u \in I_0 \cap \{u \geqslant 0\},\\
&(7.3_2) \quad & f(t, u) &\leqslant -Q\,u, & & t \in J,\ u \in I_0 \cap \{u \leqslant 0\},\\
&(7.3_3) \quad & 2(p\,v^+ + n\,v^-) \leqslant g(t, v) &\leqslant 2(P\,v^+ + N\,v^-), & & t \in J,\ v \in I_1,
\end{aligned}$$

$$u \in \mathscr{A}(J) \text{ satisfies Eq. (7.1) on } J, \tag{7.4}$$

$$u \geqslant 0 \quad (\text{or } u \leqslant 0) \quad \text{on } J. \tag{7.5}$$

Then

either $|J| \leqslant \vartheta(q, p) + \vartheta(q, n)$ *(or* $|J| \leqslant \vartheta(-Q, -P) + \vartheta(-Q, -N)$*)*
or $u \equiv 0$ *on* J.

Special cases (cf. Zuazua (1990)). If

- $u\,f(t, u) \geqslant q\,u^2, \quad t \in J,\ u \in I_0,$
- $|g(t, v)| \leqslant 2P\,|v|, \quad t \in J,\ v \in I_1,$

then $p = n = -P = -N$, $q = -Q$ *with* $P < \sqrt{q}$. *If, moreover,* g *satisfies the sign condition*

$$vg(t,v) \geqslant 0, \qquad t \in J,\ v \in I_1,$$

then $p \geqslant 0$, $N \leqslant 0$, $n = -P$ *and we can choose, in general, at least* $p = 0$, $N = 0$. *In particular cases a better choice (that is, leading to smaller values of* $\vartheta_p^q + \vartheta_n^q$ *and* $\vartheta(-Q,-P) + \vartheta(-Q,-N)$*) may be possible. For example, for the nonlinearity*

$$g(t,v) = 2(d + \varepsilon \sin v)v, \qquad d > 0, \quad |\varepsilon| < d \qquad (v \in \mathbf{R})$$

the best choice is $P = -n = d + \varepsilon$ *and* $p = -N = d - \varepsilon$.

Remark 7.1. The equation (7.1) is called conditionally conjugate on J if any non-trivial solution $u \in \mathscr{A}(J)$ vanishes at two distinct points of J. Under the assumptions of Theorem 7.1 Eq. (7.1) is conditionally conjugate on every interval the length of which is greater than $\max\{\vartheta(q,p)+\vartheta(q,n),\ \vartheta(-Q,-P)+\vartheta(-Q,-N)\}$.

8. Oscillatory properties of equations

Let

$$J_0 = [a, +\infty) \text{ for some } a \in \mathbf{R}.$$

If $u \in \mathscr{A}(J_0)$ is a solution of Eq. (7.1) then Theorem 7.1 yields that Eq. (7.1) is conditionally conjugate on every interval $[c, +\infty]$ with $c > a$. This is usually expressed in terms of the oscillation theory.

A measurable function $u\colon J_0 \to \mathbf{R}$ is called *oscillatory* (at $+\infty$) if there exists (the so-called *oscillatory time*) $\Theta > 0$ such that

$$u \geqslant 0 \quad (u \leqslant 0) \quad \text{on} \quad J \subset J_0 \Longrightarrow \begin{cases} \text{either} & u \equiv 0 \quad \text{on} \quad J_0 \\ \text{or} & |J| \leqslant \Theta. \end{cases}$$

In other words, if u is non-trivial on J_0, $J \subset J_0$, $|J| > \Theta$, then u changes the sign on J, more precisely, $\operatorname{meas}\{t \mid t \in J,\ u(t) > 0\} > 0$ and $\operatorname{meas}\{t \mid t \in J,\ u(t) < 0\} > 0$.

Theorem 8.1. *Let* $J_0 = [a, +\infty)$ *for some* $a \in \mathbf{R}$. *Let the hypotheses* (7.2) *and* (7.3) *be fulfilled with* $J = J_0$. *Then any solution* $u \in \mathscr{A}(J_0)$ *of Eq.* (7.1) *is oscillatory and*

$$\Theta = \max\{\vartheta(q,p) + \vartheta(q,n),\ \vartheta(-Q,-P) + \vartheta(-Q,-N)\}.$$

As a consequence, we state the final result concerning the equation

$$\ddot{u} + 2(p\dot{u}^+ + n\dot{u}^-) + qu = 0. \tag{8.1}$$

Theorem 8.2. *Let $(q,p),(q,n) \in \mathcal{O}$. Then any solution $u \in \mathscr{A}(\mathbf{R})$ of Eq. (8.1) is oscillatory and*

$$\Theta = \vartheta_p^q + \vartheta_n^q.$$

Remark 8.1. The oscillatory time Θ in the above theorems is optimal in the sense that for any $\Theta_1 < \Theta$ there exists an interval $J \subset J_0$, $|J| \geqslant \Theta_1$ and a solution of Eq. (7.1) with suitable f and g, for example of the form (6.7), that does not change the sign on J.

In the end, we specify the results for each of the particular equations (4.6) through (4.8). Namely, any solution $u \in \mathscr{A}(\mathbf{R})$ of the equation

$$\ddot{u} + 2p\beta(\dot{u})\dot{u} + qu = 0, \tag{8.2}$$

where $(q,p) \in \mathcal{O}$, $\beta =$ sgn function or $\beta =$ Heaviside function, is oscillatory and the oscillatory time is

$$\Theta = 2\vartheta_p^q$$

and

$$\Theta = \vartheta_p^q + \frac{\pi}{2\sqrt{q}},$$

respectively. A well-known result from the linear oscillation theory is obtained for $\beta \equiv 1$: if $|p| < q$ then any solution is oscillatory and the oscillatory time is

$$\Theta = \frac{\pi}{\sqrt{q-p^2}}.$$

References

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