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# WEIGHT MINIMIZATION OF ELASTIC BODIES WEAKLY SUPPORTING TENSION
# II. DOMAINS WITH TWO CURVED SIDES

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*Summary.* Extending the results of the previous paper [1], the authors consider elastic bodies with two design variables, i.e. "curved trapezoids" with two curved variable sides. The left side is loaded by a hydrostatic pressure. Approximations of the boundary are defined by cubic Hermite splines and piecewise linear finite elements are used for the displacements. Both existence and some convergence analysis is presented for approximate penalized optimal design problems.

*Keywords*: shape optimization, weight minimization, penalty method, masonry-like materials, finite elements

*AMS classification*: 65N30, 65K10, 73C99

## INTRODUCTION

The present paper is a continuation of the previous article [1], where the shape of the cross-section has been restricted to "trapezoids" with one curved side. The aim of the present paper is to study similar problems for domains with two curved sides. The left curved side is supposed to be loaded by a hydrostatic pressure and therefore—in contrast with the part [1]—the existence and convergence analysis now requires to introduce $C^1$-metric for the design variables, which themselves are to be smooth enough. Consequently, we approximate the boundary by cubic Hermite splines. On the other hand, we keep the simplest possible finite elements, i.e., the piecewise linear displacement model on triangulations, which are completed by triangles with one curved (cubic) side along the boundary of the cross-section. Thus we obtain piecewise constant stress field approximations, which simplify the penalty computation, as in [1].

Section 1 contains some definitions and auxiliary results. The existence of an optimal domain is proved in Section 2. Approximate problem is introduced and analyzed in Section 3. We present some sensibility analysis by means of an adjoint problem technique in Section 4. The last section contains a convergence analysis based on an intermediate penalized optimal design problem.

## 1. Assumptions and definitions

Throughout the paper we shall use the same notation as in [1]. Consider a class of domains $\{\Omega(v)\}$, where the design variable $v = (v_1, v_2)$ belongs to the following set

$$U_{\text{ad}} = U^1_{\text{ad}} \times U^2_{\text{ad}},$$
$$U^1_{\text{ad}} = \{v \in C^{(1),1}(I) \mid \alpha \leqslant v(x_2) \leqslant \beta, |v'| \leqslant C_1 \text{ in } I, |v''| \leqslant C_2 \text{ a.e. in } I\},$$
$$U^2_{\text{ad}} = \{v \in C^{(1),1}(I) \mid 0 \leqslant v(x_2) \leqslant \gamma, v(1) = 0, |v'| \leqslant \tilde{C}_1 \text{ in } I, |v''| \leqslant \tilde{C}_2 \text{ a.e. in } I\},$$

where

$$I = [0, 1], \quad v' = \frac{\mathrm{d}v}{\mathrm{d}x_2}, \quad v'' = \frac{\mathrm{d}^2 v}{\mathrm{d}x_2^2},$$
$$\Omega(v) = \{(x_1, x_2) \mid -v_2(x_2) < x_1 < v_1(x_2),\ 0 < x_2 < 1\},$$

$\alpha$, $\beta$, $\gamma$, $C_1$, $C_2$, $\tilde{C}_1$, $\tilde{C}_2$ are given positive constants, $\alpha < \beta$. Henceforth $\Gamma(v_i)$ denotes the graph of the function $v_i \in U^i_{\text{ad}}$, $(i = 1, 2)$, $\Omega_\delta = (-\delta, \delta) \times (0, 1)$, $\delta > \max\{\beta, \gamma\}$.

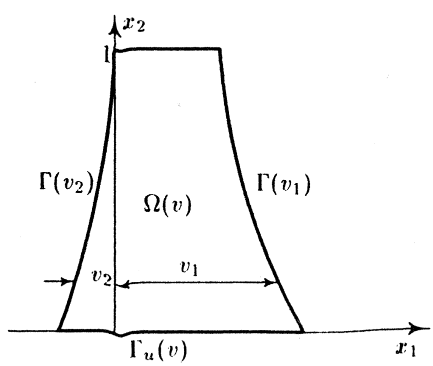


Fig. 1

Assume that the elastic body occupying the domain $\Omega(v)$ is in the state of the plane strain and the following basic relations hold:
strain-displacement

$$e_{ij}(u) = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad i,j = 1,2,$$

stress-strain

$$\sigma_{ij} = c_{ijml} e_{ml}$$

(any repeated index implies the summation over 1,2), where the coefficients $c_{ijml}$ are given in $L^\infty(\Omega_\delta)$, being symmetric and positive definite (see [1]-(1)), as usual.

Let body forces $F \in [L^2(\Omega_\delta)]^2$ and the following hydrostatic surface pressure $g \in [L^2(\Gamma(v_2))]^2$ be given

$$g = (g_1, g_2) = (H - x_2)\,(1 + (v_2')^2)^{-\frac{1}{2}}(1, v_2') \quad \text{for } x_2 \in (0, H),$$
$$g = 0 \quad \text{for } x_2 \in (H, 1) \quad \text{with some } H \in (0,1).$$

We introduce the space of virtual displacements

$$V(v) = \{w \in [H^1(\Omega(v))]^2 \mid w = 0 \text{ on } \Gamma_u(v)\},$$

where

$$\Gamma_u(v) = \{(x_1, 0) \mid -v_2(0) < x_1 < v_1(0)\}$$

and the virtual work of external forces

$$\mathcal{F}(v;\, w) = \int_{\Omega(v)} F_i w_i \,\mathrm{d}x + \int_0^H (H - x_2)(w_1 + w_2 v_2')\big|_{x_1 = -v_2(x_2)} \,\mathrm{d}x_2.$$

The weak solution of the elastostatic problem is defined by the formula (2) of [1].

Before proving the existence and uniqueness of the weak solution, we establish the following auxiliary result.

**Lemma 1.1.** *There exists a constant $C > 0$ independent of $v$ and such that*

$$\|u\|_{0,\Gamma(v_2)} = \left(\int_0^1 (u(-v_2(x_2), x_2))^2 \,\mathrm{d}x_2\right)^{\frac{1}{2}} \leqslant C\|u\|_{1,\Omega(v)}$$

*holds for all $u \in H^1(\Omega(v))$ and all $v \in U^0_{\mathrm{ad}}$, where*

$$U^0_{\mathrm{ad}} = U^{01}_{\mathrm{ad}} \times U^{02}_{\mathrm{ad}},$$
$$U^{01}_{\mathrm{ad}} = \{v \in C^{(1),1}(I) \mid \frac{\alpha}{2} \leqslant v(x_2) \leqslant 2\beta, |v'| \leqslant 2C_1, |v''| \leqslant C_2 \text{ a.e.}\},$$
$$U^{02}_{\mathrm{ad}} = \{v \in C^{(1),1}(I) \mid -\frac{\alpha}{4} \leqslant v(x_2) \leqslant 2\gamma, |v'| \leqslant 2\tilde{C}_1, |v''| \leqslant \tilde{C}_2 \text{ a.e.}\}.$$

Proof. Assume that $u \in C^\infty(\overline{\Omega(v)})$ for the time being. Then we may write

$$u(-v_2(x_2), x_2) = u\left(\frac{\alpha}{4}, x_2\right) - \int\limits_{-v_2(x_2)}^{\frac{\alpha}{4}} \frac{\partial u}{\partial x_1}\, \mathrm{d}x_1,$$

$$u^2(-v_2(x_2), x_2) \leqslant 2u^2\left(\frac{\alpha}{4}, x_2\right) + 2\left(2\gamma + \frac{\alpha}{4}\right) \int\limits_{-v_2(x_2)}^{\frac{\alpha}{4}} \left(\frac{\partial u}{\partial x_1}\right)^2 \mathrm{d}x_1,$$

$$\int_0^1 u^2(-v_2(x_2), x_2)\, \mathrm{d}x_2 \leqslant 2\int_0^1 u^2\left(\frac{\alpha}{4}, x_2\right) \mathrm{d}x_2 + 2\left(2\gamma + \frac{\alpha}{4}\right) \int\limits_{\Omega_0(v_2)} \left(\frac{\partial u}{\partial x_1}\right)^2 \mathrm{d}x,$$

where $\Omega_0(v_2)$ is the domain between $\Gamma(v_2)$ and $\Gamma(-\frac{\alpha}{4})$.

Denoting $\mathcal{R} = (\frac{\alpha}{4}, \frac{\alpha}{2}) \times (0,1)$, we have $\mathcal{R} \subset \Omega(v)$ for all $v \in U_{\mathrm{ad}}^0$, so that

$$\|u\|^2_{0,\Gamma(-\frac{\alpha}{4})} \leqslant C_{\mathcal{R}} \|u\|^2_{1,\mathcal{R}} \leqslant C_{\mathcal{R}} \|u\|^2_{1,\Omega(v)}$$

with $C_{\mathcal{R}}$ independent of $v$. Therefore, we may write

$$\begin{aligned} \|u\|^2_{0,\Gamma(v_2)} &\leqslant 2C_{\mathcal{R}} \|u\|^2_{1,\Omega(v)} + 2\left(2\gamma + \frac{\alpha}{4}\right) \int\limits_{\Omega(v)} \left(\frac{\partial u}{\partial x_1}\right)^2 \mathrm{d}x \\ &\leqslant 2\left(C_{\mathcal{R}} + 2\gamma + \frac{\alpha}{4}\right) \|u\|^2_{1,\Omega(v)}. \end{aligned}$$

Making use of the density of $C^\infty(\overline{\Omega(v)})$ in $H^1(\Omega(v))$ and the Trace Theorem, we extend the inequality to all $u \in H^1(\Omega(v))$. □

**Lemma 1.2.** *There exists a unique weak solution $u(v)$ for any $v \in U_{\mathrm{ad}}^0$. Moreover, a constant $C_5$ exists, independent of $v$ and such that*

$$\|u(v)\|_{1,\Omega(v)} \leqslant C_5 \quad \forall v \in U_{\mathrm{ad}}^0. \tag{1}$$

Proof. We may divide $\Omega(v)$ into two parts by a straight-line $x_1 = \frac{\alpha}{3}$ and apply the results of the paper [4] (Section 2.2(i)) to both parts separately to get the uniform Korn's inequality

$$\|e(w)\|_{0,\Omega(v)} \geqslant C\|w\|_{1,\Omega(v)} \tag{2}$$

with some positive constant $C$ independent of $v$.

Next we derive the following estimate for any $v \in U_{\text{ad}}^0$

$$(3) \qquad \begin{aligned} |\mathcal{F}(v;\, w)| &\leqslant \|F\|_{0,\Omega_\delta} \|w\|_{0,\Omega(v)} + H \int_0^1 (|w_1| + 2C_1|w_2|)\, \mathrm{d}x_2 \\ &\leqslant \|F\|_{0,\Omega_\delta} \|w\|_{0,\Omega(v)} + C\|w_1\|_{1,\Omega(v)} + 2C\tilde{C}_1\|w_2\|_{1,\Omega(v)} \\ &\leqslant C_4 \|w\|_{1,\Omega(v)} \end{aligned}$$

employing Lemma 1.1 twice.

Using the positive definiteness of the form $c_{ijml}e_{ij}e_{ml}$, (6) of [1], (2) and (3), we prove the unique solvability and the estimate (1) in a way parallel to that of proof of Lemma 1.1 in [1].

We introduce the set of statically admissible design variables

$$\mathcal{E}_{\text{ad}} = \{v \in U_{\text{ad}} \mid \sigma(v) \in M(\Omega(v))\},$$

where the set $M(\Omega(v))$ and $\sigma(v)$ have been defined in [1]-Sect. 1. Then we define the following Optimal Design Problem

$$(4) \qquad v_0 = \arg\min_{v \in \mathcal{E}_{\text{ad}}} j(v),$$

where

$$j(v) = \int_{\Omega(v)} p(x)\, \mathrm{d}x, \quad p \in L^2(\Omega_\delta), \quad p > 0.$$

□

## 2. Existence of an optimal domain

We shall follow the main features of the argument of Section 2 in [1]. First we establish an important

**Proposition 2.1.** *Let $\{v_n\}, n \to \infty$ be a sequence of couples $v_n \in U_{\text{ad}}^0$ such that*

$$(5) \qquad v_n \to v \text{ in } [C^{(1)}(I)]^2.$$

*Then*

$$\tilde{\sigma}(v_n) \to \tilde{\sigma}(v) \text{ in } S(\Omega_\delta),$$

*where $\tilde{\sigma}(v_n)$ and $\tilde{\sigma}(v)$ denote the extensions by zero to the domain $\Omega_\delta - \Omega(v_n)$ and $\Omega_\delta - \Omega(v)$, respectively.*

In the proof the basic role is played by the following

**Lemma 2.1.** *Let the assumptions of the Proposition 2.1 be fulfilled. Then*

$$u(v_n)\big|_{G_m} \rightharpoonup u(v)\big|_{G_m} \text{ (weakly) in } [H^1(G_m)]^2$$

*holds for $n \to \infty$ and any integer $m > \frac{8}{\alpha}$, where*

$$G_m = \{x \big| -v_2(x_2) + \frac{1}{m} < x_1 < v_1(x_2) - \frac{1}{m},\ 0 < x_2 < 1\}.$$

P r o o f is an easy modification of the proof of Lemma 2.1 in [1]. We employ the extension $\hat{u}(v_n)$ of $u(v_n)$ symmetric with respect to both $\Gamma(v_{n1})$ and $\Gamma(v_{n2})$. The only difference occurs when proving that (cf. [1]–(27)) if $v_{r2} \to v_2$ in $C^1(I)$, then

$$\lim_{r\to\infty} \mathcal{F}(v_r\,;\,\hat{w}) = \mathcal{F}(v\,;\,w). \tag{6}$$

Here we may write, denoting $\Gamma_r \equiv \Gamma(v_{r2})$, $\Gamma \equiv \Gamma(v_2)$,

$$\begin{aligned}
|\mathcal{F}(v_r\,;\,\hat{w}) - \mathcal{F}(v\,;\,w)| &\leqslant \Big|\int_{\Omega(v_r)} F_i \hat{w}_i \,\mathrm{d}x - \int_{\Omega(v)} F_i w_i \,\mathrm{d}x\Big| \\
&+ \Big|\int_0^H (H - x_2)(\hat{w}_1 + \hat{w}_2 v'_{r2})\big|_{\Gamma_r} \mathrm{d}x_2 - \int_0^H (H - x_2)(w_1 + w_2 v'_2)\big|_{\Gamma} \mathrm{d}x_2\Big| \\
&\leqslant \int_{\Delta(\Omega(v_r),\Omega(v))} |F_i \hat{w}_i| \,\mathrm{d}x + \int_0^H |\hat{w}_1\big|_{\Gamma_r} - w_1\big|_{\Gamma}| \,\mathrm{d}x_2 \\
&+ \int_0^H |v'_{r2}\hat{w}_2\big|_{\Gamma_r} - v'_2 w_2\big|_{\Gamma}| \,\mathrm{d}x_2 \equiv A_r + B_r + C_r\,;
\end{aligned}$$

$$\lim_{r\to\infty} A_r = 0$$

by virtue of the uniform convergence of $v_{r2}$ to $v_2$. Furthermore, we may write

$$B_r \leqslant \int_0^H \mathrm{d}x_2 \int\limits_{\min(-v_{r2},-v_2)}^{\max(-v_{r2},-v_2)} \left|\frac{\partial \hat{w}_1}{\partial x_1}\right| \mathrm{d}x_1 \to 0,$$

$$C_r \leqslant \int_0^H |v'_{r2}(\hat{w}_2\big|_{\Gamma_r} - w_2\big|_{\Gamma})| \,\mathrm{d}x_2 + \int_0^H |(v'_{r2} - v'_2) w_2\big|_{\Gamma}| \,\mathrm{d}x_2 \equiv C_{r1} + C_{r2},$$

$$C_{r1} \leqslant 2\tilde{C}_1 \int_0^H \mathrm{d}x_2 \int\limits_{\min(-v_{r2},-v_2)}^{\max(-v_{r2},-v_2)} \left|\frac{\partial \hat{w}_2}{\partial x_1}\right| \mathrm{d}x_1 \to 0,$$

$$C_{r2} \leqslant \|v'_{r2} - v'_2\|_{C(I)} \int_0^H |w_2\big|_{\Gamma}| \,\mathrm{d}x_2 \to 0$$

by virtue of the uniform convergence of the derivatives $v'_{r2}$ to $v'_2$. Combining the above results, we arrive at (6). □

Proof of Proposition 2.1. On the basis of Lemma 2.1 we derive that

$$\tilde{\sigma}_n \rightharpoonup \tilde{\sigma} \text{ (weakly) in } S(\Omega_\delta), \tag{7}$$

where $\sigma_n = \sigma(v_n)$, $\sigma = \sigma(v)$ (cf. the derivation of (29) in [1]). Then we have to show that

$$\lim_{n\to\infty} \mathcal{F}(v_n\,;\,u_n) = \mathcal{F}(v\,;\,u). \tag{8}$$

To this end, we may write (denoting $\Omega_n = \Omega(v_n)$, $\Omega = \Omega(v)$, $\Gamma_n = \Gamma(v_{n2})$, $\Gamma = \Gamma(v_2)$),

$$\begin{aligned}
(9)\quad |\mathcal{F}(v_n\,;\,u_n) - \mathcal{F}(v\,;\,u)| &\leqslant \Big|\int_{\Omega_n} F\cdot u_n\,\mathrm{d}x - \int_\Omega F\cdot u\,\mathrm{d}x\Big| \\
&+ \Big|\int_0^H (H-x_2)(u_{n1}|_{\Gamma_n} - u_1|_\Gamma + v'_{n2}u_{n2}|_{\Gamma_n} - v'_2u_2|_\Gamma)|\,\mathrm{d}x_2\Big| = A+B,
\end{aligned}$$

$$\begin{aligned}
A &\leqslant \Big|\int_{G_m} F\cdot(u_n-u)\,\mathrm{d}x\Big| + \Big|\int_{\Omega_n - G_m} F\cdot u_n\,\mathrm{d}x\Big| + \Big|\int_{\Omega - G_m} F\cdot u\,\mathrm{d}x\Big| \\
&= A_1 + A_2 + A_3,
\end{aligned}$$

$A_1 \to 0$ as $n\to\infty$ by virtue of Lemma 2.1 ;

$A_2 \leqslant \|F\|_{0,\Omega_n - G_m}\, C_5 \to 0$

by virtue of (1) and the limit

$$\operatorname{meas}(\Omega_n - G_m) \to 0 \text{ as } n\to\infty, m\to\infty, n > n_0(m);$$

$$A_3 \to 0 \text{ as } m\to\infty.$$

Combining these results, we deduce that

$$A \to 0 \text{ as } n\to\infty. \tag{10}$$

An estimate of the term $B$ is more difficult. We may write

$$\begin{aligned}
(11)\quad B &\leqslant \int_0^1 |u_{n1}|_{\Gamma_n} - u_1|_\Gamma|\,\mathrm{d}x_2 + \int_0^1 |v'_{n2}u_{n2}|_{\Gamma_n} - v'_2u_2|_\Gamma|\,\mathrm{d}x_2 = B_1 + B_2; \\
B_1 &\leqslant \int_0^1 |u_{n1}|_{\Gamma_n} - \hat{u}_{n1}|_\Gamma|\,\mathrm{d}x_2 + \int_0^1 |\hat{u}_{n1}|_\Gamma - u_1|_\Gamma|\,\mathrm{d}x_2 = B_{11} + B_{12},
\end{aligned}$$

where $\hat{u}_{n1}$ denotes the extension of $u_{n1}$ symmetric with respect to $\Gamma_n$,

$$\begin{aligned}
B_{11} &\leqslant \int_0^1 \mathrm{d}x_2 \int_{\min(\Gamma_n,\Gamma)}^{\max(\Gamma_n,\Gamma)} \Big|\frac{\partial \hat{u}_{n1}}{\partial x_1}\Big|\,\mathrm{d}x_1 = \int_{\Omega(\Gamma_n,\Gamma)} \Big|\frac{\partial u_{n1}}{\partial x_1}\Big|\,\mathrm{d}x \\
&\leqslant \big(\operatorname{meas}\Omega(\Gamma_n,\Gamma)\big)^{\frac12}\|\hat{u}_{n1}\|_{1,\Omega_\alpha},
\end{aligned}$$

where

$$\min(\Gamma_n, \Gamma) = \min(-v_{n2}(x_2), -v_2(x_2)),$$
$$\max(\Gamma_n, \Gamma) = \max(-v_{n2}(x_2), -v_2(x_2)),$$

$\Omega(\Gamma_n, \Gamma)$ denotes the set "between" $\Gamma_n$ and $\Gamma$,

$$\Omega_\alpha = \Omega\left(-v_2 - \frac{\alpha}{4}, v_1 + \frac{\alpha}{4}\right).$$

Recall that (cf. (22) in [1])

(12) $$\|\hat{u}_{nj}\|_{1,\Omega_\alpha} \leqslant C \quad \forall n > n_0(\alpha),\ j = 1, 2,$$

and

(13) $$\hat{u}_n\big|_\Omega \rightharpoonup u \text{ (weakly) in } [H^1(\Omega)]^2$$

can be derived as in the proof of Lemma 2.1 in [1].

Using (12) and the uniform convergence of $v_{n2}$ to $v_2$, we obtain

$$B_{11} \to 0 \text{ as } n \to \infty.$$

Next we have

$$B_{12} \leqslant \|u_{n1} - u_1\|_{0,\Gamma} \to 0,$$

since the trace operator $T\colon H^1(\Omega) \to L^2(\Gamma)$ is compact and (13) holds.

We also have

$$B_2 \leqslant \int_0^1 |v'_{n2} u_{n2}\big|_{\Gamma_n} - v'_2 u_{n2}\big|_{\Gamma_n}| \, dx_2 + \int_0^1 |v'_2(u_{n2}\big|_{\Gamma_n} - \hat{u}_{n2}\big|_\Gamma)| \, dx_2$$
$$+ \int_0^1 |v'_2(u_{n2}\big|_\Gamma - u_2\big|_\Gamma)| \, dx_2 = B_{21} + B_{22} + B_{23};$$
$$B_{21} \leqslant \|v'_{n2} - v'_2\|_{C(I)} \|u_{n2}\|_{0,\Gamma_n} \to 0,$$

since

$$\|u_{n2}\|_{0,\Gamma_n} \leqslant C'\|u_{n2}\|_{1,\Omega_n} \leqslant C'C_5$$

by virtue of Lemma 1.1 and (1);

$$B_{22} \leqslant 2\tilde{C}_1 \int_0^1 |u_{n2}\big|_{\Gamma_n} - u_{n2}\big|_\Gamma| \, dx_2 \leqslant 2\tilde{C}_1 \int_{\Omega(\Gamma_n,\Gamma)} \left|\frac{\partial u_{n2}}{\partial x_1}\right| dx$$
$$\leqslant 2\tilde{C}_1(\operatorname{meas}\Omega(\Gamma_n, \Gamma))^{\frac{1}{2}} \|u_{n2}\|_{1,\Omega_n} \to 0$$

using (12);

$$B_{23} \leqslant 2\tilde{C}_1 \int_0^1 \left| \hat{u}_{n2}\big|_\Gamma - u_2\big|_\Gamma \right| \mathrm{d}x_2 \leqslant 2\tilde{C}_1 \|\hat{u}_{n2} - u_2\|_{0,\Gamma} \to 0$$

(cf. the case of $B_{12}$) by virtue of (13).

Altogether, we obtain that $B \to 0$ as $n \to \infty$. Consequently, (8) is verified. The rest of the proof goes through as in [1]. □

A consequence of the Proposition 2.1 is the following

**Lemma 2.2.** *The set $\mathcal{E}_{\mathrm{ad}}$ is compact in $[C^{(1)}(I)]^2$.*

Proof. Both $U^1_{\mathrm{ad}}$ and $U^2_{\mathrm{ad}}$ are compact in $C^{(1)}(I)$ (cf. [3], Lemma 2). Therefore $U_{\mathrm{ad}} = U^1_{\mathrm{ad}} \times U^2_{\mathrm{ad}}$ is compact in $[C^{(1)}(I)]^2$. The argument is then the same as that for Lemma 2.2 in [1], except that the scalar $v_n$ is replaced by a vector $(v_{n1}, v_{n2})$ and the convergence in $C^{(1)}(I)$ by the convergence in $[C^{(1)}(I)]^2$. □

Note that both Remarks 2.1 and 2.2 of [1] remain true.

**Theorem 2.1.** *Let the set $\mathcal{E}_{\mathrm{ad}}$ be non-empty. Then there exists at least one solution of the Optimal Design Problem (4).*

Proof follows from Lemma 2.2 and the continuity of $j(v)$ in $[C^{(1)}(I)]^2$. □

Remark 2.1. Since the trapezoidal shapes of $\Omega(v)$ can be embedded into our class of domains with $v \in U_{\mathrm{ad}}$, the assumption $\mathcal{E}_{\mathrm{ad}} \neq \emptyset$ can be satisfied by means of some results of [5].

## 3. Approximate solution

In the present section we propose an approximate solution of the problem (4), which is based on piecewise cubic approximations of the unknown boundary, a simple piecewise linear finite element model and a penalty approach, similar to that of [1].

Let $N$ be a positive integer, $h = \frac{1}{N}$ and $\Delta_j = [(j-1)h, jh]$, $j = 1, 2, \ldots, N$. We define

$$\begin{aligned} U^{h1}_{\mathrm{ad}} = \{ & v_h \in C^{(1),1}(I) \mid v_h\big|_{\Delta_j} \in P_3(\Delta_j),\ j = 1, \ldots, N, \\ & \alpha \leqslant v_h(jh) \leqslant \beta,\ |v_h'(jh)| \leqslant C_1,\ |v_h''(jh\pm)| \leqslant C_2, \\ & j = 0, 1, \ldots, N\}, \\ U^{h2}_{\mathrm{ad}} = \{ & v_h \in C^{(1),1}(I) \mid v_h\big|_{\Delta_j} \in P_3(\Delta_j),\ j = 1, \ldots, N, \\ & 0 \leqslant v_h(jh) \leqslant \gamma, v_h(1) = 0, |v_h'(jh)| \leqslant \tilde{C}_1, \\ & |v_h''(jh\pm)| \leqslant \tilde{C}_2,\ j = 0, 1, \ldots, N\}, \end{aligned}$$

where

$$v_h''(jh\pm) = \lim_{x_2 \to jh\pm} v_h''(x_2),$$

$P_3(\Delta_j)$ is the space of cubic polynomials on $\Delta_j$,

$$U_{\text{ad}}^h = U_{\text{ad}}^{h1} \times U_{\text{ad}}^{h2}.$$

In what follows, we shall need and estimate of the "distance" between $U_{\text{ad}}^h$ and $U_{\text{ad}}$, since $U_{\text{ad}}^h \not\subset U_{\text{ad}}$. To this end we establish the following

**Lemma 3.1.** *Let $v_h \in U_{\text{ad}}^{h1}$. Then*

$$\begin{aligned} &|v_h''| \leqslant C_2 \quad \text{for a.a. } x_2 \in I, \\ &\|v_h'\|_{C(I)} \leqslant C_1 + \frac{1}{2}C_2 h, \end{aligned} \tag{14}$$

$$\alpha - \frac{1}{2}C_1 h - \frac{1}{4}C_2 h^2 \leqslant v_h(x_2) \leqslant \beta + \frac{1}{2}C_1 h + \frac{1}{4}C_2 h^2, \quad x_2 \in I. \tag{15}$$

*Let $v_h \in U_{\text{ad}}^{h2}$. Then*

$$\begin{aligned} &|v_h''| \leqslant \tilde{C}_2 \text{ for a.a. } x_2 \in I, \\ &\|v_h'\|_{C(I)} \leqslant \tilde{C}_1 + \frac{1}{2}\tilde{C}_2 h, \\ &-\frac{1}{2}\tilde{C}_1 h - \frac{1}{4}\tilde{C}_2 h^2 \leqslant v_h(x_2) \leqslant \gamma + \frac{1}{2}C_1 h + \frac{1}{4}C_2 h^2, \quad x_2 \in I. \end{aligned}$$

Proof. The estimate (14) follows from the linearity of $v_h''$ in $\Delta_j$. In any interval $\Delta_j$ we have ($z \in \Delta_j, z_j = jh$ or $z_j = jh - h$)

$$|v_h'(z)| \leqslant |v_h'(z_j)| + \left| \int_{z_j}^{z} v_h''(t)\mathrm{d}t \right| \leqslant C_1 + |z - z_j| C_2 \leqslant C_1 + \frac{1}{2} h C_2.$$

The derivation of (15) is analogous. □

**Lemma 3.2.** *There exists a positive constant $h_0$ such that*

$$U_{\text{ad}}^h \subset U_{\text{ad}}^0 \quad \forall h < h_0.$$

Proof is immediate consequence of Lemma 3.1. □

Let $v_h \in U_{\mathrm{ad}}^h$ and denote $\Omega_h = \Omega(v_h)$. The domain $\Omega_h$ will be divided into triangles and curved triangles as follows, by means of a reference domain triangulation $\mathcal{T}_h(v_h^0)$.

Let us choose an initial function $v_h^0 \in [P_1(I)]^2 \cap U_{\mathrm{ad}}^h$ and construct a suitable uniform triangulation $\mathcal{T}_h(v_h^0)$. For a general $v_h \in U_{\mathrm{ad}}^h$ let us construct $\mathcal{T}_h(v_h)$ as a "distortion" of the initial triangulation $\mathcal{T}_h(v_h^0)$, preserving the number of nodes on any straight line segment

$$\{x \mid x_1 \in [-v_{h2}(jh), v_{h1}(jh)],\ x_2 = jh,\ j = 0, 1, \dots, N\}$$

and the uniform partition of these segments. The elements adjacent to $\Gamma(v_{h1})$, $\Gamma(v_{h2})$ may have one curved side. We assume that the parameters $C_i$ and $\tilde{C}_i$, $i = 1, 2$, are sufficiently small so that the curved sides of the triangles cannot cross the other sides.

We shall employ finite element spaces with linear polynomials on any triangle or "curved triangle" $K \in \mathcal{T}_h(v_h)$:

$$V_h(v_h) = \{w_h \in [C(\bar{\Omega}_h)]^2 \mid w_h|_K \in [P_1(K)]^2 \quad \forall K \in \mathcal{T}_h(v_h),\ w_h = 0 \text{ on } \Gamma_u(v_h)\}. \tag{16}$$

The finite element solution of the elastostatic problem (cf. (2) in [1]) will be defined as $u_h(v_h) \in V_h(v_h)$ such that

$$a(v_h\,;\, u_h(v_h), w_h) = \mathcal{F}(v_h\,;\, w_h) \quad \forall w_h \in V_h(v_h). \tag{17}$$

Since $V_h(v_h) \subset V(v_h)$ and $v_h \in U_{\mathrm{ad}}^0$ for $h < h_0$ by virtue of Lemma 3.2, we can use the positive definiteness of the form $c_{ijml}e_{ij}e_{ml}$ (cf. (1) from [1]), (6) of [1], (2) and (3) to verify the existence and uniqueness of $u_h(v_h)$, provided $h < h_0$. Moreover, we obtain that

$$\|u_h(v_h)\|_{1,\Omega(v_h)} \leqslant C_5 \quad \forall v_h \in U_{\mathrm{ad}}^h,\ \forall h < h_0 \tag{18}$$

with the constant $C_5$ independent of $v_h$ and of $h$.

We define the approximate stress field $\sigma^h(v_h)$ by the following formula

$$\sigma_{ij}^h(v_h) = c_{ijml}e_{ml}(u_h(v_h)), \quad i, j = 1, 2. \tag{19}$$

Note that the stress field $\sigma^h(v_h)$ is piecewise constant, if the coefficients $c_{ijml}$ are constant in every triangle $K \in \mathcal{T}_h(v_h)$.

We introduce the penalized cost functional

$$j_\varepsilon(v, \sigma) = j(v) + \frac{1}{\varepsilon}\sum_{i=1}^{3} f_i(v, \sigma),$$

where

$$f_i(v,\sigma) = \int_{\Omega(v)} (\sigma_{ii} - k)^+ \mathrm{d}x \quad \text{(no sum)}\ , i = 1, 2,$$

$$f_3(v,\sigma) = \int_{\Omega(v)} (\det(\sigma - \varkappa))^- \mathrm{d}x, \quad \varkappa_{ij} = k\delta_{ij},\ k \in L^2(\Omega_\delta),\ k \geqslant 0$$

and the Approximate Optimal Design Problem

$$v_h^\varepsilon = \arg\min_{v^h \in U^h_{\mathrm{ad}}} j_\varepsilon(v_h, \sigma^h(v_h)). \tag{20}$$

**Theorem 3.1.** *The Approximate Optimal Design Problem (20) has at least one solution $v_h^\varepsilon$ for any fixed $h = \frac{1}{N}$ and any positive parameter $\varepsilon$.*

In the proof the following lemmas play the crucial role.

**Lemma 3.3.** *Let $h = \frac{1}{N}$ be fixed and let $v_h^n \in U^h_{\mathrm{ad}}$,*

$$v_h^n \to v_h \text{ in } [C^{(1)}(I)]^2 \quad \text{as } n \to \infty.$$

*Then*

$$\tilde{\sigma}^h(v_h^n) \to \tilde{\sigma}^h(v_h) \text{ in } S(\Omega_\delta),$$

*where $\tilde{\sigma}^h$ denote the extensions of $\sigma^h$ by zero.*

Proof is analogous to that of Lemma 3.1 in [1]. □

**Lemma 3.4.** *Let the assumptions of Lemma 3.3 be fulfilled. Then for $i = 1, 2, 3$*

$$f_i(v_h^n, \sigma^h(v_h^n)) \to f_i(v_h, \sigma^h(v_h)) \quad \text{as } n \to \infty.$$

Proof is the same as that of Lemma 3.2 in [1]. □

Proof of Theorem 3.1. From Lemma 3.4 we conclude that

$$\begin{aligned} &j_\varepsilon(v_h^n, \sigma^h(v_h^n)) \to j_\varepsilon(v_h, \sigma^h(v_h)) \quad \text{as } n \to \infty, \\ &\text{if } v_h^n \to v_h \text{ in } [C^{(1)}(I)]^2. \end{aligned} \tag{21}$$

Any function $v_h \in U^h_{\mathrm{ad}}$ is uniquely determined by a vector $a$ of $4(N+1)-1$ parameters

$$v_{h1}(jh), v'_{h1}(jh) \text{ and } v_{h2}(jh), v'_{h2}(jh), \quad j = 0, 1, \ldots, N \text{ (excepting } v_{h2}(1)).$$

It is therefore easy to realize that

$$v_h \in U^h_{\mathrm{ad}} \Leftrightarrow a \in \mathcal{A} \subset R^{4N+3},$$

where $\mathcal{A}$ is a compact subset, being bounded and closed. Since

$$v_h^n(a^n) \to v_h(a) \text{ in } [C^{(1)}(I)]^2 \text{ as } n \to \infty$$

iff

$$a^n \to a \text{ in } R^{4N+3},$$

the function $a \mapsto j_\varepsilon(v_h(a), \sigma^h(v_h(a)))$ is continuous in $\mathcal{A}$ by virtue of (21). Hence it attains a minimum in $\mathcal{A}$. □

## 4. Gradient of the cost functional—Adjoint Problem

To calculate the gradient of the penalized cost functional we can use the method of Adjoint Problem. Let us denote

$$v_{h1} = \sum_{j=1}^{N+1} a_j \varphi_j(x_2) + \sum_{j=1}^{N+1} a'_j \Psi_j(x_2),$$

$$v_{h2} = \sum_{j=N+2}^{2N+1} a_j \varphi_j(x_2) + \sum_{j=N+2}^{2N+2} a'_j \Psi_j(x_2),$$

where $\varphi_j$ and $\Psi_j$ are cubic basis functions, the support of which consists of the subintervals containing the node $x_2 = (j-1)h$ or $x_2 = (j-N-2)h$, respectively,

$$a_j = \begin{cases} v_{h1}(jh-h), & 1 \leqslant j \leqslant N+1, \\ v_{h2}((j-N-2)h), & N+2 \leqslant j \leqslant 2N+1, \end{cases}$$

$$a'_j = \begin{cases} v'_{h1}(jh-h), & 1 \leqslant j \leqslant N+1, \\ v'_{h2}((j-N-2)h), & N+2 \leqslant j \leqslant 2N+2, \end{cases}$$

$$a = (a_1, a_2, \dots, a_{2N+1}, a'_1, a'_2, \dots, a'_{2N+2})^T$$

(cf. the proof of Theorem 3.1.).

Denoting the basis functions of the space $V_h(v_h)$ by $\{w_r(a)\}$, $r = 1, 2, \dots, d$, we may proceed as in Section 5.1 of [1] to arrive at Lemma 5.1 of [1] without any formal change. Some changes will occur in Lemma 5.2 of [1], where the formula (71) remains valid, but for the derivatives $\frac{\partial J}{\partial a_j}$ and $\frac{\partial J}{\partial a'_j}$ we get new expressions as follows.

Let $j = 1, 2, \dots, N+1$. Then

$$\begin{aligned}
\frac{J(a,U)}{\partial a_j} = {} & \int_0^1 \varphi_j(x_2) p(v_{h1}(x_2), x_2)\, \mathrm{d}x_2 \\
& + \frac{1}{\varepsilon} \sum_{i=1}^{2} \Big[ \int_{\Omega(a)} H(\sigma_{ii}^h(a) - k) \sum_{r=1}^{d} U_r \frac{\partial \sigma_{ii}^{(r)}(a)}{\partial a_j}\, \mathrm{d}x \\
& + \int_0^1 \varphi_j(x_2) \big( (\sigma_{ii}^h(a) - k)\big|_{x_1 = v_{h1}(x_2)} \big)^+ \mathrm{d}x_2 \Big] \\
& - \frac{1}{\varepsilon} \int_{\Omega(a)} H(-\det(\sigma^h(a) - \varkappa)) \sum_{r=1}^{d} U_r \Big( \frac{\partial \sigma_{11}^{(r)}}{\partial a_j} (\sigma_{22}^h(a) - k) \\
& + \frac{\partial \sigma_{22}^{(r)}}{\partial a_j} (\sigma_{11}^h(a) - k) - \frac{2 \partial \sigma_{12}^{(r)}}{\partial a_j} \sigma_{12}^h(a) \Big)\, \mathrm{d}x \\
& + \frac{1}{\varepsilon} \int_0^1 \varphi_j(x_2) \big( \det(\sigma^h(a) - \varkappa)\big|_{x_1 = v_{h1}(x_2)} \big)^- \mathrm{d}x_2.
\end{aligned}$$

For $j = N+2, \ldots, 2N+1$ we have

$$\frac{\partial J(a,U)}{\partial a_j} = \int_0^1 \varphi_j(x_2) p(-v_{h2}(x_2), x_2)\,\mathrm{d}x_2$$

$$+ \frac{1}{\varepsilon} \sum_{i=1}^{2} \Big[ \int_{\Omega(a)} H(\sigma_{ii}^h(a) - k) \sum_{r=1}^{d} U_r \frac{\partial \sigma_{ii}^{(r)}(a)}{\partial a_j}\,\mathrm{d}x$$

$$- \int_0^1 \varphi_j(x_2) \big( (\sigma_{ii}^h(a) - k)\big|_{x_1 = -v_{h2}(x_2)} \big)^+ \mathrm{d}x_2 \Big]$$

$$- \frac{1}{\varepsilon} \int_{\Omega(a)} H(-\det(\sigma^h(a) - \varkappa)) \sum_{r=1}^{d} U_r \Big( \frac{\partial \sigma_{11}^{(r)}}{\partial a_j} (\sigma_{22}^h(a) - k)$$

$$+ \frac{\partial \sigma_{22}^{(r)}}{\partial a_j} (\sigma_{11}^h(a) - k) - \frac{2\partial \sigma_{12}^{(r)}}{\partial a_j} \sigma_{12}^h(a) \Big)\,\mathrm{d}x$$

$$- \frac{1}{\varepsilon} \int_0^1 \varphi_j(x_2) \big( \det(\sigma^h(a) - \varkappa)\big|_{x_1 = -v_{h2}(x_2)} \big)^- \mathrm{d}x_2.$$

$$\frac{\partial J(a,U)}{\partial a_j'} = \int_0^1 \Psi_j(x_2) p(v_{h1}(x_2), x_2)\,\mathrm{d}x_2$$

$$+ \sum_{i=1}^{2} \frac{1}{\varepsilon} \int_0^1 \Psi_j(x_2) \big( (\sigma_{ii}^h(a) - k)\big|_{x_1 = v_{h1}(x_2)} \big)^+ \mathrm{d}x_2$$

$$+ \frac{1}{\varepsilon} \int_0^1 \Psi_j(x_2) \big( \det(\sigma^h(a) - \varkappa)\big|_{x_1 = v_{h1}(x_2)} \big)^- \mathrm{d}x_2$$

for $j = 1, 2, \ldots, N+1$;

$$\frac{\partial J(a,U)}{\partial a_j'} = \int_0^1 \Psi_j(x_2) p(-v_{h2}(x_2), x_2)\,\mathrm{d}x_2$$

$$- \frac{1}{\varepsilon} \sum_{i=1}^{2} \int_0^1 \Psi_j(x_2) \big( (\sigma_{ii}^h(a) - k)\big|_{x_1 = -v_{h2}(x_2)} \big)^+ \mathrm{d}x_2$$

$$- \frac{1}{\varepsilon} \int_0^1 \Psi_j(x_2) \big( \det(\sigma^h(a) - \varkappa)\big|_{x_1 = -v_{h2}(x_2)} \big)^- \mathrm{d}x_2$$

for $j = N+2, \ldots, 2N+2$.

## 5. Convergence analysis

First we shall introduce an intermediate Penalized Optimal Design Problem (cf. Section 6 of [1], (84))

$$v_\varepsilon = \arg\min_{v \in U_{\mathrm{ad}}} j_\varepsilon(v, \sigma(v)). \tag{22}$$

**Proposition 5.1.** *There exists at least one solution of the problem (22) for any positive parameter $\varepsilon$.*

The proof is based on the following lemma.

**Lemma 5.1.** *The functions*

$$v \mapsto f_i(v, \sigma(v)), \quad i = 1, 2, 3$$

*are continuous in the subset $U^0_{\mathrm{ad}} \subset [C^{(1)}(I)]^2$.*

P r o o f follows from Proposition 2.1 in the same way as in the proof of Lemma 6.1 of [1]. □

P r o o f of Proposition 5.1. The set $U_{\mathrm{ad}}$ is compact in $[C^{(1)}(I)]^2$ and the functionals $j(v)$, $f_i(v, \sigma(v))$ are continuous in the latter space by Lemma 5.1. Consequently, a minimizer $v_\varepsilon$ of $j_\varepsilon(v, \sigma(v))$ exists. □

**Theorem 5.1.** *Let $\mathcal{E}_{\mathrm{ad}}$ be non-empty, let $\{v_\varepsilon\}$, $\varepsilon \to 0$, be a sequence of solutions of the problem (22), $\{\sigma(v_\varepsilon)\}$ the sequence of corresponding stress fields.*

*Then there exist a subsequence $\{v_{\tilde{\varepsilon}}\} \subset \{v_\varepsilon\}$ and an element $v^* \in U_{\mathrm{ad}}$ such that*

$$v_{\tilde{\varepsilon}} \to v^* \text{ in } [C^{(1)}(I)]^2,$$
$$\tilde{\sigma}(v_{\tilde{\varepsilon}}) \to \tilde{\sigma}(v^*) \text{ in } S(\Omega_\delta),$$

*where $v^*$ is a solution of the Optimal Design Problem (4) and $\tilde{\sigma}$ denote the extensions of the stress fields by zero.*

P r o o f is the same as that of Theorem 6.1 in [1]. □

**Proposition 5.2.** *Let $\{v_h\}, h \to 0$, be a sequence of $v_h \in U^h_{\mathrm{ad}}$ such that*

$$v_h \to v \text{ in } [C^{(1)}(I)]^2. \tag{23}$$

*Then*

$$(24) \qquad \tilde{\sigma}^h(v_h) \to \tilde{\sigma}(v) \text{ in } S(\Omega_\delta),$$

*where $\tilde{\sigma}^h$ denote the extensions of $\sigma^h$ by zero.*

First of all we shall establish an auxiliary lemma.

**Lemma 5.2.** *Let the assumptions of Proposition 5.2 be fulfilled. Then*

$$u_h(v_h)\big|_{G_m} \rightharpoonup u(v)\big|_{G_m} \text{ (weakly) in } [H^1(G_m)]^2 \text{ as } h \to 0$$

*holds for any $m > \dfrac{8}{\alpha}$ (see Lemma 2.1 for the definition of $G_m$).*

P r o o f. Let us denote $\Omega_h = \Omega(v_h)$, $\Omega = \Omega(v)$ and define the extension $\hat{u}_h$ of $u_h = u_h(v_h)$, symmetric with respect to both $\Gamma(v_{h1})$ and $\Gamma(v_{h2})$ in the $x_1$-direction. Then $\hat{u}_h$ is defined in the domain

$$\Omega_\alpha = \Omega\left(-v_2 - \frac{\alpha}{4}, v_1 + \frac{\alpha}{4}\right)$$

and since the derivatives $v'_{h1}$ and $v'_{h2}$ are bounded, we obtain

$$(25) \qquad \|\hat{u}_h\|_{1,\Omega_\alpha} \leqslant C \quad \forall h < h_0$$

(cf. the proof of Lemma 2.1 in [1]). Then a subsequence of $\{\hat{u}_h\}$ exists (and we shall denote it by the same symbol), such that

$$(26) \qquad \hat{u}_h \rightharpoonup \bar{u} \text{ (weakly) in } [H^1(\Omega_\alpha)]^2$$

with some $\bar{u} \in [H^1(\Omega_\alpha)]^2$. Since $\hat{u}_h \in V(-v_2 - \frac{\alpha}{4}, v_1 + \frac{\alpha}{4})$, we obtain $\bar{u} \in V(-v_2 - \frac{\alpha}{4}, v_1 + \frac{\alpha}{4})$.

Let any $w \in V(v)$ be given. We construct an extension $\hat{w} \in V(\delta)$ by means of (possibly repeated) extensions symmetric with respect to $\Gamma(v_1)$ and $\Gamma(v_2)$. There exists a sequence $\{w_\eta\}, \eta \to 0$, such that $w_\eta \in [C^\infty(\bar{\Omega}_\delta)]^2$, $w_\eta = 0$ in a neighbourhood of the $x_1$-axis and

$$w_\eta \to \hat{w} \text{ in } [H^1(\Omega_\delta)]^2.$$

Let $\pi_h w_\eta$ be the linear Lagrange interpolate of $w_\eta$ over the triangulation $\mathcal{T}_h(v_h)$, $\pi_h w_\eta \in V_h(v_h)$. Let us define the rectangle $\Omega^*_\delta = (-\delta, \delta) \times (-\frac{1}{2}, \frac{3}{2})$. There exists an extension $Ew_\eta \in [H^2(\Omega^*_\delta)]^2$ of the function $w_\eta$. We shall prove that

$$(27) \qquad \|\pi_h w_\eta - w_\eta\|_{1,\Omega_h} \leqslant Ch\|Ew_\eta\|_{2,\Omega^*_\delta}$$

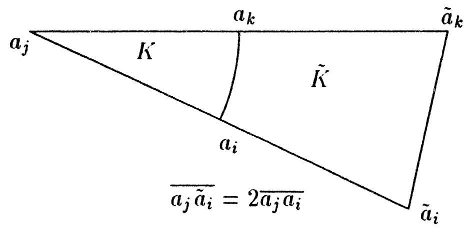


Fig. 2

holds with some constant $C$ independent of both $h$ and $v_h$. Indeed, for any curved triangle $K$ adjacent to $\Gamma(v_{hi})$, $i = 1, 2$, we define a "twice enlarged" triangle $\tilde{K} = \Delta a_j \tilde{a}_i \tilde{a}_k$. For the remaining triangles let $\tilde{K} = K$.

Let $\pi_2$ denote the linear interpolation on $\tilde{K}$ with the nodes $a_i$, $a_j$, $a_k$. Making use of the affine equivalence and the regularity of the family $\{\mathcal{T}_h(v_h)\}$, $h \to 0$, $v_h \in U^h_{\text{ad}}$ (which can be verified by a direct calculation), we arrive at the following estimate

$$\|\pi_2 Ew_\eta - Ew_\eta\|_{1,\tilde{K}} \leqslant Ch|Ew_\eta|_{2,\tilde{K}},$$

where $C$ is independent of $h$ and $v_h$. Since $\pi_2 Ew = \pi_h w$ holds on any $K$, using also the standard esimate on the remaining straight triangles, we may write

$$\begin{aligned}\|w_\eta - \pi_h w_\eta\|^2_{1,\Omega_h} &= \sum_{K \in \mathcal{T}_h(v_h)} \|w_\eta - \pi_h w_\eta\|^2_{1,K} \leqslant \sum_{K \in \mathcal{T}_h(v_h)} \|Ew_\eta - \pi_2 Ew\|^2_{1,\tilde{K}} \\ &\leqslant C^2 h^2 \sum_{K \in \mathcal{T}_h(v_h)} |Ew_\eta|^2_{2,\tilde{K}} \leqslant \tilde{C}h^2 \|Ew_\eta\|^2_{2,\Omega^*_\delta}\end{aligned}$$

and (27) follows. We may write

$$a(v_h\,;\,u_h, \pi_h w_\eta) = \mathcal{F}(v_h\,;\,\pi_h w_\eta). \tag{28}$$

Let us pass to the limit with $h \to 0$. We have

$$\begin{aligned}&|a(v_h\,;\,u_h, \pi_h w_\eta) - a(v\,;\,\bar{u}, w_\eta)| \leqslant |a(v_h\,;\,u_h, \pi_h w_\eta - w_\eta)| \\ &+ |a(v_h\,;\,u_h, w_\eta) - a(v\,;\,\hat{u}_h, w_\eta)| + |a(v\,;\,\hat{u}_h - \bar{u}, w_\eta)| = \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3;\end{aligned}$$

$$\mathcal{K}_1 \leqslant C_3 \|u_h\|_{1,\Omega_h} \|\pi_h w_\eta - w_\eta\|_{1,\Omega_h} \leqslant C_3 C_5 Ch \|Ew_\eta\|_{2,\Omega^*_\delta} \to 0$$

by virtue of (18) and (27);

$$\mathcal{K}_2 \leqslant \int_{\Delta(\Omega_h,\Omega)} |c_{ijml} e_{ij}(u_h) e_{ml}(w_\eta)|\,\mathrm{d}x \leqslant C_3 \|u_h\|_{1,\Omega_\alpha} \|w_\eta\|_{1,\Delta(\Omega_h,\Omega)} \to 0,$$

making use of (25) and

$$\text{(29)} \qquad \operatorname{meas}\Delta(\Omega_h, \Omega) \to 0;$$

$\mathcal{K}_3 \to 0$ follows from the weak convergence (26).

Conseguently, we have

$$\text{(30)} \qquad a(v_h\,;\, u_h, \pi_h w_\eta) \to a(v\,;\, \bar{u}, w_\eta).$$

Next we derive that

$$\text{(31)} \qquad \mathcal{F}(v_h\,;\, \pi_h w_\eta) \to \mathcal{F}(v\,;\, w_\eta).$$

Indeed, we may write (denoting $\Gamma_h = \Gamma(v_{h2})$, $\Gamma = \Gamma(v_2)$)

$$\begin{aligned}
|\mathcal{F}(v_h\,;\, \pi_h w_\eta) - \mathcal{F}(v\,;\, w_\eta)| &\leqslant \left| \int_{\Omega_h} F \cdot \pi_h w_\eta \, dx - \int_\Omega F \cdot w_\eta \, dx \right| \\
&\quad + \left| \int_0^H (H - x_2)(\pi_h w_{\eta 1}|_{\Gamma_h} - w_{\eta 1}|_\Gamma + v'_{h2}\pi_h w_{\eta 2}|_{\Gamma_h} - v'_2 w_{\eta 2}|_\Gamma) \, dx_2 \right| \\
&= A + B, \\
A &\leqslant \left| \int_{\Omega_h} F \cdot (\pi_h w_\eta - w_\eta) \, dx \right| + \left| \int_{\Omega_h} F \cdot w_\eta \, dx - \int_\Omega F \cdot w_\eta \, dx \right| \\
&\leqslant \|F\|_{0,\Omega_\delta} \|\pi_h w_\eta - w_\eta\|_{0,\Omega_h} + \int_{\Delta(\Omega_h, \Omega)} |F \cdot w_\eta| \, dx \to 0,
\end{aligned}$$

using (27) and (29). Furthermore, we have

$$B \leqslant \int_0^1 |\pi_h w_{\eta 1}|_{\Gamma_h} - w_{\eta 1}|_\Gamma| \, dx_2 + \int_0^1 |v'_{h2}\pi_h w_{\eta 2}|_{\Gamma_h} - v'_2 w_{\eta 2}|_\Gamma| \, dx_2 = B_1 + B_2,$$

$$\text{(32)} \quad B_1 \leqslant \int_0^1 |\pi_h w_{\eta 1}|_{\Gamma_h} - w_{\eta 1}|_{\Gamma_h}| \, dx_2 + \int_0^1 |w_{\eta 1}|_{\Gamma_h} - w_{\eta 1}|_\Gamma| \, dx_2 = B_{11} + B_{12},$$

$$\text{(33)} \qquad B_{11} \leqslant \|\pi_h w_{\eta 1} - w_{\eta 1}\|_{0,\Gamma_h} \leqslant C\|\pi_h w_{\eta 1} - w_{\eta 1}\|_{1,\Omega_h} \to 0$$

by virtue of Lemma 1.1, Lemma 3.2 and (27);

$$\text{(34)} \qquad B_{12} \leqslant \int_{\Delta(\Gamma_h, \Gamma)} \left| \frac{\partial w_{\eta 1}}{\partial x_1} \right| \, dx \leqslant (\operatorname{meas} \Omega(\Gamma_h, \Gamma))^{\frac{1}{2}} \|w_{\eta 1}\|_{1,\Omega_\delta},$$

where $\Omega(\Gamma_h, \Gamma)$ denotes the set "between" $\Gamma_h$ and $\Gamma$. Since

$$\operatorname{meas} \Omega(\Gamma_h, \Gamma) \to 0,$$

$B_{12}$ tends to zero. Combining (32), (33) and (34), we arrive at

$$B_1 \to 0. \tag{35}$$

Next we have

$$B_2 \leqslant \int_0^1 |(v'_{h2} - v'_2)\pi_h w_{\eta 2}|_{\Gamma_h}|\,\mathrm{d}x_2 + \int_0^1 |v'_2(\pi_h w_{\eta 2}|_{\Gamma_h} - w_{\eta 2}|_{\Gamma_h})|\,\mathrm{d}x_2$$
$$+ \int_0^1 |v'_2(w_{\eta 2}|_{\Gamma_h} - w_{\eta 2}|_{\Gamma})|\,\mathrm{d}x_2 = B_{21} + B_{22} + B_{23};$$
$$B_{21} \leqslant \|v'_{h2} - v'_2\|_{C(I)}\|\pi_h w_{\eta 2}\|_{0,\Gamma_h} \leqslant C\|v_h - v\|_{C^1(I)}\|\pi_h w_{\eta 2}\|_{1,\Omega_h} \to 0$$

by virtue of Lemma 1.1, Lemma 3.2 and since

$$\|\pi_h w_{\eta 2}\|_{1,\Omega_h} \leqslant \|w_{\eta 2}\|_{1,\Omega_\delta} + Ch\|Ew_{\eta 2}\|_{2,\Omega_\delta^*} \leqslant \tilde{C} \quad \forall h < h_0;$$
$$B_{22} \leqslant \tilde{C}_1\|\pi_h w_{\eta 2} - w_{\eta 2}\|_{0,\Gamma_h} \leqslant \tilde{C}_1 C\|\pi_h w_{\eta 2} - w_{\eta 2}\|_{1,\Omega_h} \to 0$$

(cf. (33)),

$$B_{23} \leqslant \tilde{C}_1\big(\operatorname{meas}\Omega(\Gamma_h,\Gamma)\big)^{\frac{1}{2}}\|w_{\eta 2}\|_{1,\Omega_\delta} \to 0$$

(cf. (34)).
Combining the last three results, we arrive at

$$B_2 \to 0.$$

Altogether, (31) is verified. Consequently, from (28), (30) and (31) we obtain

$$a(v;\, \bar{u}, w_\eta) = \mathcal{F}(v;\, w_\eta).$$

Passing to the limit with $\eta \to 0$, we arrive at

$$a(v;\, \bar{u}, w) = \mathcal{F}(v;\, w).$$

Since $v \in U^0_{\mathrm{ad}}$, we can use Lemma 1.2 to obtain that

$$\bar{u}|_{\Omega} = u(v) \tag{36}$$

follows from the uniqueness of $u(v)$.

Since $G_m \subset \Omega_h \subset \Omega_\alpha$ holds for all $h < h_0(m)$, (26) and (36) imply the assertion of our Lemma. □

Proof of Proposition 5.2. Let us denote $\sigma = \sigma(v)$. From Lemma 5.2 and the definition (19) of $\sigma^h$ it follows that

$$\sigma^h\big|_{G_m} \rightharpoonup \sigma\big|_{G_m} \text{ (weakly) in } S(G_m) \quad \forall m > \frac{8}{\alpha}, \qquad \text{as } h \to 0. \tag{37}$$

Then we show that

$$\tilde{\sigma}^h \rightharpoonup \tilde{\sigma} \text{ (weakly) in } S(\Omega_\delta), \tag{38}$$

using (37) and the boundedness of all $\sigma^h$ in $S(\Omega_h)$, which follows from (18), (19), (for detail, see the proof of (102) in [1]).

Making use of the inverse relation

$$e(u_h) = b\sigma^h,$$

we may write

$$\left\langle \tilde{\sigma}^h, b\tilde{\sigma}^h \right\rangle_{\Omega_\delta} = a(v_h\,;\, u_h, u_h) = \mathcal{F}(v_h\,;\, u_h). \tag{39}$$

Next we can show that

$$\mathcal{F}(v_h\,;\, u_h) \to \mathcal{F}(v\,;\, u). \tag{40}$$

The proof is quite analogous to that of (8). It is sufficient to replace $u_n$ by $u_h$, $\Gamma_n$ by $\Gamma_h$ and to use (18), (25) and (26).

Since

$$\mathcal{F}(v\,;\, u) = a(v\,;\, u, u) = \langle \tilde{\sigma}, b\tilde{\sigma} \rangle_{\Omega_\delta},$$

we are led to the limit of the energy norms

$$\left\langle \tilde{\sigma}^h, b\tilde{\sigma}^h \right\rangle_{\Omega_\delta} \to \langle \tilde{\sigma}, b\tilde{\sigma} \rangle_{\Omega_\delta} \tag{41}$$

The equivalence of the energy norm with the standard norm of $S(\Omega_\delta)$, the weak convergence of (38) and (41) yield that

$$\|\tilde{\sigma}^h - \tilde{\sigma}\|_{0,\Omega_\delta} \to 0 \text{ as } h \to 0.$$

□

**Proposition 5.3.** *Let the assumptions of Proposition 5.2 be satisfied. Then*

$$j_\varepsilon(v_h, \sigma^h(v_h)) \to j_\varepsilon(v, \sigma(v)), \text{ as } h \to 0.$$

Proof follows from Proposition 5.2 if we argue as in the proof of Lemma 6.1 of [1]. □

**Theorem 5.2.** *Let $\{v^\varepsilon_h\}$, $h \to 0$, be a sequence of solutions of the Approximate Optimal Design Problem* (20) *for a fixed $\varepsilon > 0$.*

*Then there exist a subsequence $\{v^\varepsilon_{\tilde h}\} \subset \{v^\varepsilon_h\}$ and an element $v_\varepsilon$ such that*

$$
\begin{aligned}
&v^\varepsilon_{\tilde h} \to v_\varepsilon \text{ in } [C^{(1)}(I)]^2,\\
&\tilde\sigma^{\tilde h}(v^\varepsilon_{\tilde h}) \to \tilde\sigma(v_\varepsilon) \text{ in } S(\Omega_\delta)
\end{aligned} \tag{42}
$$

*as $\tilde h \to 0$ and $v_\varepsilon$ is a solution of the Penalized Optimal Design Problem* (22).

Before proving the above Theorem we shall establish an important

**Lemma 5.3.** *Given any $v \in U^i_{\text{ad}}$, $i = 1, 2$, and any sequence $\{h_n\}_{n=1}^\infty$, $h_n = \frac{1}{N_n}$, $h_n \to 0$, there exist a subsequence $\{h_k\}_{k=K_1}^\infty$, a sequence $\{v_{h_k}\}$ and a positive constant $h_0(v)$ such that*

$$
\begin{aligned}
&v_{h_k} \in U^{h_k i}_{\text{ad}} \quad \forall h_k < h_0(v),\\
&v_{h_k} \to v \text{ in } C^1(I) \text{ as } k \to \infty.
\end{aligned}
$$

P r o o f. Let us show the detailed proof for $i = 2$. The case $i = 1$ is analogous (cf. also [3], proof of Lemma 11).

1° Let us extend $v(x_2)$ for $x_2 > 1$ in an antisymmetric way, i.e., let $s = x_2 - 1$, $\tilde v(s) = v(1+s)$ for $s \in [-1, 0]$ and $\tilde v(s) = -\tilde v(-s)$ for $s \in (0, 1]$.
Define $v_\lambda(s) = \tilde v((1-\lambda)s)$, $s \in I_\lambda \equiv [-(1-\lambda)^{-1}, (1-\lambda)^{-1}]$, where $\lambda \in (0, 1)$. Denote also $u^{(i)} \equiv \dfrac{\mathrm{d}^i u}{\mathrm{d}s^i}$, $i = 1, 2$, in what follows.

Obviously, we have

$$v_\lambda \in C^{(1),1}(I_\lambda),\ 0 \leqslant v_\lambda(s) \leqslant \gamma \quad \text{for } s \in [-(1-\lambda)^{-1}, 0], \tag{43}$$

$$|v_\lambda{}^{(i)}(s)| \leqslant (1-\lambda)^i \tilde C_i, \quad i = 1, 2 \text{ for a.a. } s \in I_\lambda, \tag{44}$$

$$|v_\lambda - \tilde v| \leqslant \tilde C_1 \lambda, \tag{45}$$

$$|v'_\lambda - \tilde v'| \leqslant (\tilde C_1 + \tilde C_2)\lambda \quad \text{for } s \in [-1, 0]. \tag{46}$$

2° We apply the regularization $R_H$ as follows:

$$R_H f(s) = (\varkappa H)^{-1} \int_{-\infty}^{\infty} \omega_1(s - y, H) f(y)\, dy,$$

where $H = \text{const} > 0$,

$$
\omega_1(z, H) = \begin{cases} \exp \dfrac{z^2}{z^2 - H^2} & \text{if } |z| < H,\\ 0 & \text{if } |z| \geqslant H, \end{cases}
$$

$$\varkappa = H^{-1} \int\limits_{|z|<H} \omega_1(z, H)\, dz.$$

Thus we obtain

$$(47) \qquad R_H v_\lambda \in C^\infty([-1,0]), \quad 0 \leqslant R_H v_\lambda(s) \leqslant \gamma,$$

$$(48) \qquad |(R_H v_\lambda)^{(i)}(s)| \leqslant \tilde{C}_i (1-\lambda)^i, \quad i = 1,2 \text{ for } s \in [-1,0]$$

and $H < \frac{\lambda}{1-\lambda}$, using (43), (44) and the odd character of $v_\lambda$.

Moreover,

$$(49) \qquad R_H v_\lambda \rightarrow v_\lambda \text{ in } C^1([-1,0]) \text{ as } H \rightarrow 0.$$

In fact, for $i = 0, 1$ and any $p > 1$ we may write

$$\begin{aligned} \|R_H - v_\lambda{}^{(i)}\|_{C([-1,0])} &\leqslant C_0 \|R_H v_\lambda{}^{(i)} - v_\lambda{}^{(i)}\|_{W^{1,p}([-1,0])} \\ &\leqslant C_0 \|R_H v_\lambda - v_\lambda\|_{W^{2,p}([-1,0])}, \end{aligned}$$

using the continuous embedding

$$W^{1,p}(J) \hookrightarrow C(J), \quad J = [-1,0]$$

and

$$v_\lambda \in C^{(1),1}(I_\lambda) \subset W^{2,p}(I_\lambda).$$

The last norm in $W^{2,p}(J)$ tends to zero as $H \rightarrow 0$.

3° Let us denote

$$(50) \qquad \mu = \frac{5\omega(h, (R_H v_\lambda)'')}{\tilde{C}_2},$$

where $\omega(h, f)$ is the modulus of continuity of the function $f$. Note that

$$(51) \qquad \mu \rightarrow 0 \text{ as } h \rightarrow 0 \text{ for any fixed } \lambda,\ H < \lambda(1-\lambda)^{-1}.$$

Let us introduce the cubic spline interpolation $\operatorname{Sp} f$ of the function $f \in C^2(I)$ on the mesh $\{s_j = -jh,\ j = 0, 1, 2, \dots, N\}$ with $(\operatorname{Sp} f)'' = f''$ at the endpoints and define

$$v_h = (1-\mu) \operatorname{Sp}(R_H v_\lambda)$$

for

$$(52) \qquad h < \tilde{C}_1 (6\tilde{C}_2)^{-1} \lambda.$$

We shall employ the following error estimates (see [2], Chapt. II, Th. 9 and Th. 10)

$$\|(\operatorname{Sp} f)'' - f''\|_{C(I)} \leqslant 5\omega(h, f''), \tag{53}$$

$$\|(\operatorname{Sp} f^{(i)} - f^{(i)}\|_{C(I)} \leqslant 12 \cdot 2^{i-2} h^{2-i} \|f''\|_{C(I)}, \quad i = 0, 1. \tag{54}$$

Then we may write, making use of (50), (48), (52) and (53)

$$\|v_h'' - (1-\mu)(R_H v_\lambda)''\|_{C(I)} \leqslant (1-\mu) 5\omega(h, (R_H v_\lambda)'') \leqslant \tilde{C}_2 \mu,$$

$$|v_h''| \leqslant (1-\mu)|(R_H v_\lambda)''| + \tilde{C}_2\mu \leqslant \tilde{C}_2(1-\lambda)^2(1-\mu) + \tilde{C}_2\mu \leqslant \tilde{C}_2; \tag{55}$$

$$\|v_h' - (1-\mu)(R_H v_\lambda)'\|_{C(I)} \leqslant 6h(1-\mu)\|(R_H v_\lambda)''\|_{C(I)} \leqslant 6h\tilde{C}_2,$$

$$|v_h'| \leqslant (1-\mu)|(R_H v_\lambda)'| + 6h\tilde{C}_2 \leqslant (1-\mu)(1-\lambda)\tilde{C}_1 + \tilde{C}_1\lambda \leqslant \tilde{C}_1, \tag{56}$$

which hold for all $s \in I$, provided $\mu < 1$, i.e., if $h < \bar{h}_0(\lambda, H)$.

At the nodal points (for $s = jh$) we then have

$$0 \leqslant v_h = (1-\mu) R_H v_\lambda \leqslant (1-\mu)\gamma \leqslant \gamma \tag{57}$$

on the basis of (47).

Since $v_\lambda$ is an odd function, we have

$$v_h(0) = (1-\mu) R_H v_\lambda(0) = 0. \tag{58}$$

From (55)–(58) we obtain that $v_h \in U_{\mathrm{ad}}^{(2)h}$ if $h < \bar{h}_0(\lambda, H)$ and if (52) is satisfied.

Let us estimate the distance between $v_h$ and $v$ as follows

$$\begin{aligned} \|v_h - v\| &\leqslant \|(1-\mu)\operatorname{Sp}(R_H v_\lambda) - (1-\mu)R_H v_\lambda\| + \|(1-\mu)R_H v_\lambda - R_H v_\lambda\| \\ &\quad + \|R_H v_\lambda - v_\lambda\| + \|v_\lambda - v\| \\ &\leqslant (1-\mu)3h^2\|(R_H v_\lambda)''\| + \mu\|R_H v_\lambda\| + \|R_H v_\lambda - v_\lambda\| + \tilde{C}_1\lambda \\ &\leqslant 3h^2(1-\mu)(1-\lambda)^2\tilde{C}_2 + \mu\gamma + \|R_H v_\lambda - v_\lambda\| + \tilde{C}_1\lambda, \end{aligned} \tag{59}$$

where all the norms are those of $C(I)$ and (54), (45), (48) has been used.

Given any positive integer $k$, we can choose fixed $\lambda_k$ and $H_k$ as follows

$$\lambda_k = (4k\tilde{C}_1)^{-1}, \quad H_k \leqslant \lambda_k \quad \text{and such that (cf. (49))}$$

$$\|R_{H_k} v_{\lambda k} - v_{\lambda k}\| \leqslant (4k)^{-1}.$$

If we choose $h$ such that

$$h < (12\tilde{C}_2 k)^{-\frac{1}{2}} \text{ and } \omega(h, (R_{H_k} v_{\lambda k})'') \leqslant \tilde{C}_2(20\gamma k)^{-1} \tag{60}$$

(cf. (51)), then (59) yields the estimate

$$\|v_h - v\| \leqslant \frac{1}{k}.$$

From $h_n \to 0$ we deduce that $h_k \equiv h_{n_k}$ exist, which satisfy (52) and (60). Consequently, we obtain

$$\|v_{h_k} - v\|_{C(K)} \leqslant \frac{1}{k}, \quad k \to \infty.$$

Since the above argument can be repeated for the difference $(v_h' - v')$ with similar estimates, we arrive at the convergence in $C^1([0, 1])$. □

Proof of Theorem 5.2. Since $U_{\text{ad}}^{h} \subset U_{\text{ad}}^{0}$ for any $h < h_0$ by Lemma 3.2 and the set $U_{\text{ad}}^{0}$ is compact in $[C^1(I)]^2$, there exists a subsequence $\{v_{\tilde{h}}^{\varepsilon}\}$ such that (42) holds with some $v_\varepsilon \in U_{\text{ad}}^{0}$. Using Lemma 3.1, we deduce that $v_\varepsilon \in U_{\text{ad}}$.

Let us consider an arbitrary $v \in U_{\text{ad}}$. By Lemma 5.3, there exists a subsequence of $\{\tilde{h}\}$ (and we shall denote it by the same symbol) such that

$$v_{\tilde{h}} \in U_{\text{ad}}^{\tilde{h}}, \; v_{\tilde{h}} \to v \text{ in } [C^{(1)}(I)]^2 \text{ as } \tilde{h} \to 0.$$

By definition, we may write

$$j_\varepsilon(v_{\tilde{h}}^{\varepsilon}, \sigma^h(v_{\tilde{h}}^{\varepsilon})) \leqslant j_\varepsilon(v_{\tilde{h}}, \sigma^{\tilde{h}}(v_{\tilde{h}})).$$

Let us pass to the limit with $\tilde{h} \to 0$ and apply Proposition 5.3 to both sides of the inequality. Thus we obtain

$$j_\varepsilon(v_\varepsilon, \sigma(v_\varepsilon)) \leqslant j_\varepsilon(v, \sigma(v)).$$

Consequently, $v_\varepsilon$ is a solution of the problem (22). The convergence of stress fields follows from Proposition 5.2. □

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Souhrn

## MINIMALIZACE VÁHY PRUŽNÝCH TĚLES, KTERÁ NEVZDORUJÍ VĚTŠÍM TAHOVÝM NAPĚTÍM

## II. OBLASTI SE DVĚMA ZAKŘIVENÝMI STRANAMI


IVAN HLAVÁČEK, MICHAL KŘÍŽEK



Výsledky článku [1] jsou rozšířeny na případ tělesa tvaru „křivočarého lichoběžníka“ se dvěma zakřivenými proměnnými stranami. Levá část tělesa je zatížena hydrostatickým tlakem. Hranice tělesa je aproximována hermitovými kubickými splajnovými funkcemi. Pro aproximaci posunutí jsou použity lineární konečné prvky. Dokazuje se existence optimální hranice a konvergence přibližných řešení, definovaných na základě metody penalizace.



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