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WEIGHT MINIMIZATION OF ELASTIC BODIES WEAKLY SUPPORTING TENSION
I. DOMAINS WITH ONE CURVED SIDE

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Summary. Shape optimization of a two-dimensional elastic body is considered, provided the material is weakly supporting tension. The problem generalizes that of a masonry dam subjected to its own weight and to the hydrostatic pressure. Existence of an optimal shape is proved. Using a penalty method and finite element technique, approximate solutions are proposed and their convergence is analyzed.

Keywords: shape optimization, weight minimization, penalty method, masonry-like materials, finite elements

AMS classification: 65N30, 65K10, 73C99

INTRODUCTION

A new attention has been drawn recently to elastic materials which are not supporting tension, such as masonry-like materials (see e.g. [5, 1]). In connection with the optimal design of a masonry dam, minimization of a cross-section of triangular shape was solved in the paper [3], whereas trapezoidal shapes were considered in [2]. In the latter papers, the masonry dam is subjected to its own weight and to hydrostatic pressure in a state of plane strain. Approximate solutions of the elastostatic boundary value problems by means of the Airy stress function were used in the stress analysis.

The aim of the present paper is to propose another approximate solution of the weight minimization problem, taking more general than trapezoidal shapes of the cross-section into consideration. To solve the elastostatic problems, a primal or dual finite element method is used with piecewise constant stress fields. Then a penalty approach enables us to remove the constraint that only small tensions are allowed and thus to simplify the resulting nonlinear programming problem.

In Sections 2 and 3, proofs of the existence of at least one solution to both the continuous (original) and the discretized (approximate) optimal design problems are given in detail. In Section 4, we apply a dual approach to the state problem. Moreover, a comparison of our model with that of Giaquinta and Giusti [5] is presented. Section 5 contains definitions of adjoint problems, which are useful in the sensitivity analysis, i.e., for calculations of the gradient of the penalized cost functional with respect to the design variable. In the last section, a convergence analysis is presented.

1. Assumptions and definitions

Throughout the paper we shall consider a class of two-dimensional domains (cross-sections of elastic bodies) $\{\Omega(v)\}$, where the design variable v belongs to the following set

$$\mathscr{U}_{\text{ad}} = \Big\{ v \in C^{(1),1}([0,1]) \mid \alpha \leqslant v(x_2) \leqslant \beta,$$
$$\left|\frac{\mathrm{d}v}{\mathrm{d}x_2}\right| \leqslant C_1 \text{ in } [0,1], \ \left|\frac{\mathrm{d}^2 v}{\mathrm{d}x_2^2}\right| \leqslant C_2 \text{ a.e. in } [0,1] \Big\}.$$

Here $C^{(1),1}$ denotes the space of continuously differentiable functions with Lipschitz derivatives, α, β, C_1, C_2 are given positive constants, $\alpha < \beta$.

Let the domain $\Omega(v)$ be defined as follows

$$\Omega(v) = \{x = (x_1, x_2) \mid 0 < x_1 < v(x_2), 0 < x_2 < 1\}$$

and let $\Gamma(v)$ denote the graph of the function v. Let the elastic body be in the state of plane strain. Assume that the following basic relations hold:

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial u_i}\right), \quad i,j = 1,2, \quad \text{(strain-displacement)}$$
$$\sigma_{ij} = c_{ijml} e_{ml}, \quad \text{(stress-strain)}$$

(any repeated index implies summation over the range 1,2), where the coefficients c_{ijml} are bounded and measurable in a rectangle

$$\Omega_\delta = (0,\delta) \times (0,1), \quad \delta > \beta.$$

Moreover, we assume that

$$c_{ijml} = c_{jiml} = c_{mlij}$$

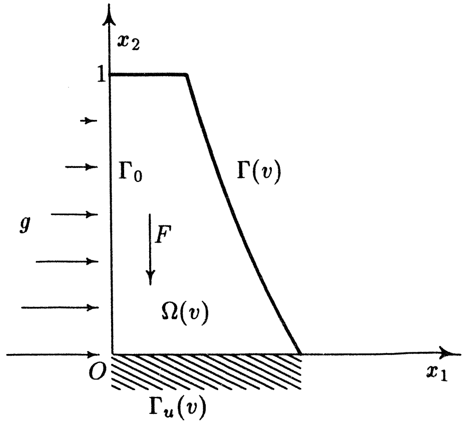

Fig. 1.

and a positive constant c_0 exists, such that

$$c_{ijml}(x)e_{ij}e_{ml} \geqslant c_0 e_{ij}e_{ij} \tag{1}$$

holds for almost all $x \in \Omega_\delta$ and for all $e \in \mathbf{R}_{\text{sym}}$. Here $\mathbf{R}_{\text{sym}}$ denotes the space of symmetric real 2×2 matrices.

Let the body be subjected to given body forces $F \in [L^2(\Omega_\delta)]^2$ and surface tractions $g \in [L^2(\Gamma_0)]^2$, where Γ_0 is a segment of the x_2-axis. (For example of a dam—see Fig. 1). Finally, let us consider the following boundary conditions .

$$u = 0 \quad \text{on} \quad \Gamma_u(v) = \{x \mid 0 < x_1 < v(0),\ x_2 = 0\}$$

(the body is fixed on $\Gamma_u(v)$) and

$$\begin{aligned} T &= g \quad \text{on} \quad \Gamma_0, \\ T &= 0 \quad \text{on} \quad \partial\Omega(v) - \Gamma_u(v) - \Gamma_0, \end{aligned}$$

where $T_i = \sigma_{ij}\nu_j$ denote components of the vector of surface tractions, ν_j are components of the unit outer normal to $\partial\Omega(v)$.

We introduce the space of virtual displacements

$$V(v) = \{w \in [H^1(\Omega)]^2 \mid w = 0 \text{ on } \Gamma_u(v)\},$$

$(H^1(\Omega)$ being the standard Sobolev space $W^{1,2}(\Omega))$, the virtual work of external forces

$$\mathscr{F}(v; w) = \int\limits_{\Omega(v)} F_i w_i \,\mathrm{d}x + \int\limits_{\Gamma_0} g_i w_i \,\mathrm{d}\Gamma$$

and the "internal virtual work"

$$a(v; w, y) = \int_{\Omega(v)} c_{ijml} e_{ij}(w) e_{ml}(y) \, dx, \quad w, y \in V(v).$$

Let us introduce the space of tensor fields defined on a domain Ω:

$$S(\Omega) = \{\sigma : \Omega \to \mathbf{R}_{\mathrm{sym}} \mid \sigma_{ij} \in L^2(\Omega), \quad i, j = 1, 2\}$$

with the inner product

$$\langle \sigma, \tau \rangle_\Omega = \int_\Omega \sigma_{ij} \tau_{ij} \, dx$$

and the associated norm

$$\|\sigma\|_{0,\Omega} = < \sigma, \sigma >_\Omega^{\frac{1}{2}} .$$

We shall use also the standard norm in $[H^k(\Omega)]^2, k = 0, 1, \ldots,$

$$\|w\|_{k,\Omega} = \Big(\sum_{i=1}^{2} \|w_i\|^2_{H^k(\Omega)} \Big)^{1/2}, \quad (H^0(\Omega) = L^2(\Omega)).$$

A (weak) solution of the elastostatic problem is defined as an element $u(v) \in V(v)$ such that

$$(2) \qquad a\big(v; u(v), w\big) = \mathscr{F}(v; w) \quad \forall w \in V(v).$$

Lemma 1.1. *There exists a unique solution $u(v)$ of the problem (2) for any $u \in \mathscr{U}^0_{\mathrm{ad}}$, where*

$$\mathscr{U}^0_{\mathrm{ad}} = \{v \in C^{(0),1}([0,1]) \text{ (i.e. Lipschitz functions)} \mid \alpha \leqslant v \leqslant \beta, \left|\frac{dv}{dx_2}\right| \leqslant C_1 \text{ a.e. in } [0,1]\}.$$

Moreover, a constant C_5 exists, independent of v, and such that

$$(3) \qquad \|u(v)\|_{1,\Omega(v)} \leqslant C_5 \quad \forall v \in \mathscr{U}^0_{\mathrm{ad}}.$$

Proof. There exists a positive constant C, indepenedent of $v \in \mathscr{U}^0_{\mathrm{ad}}$, such that

$$(4) \qquad \|e(w)\|_{0,\Omega(v)} \geqslant C \|w\|_{1,\Omega(v)} \quad \forall w \in V(v).$$

(For the proof see [7], Section 2.2 (i)). Using the property (1) of the coefficients c_{ijml}, we obtain

$$(5) \qquad a(v; w, y) \geqslant c_0 C^2 \|w\|^2_{1,\Omega(v)} \quad \forall w \in V(v).$$

In addition to that, we easily derive that

$$|a(v;w,y)| \leqslant C_3\|w\|_{1,\Omega(v)}\|y\|_{1,\Omega(v)}, \tag{6}$$

$$|\mathscr{F}(v;w)| \leqslant C_4\|w\|_{1,\Omega(v)} \quad \forall w,y \in V(v) \tag{7}$$

holds with constants C_3, C_4 independent of $v \in \mathscr{U}^0_{\text{ad}}$.

In proving (7), we write

$$|\mathscr{F}(v;w)| \leqslant \|F\|_{0,\Omega_\delta}\|w\|_{0,\Omega(v)} + \|g\|_{0,\Gamma_0}\|w\|_{0,\Gamma_0}$$

and use the estimate

$$\|w_j\|_{0,\Gamma_0} \leqslant C_0\|w_j\|_{1,\mathscr{R}_0} \leqslant C_0\|w_j\|_{1,\Omega(v)}, \quad j=1,2,$$

where $\mathscr{R}_0$ denotes the rectangle $(0,\alpha)\times(0,1)$ and the constant C_0 is indpendent of v.

The unique solvability of the problem (2) now follows from the Riesz Theorem, on the basis of (5),(6) and (7).

Inserting $u(v)$ for the test function in (2) and using (5), (7), we obtain

$$c_0C^2\|u(v)\|^2_{1,\Omega(v)} \leqslant \mathscr{F}(v;u(v)) \leqslant C_4\|u(v)\|_{1,\Omega(v)}.$$

Consequently, the estimate (3) follows with $C_5 = C_4/(c_0C^2)$. □

A *masonry-like elastic material* is weakly supporting tension, i.e., for a "small" parameter $k \geqslant 0$ we have

$$\sigma_{ij}t_it_j \leqslant k \quad \forall t \in \mathbf{R}^2, \quad \|t\| = 1, \tag{8}$$

where $\|.\|$ stands for the Euclidean norm. This requirement leads by the next Lemma 1.2 to the definition of a new set of admissible stress fields

$$M(\Omega) = \{\sigma \in S(\Omega) \mid \sigma_{11} \leqslant k,\ \sigma_{22} \leqslant k,\ \det(\sigma - \varkappa) \geqslant 0 \text{ a.e. in } \Omega\}, \tag{9}$$

where $k \in L^2(\Omega)$, $k \geqslant 0$ and

$$\varkappa = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}. \tag{10}$$

Lemma 1.2. *An equivalent definition of $M(\Omega)$ is*

$$M(\Omega) = \{\sigma \in S(\Omega) \mid \sigma_{ij}t_it_j \leqslant k \quad \forall t \in \mathbf{R}^2, \quad \|t\| = 1\}. \tag{11}$$

Proof. Let $\sigma \in \mathbf{R}_{\rm sym}$ and $k \in \mathbf{R}^1$ be arbitrary. As σ is symmetric, its characteristic polynomial

$$q(k) = (\sigma_{11} - k)(\sigma_{22} - k) - \sigma_{12}^2 = \det(\sigma - \varkappa) \tag{12}$$

has two real roots $\lambda_1(\sigma), \lambda_2(\sigma)$. Assuming that $\lambda_1(\sigma) \leqslant \lambda_2(\sigma)$, we may write

$$\lambda_1(\sigma) \leqslant \frac{\sigma_{11} + \sigma_{22}}{2} \leqslant \lambda_2(\sigma) = \max_{\|t\|=1} \sigma_{ij} t_i t_j, \tag{13}$$

since q attains its minimum just at the point $\frac{1}{2}(\sigma_{11} + \sigma_{22})$.

To prove the lemma it clearly suffices to show that

$$\lambda_2(\sigma) \leqslant k \tag{14}$$

is equivalent to

$$\sigma_{11} \leqslant k, \quad \sigma_{22} \leqslant k, \quad q(k) \geqslant 0. \tag{15}$$

So let (14) be valid. Then obviously $q(k) \geqslant 0$ as q is a convex parabola and $\lambda_2(\sigma)$ its greatest root. From (13) and (14) we have

$$\sigma_{11} + \sigma_{22} \leqslant 2\lambda_2(\sigma) \leqslant 2k. \tag{16}$$

Suppose for a moment that $\sigma_{11} > k$. Then by (16), $\sigma_{22} < k$ and from (12) we find that $q(k) < 0$, which is a contradiction. Hence, $\sigma_{11} \leqslant k$ and similarly we derive that $\sigma_{22} \leqslant k$.

Conversely, let (15) be valid. Then from (13) we get

$$\lambda_1(\sigma) \leqslant \max\{\sigma_{11}, \sigma_{22}\} \leqslant k. \tag{17}$$

But k cannot lie in the interval $(\lambda_1(\sigma), \lambda_2(\sigma))$ due to the condition $q(k) \geqslant 0$. Hence (14) holds. □

Lemma 1.3. *The set $M(\Omega)$ is convex and closed in $S(\Omega)$ for any domain $\Omega \subset \mathbf{R}^2$.*

Proof. 1°. Let $\xi \in [0,1]$ and $\tau, \sigma \in M(\Omega)$ be arbitrary. Then for any $t \in \mathbf{R}^2$, $\|t\| = 1$, we may write

$$(\xi\tau + (1-\xi)\sigma)_{ij} t_i t_j = \xi \tau_{ij} t_i t_j + (1-\xi)\sigma_{ij} t_i t_j \leqslant k \quad \text{a.e. in } \Omega,$$

making use of Lemma 1.2. Consequently, the set $M(\Omega)$ is convex.

2°. Let $\tau^n \to \tau$ in $S(\Omega)$, $\tau^n \in M(\Omega)$. Using the Lebesgue Theorem, we obtain a subsequence of $\{\tau^n\}$, still denoted by $\{\tau^n\}$, such that

$$\tau_{ij}^n(x) \to \tau_{ij}(x) \quad \text{for a.a.} \quad x \in \Omega, \quad i,j = 1,2.$$

Therefore, for any unit vector $t \in \mathbf{R}^2$ and a.a. $x \in \Omega$ we have

$$\lim_{n\to\infty} \tau_{ij}^n(x) t_i t_j = \tau_{ij}(x) t_i t_j \leqslant k,$$

using again Lemma 1.2 for $\tau^n \in M(\Omega)$. Thus $\tau \in M(\Omega)$ and $M(\Omega)$ is closed. □

Henceforth we assume that a function $k \in L^2(\Omega_\delta)$ is given, $k \geqslant 0$. Let $\sigma(v) \in S(\Omega(v))$ be defined by means of the stress-strain relation and strain-displacement relation:

$$\sigma_{ij}(v) = c_{ijml} e_{ml}\big(u(v)\big), \quad i, j = 1, 2, \tag{18}$$

where $u(v)$ is the solution of (2). We introduce the set of statically admissible design variables

$$\mathscr{E}_{\text{ad}} = \{v \in \mathscr{U}_{\text{ad}} \mid \sigma(v) \in M\big(\Omega(v)\big)\}$$

and the *Optimal Design Problem*

$$v_0 = \operatorname*{argmin}_{v \in \mathscr{E}_{\text{ad}}} j(v), \tag{19}$$

where

$$j(v) = \int_{\Omega(v)} p(x)\,\mathrm{d}x$$

represents the weight of the body, p is the specific weight, $p \in L^2(\Omega_\delta)$, $p > 0$. Note that the body forces are—in case of the dam— $F_1 = 0$, $F_2 = -p$ and p is constant or piecewise constant.

2. Existence of an optimal domain

In order to prove that the Optimal Design Problem (19) has at least one solution, we shall need the following assertion.

Proposition 2.1. *Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of functions $v_n \in \mathscr{U}_{\text{ad}}^0$ such that*

$$\lim_{n\to\infty} v_n = v \quad \textit{in} \quad C([0,1]).$$

Then

$$\tilde{\sigma}(v_n) \to \tilde{\sigma}(v) \quad \textit{in} \quad S(\Omega_\delta), \tag{20}$$

where $\tilde{\sigma}(v_n)$ and $\tilde{\sigma}(v)$ denote the extension of $\sigma(v_n)$ and $\sigma(v)$ by zero to the set $\Omega_\delta - \Omega(v_n)$ and $\Omega_\delta - \Omega(v)$, respectively.

In the proof the basic role is played by the following result.

Lemma 2.1. *Let the assumptions of Proposition 2.1 be fulfilled. Then*

$$u(v_m)|_{G_m} \rightharpoonup u(v)|_{G_m} \quad \textit{(weakly) in } [H^1(G_m)]^2$$

holds for $n \to \infty$ and any integer $m > 1/\alpha$, where

$$G_m = \Big\{x \mid 0 < x_1 < v(x_2) - \frac{1}{m},\ 0 < x_2 < 1\Big\}.$$

P r o o f. Let us set $u_n = u(v_n)$, $\Omega_n = \Omega(v_n)$, $\Omega = \Omega(v)$, $u = u(v)$. We introduce an extension of u_n as follows

$$\hat{u}_{nj}(x) = u_{nj}(x^*), \quad j = 1, 2,$$

where

$$x^* = \big(2v_n(x_2) - x_1, x_2\big) \quad \text{for} \quad x \in \Omega_\alpha^n - \Omega_n,$$
$$\Omega_\alpha^n = \big\{(x_1, x_2) \mid 0 < x_1 < v_n(x_2) + \alpha,\ 0 < x_2 < 1\big\} = \Omega(v_n + \alpha).$$

Since for a.a. points $x \in \Omega_\alpha^n - \Omega_n$ we have

$$|\nabla \hat{u}_{nj}(x)|^2 \leqslant \big(2 + 4(v_n')^2\big)|\nabla u_{nj}(x^*)|^2, \quad j = 1, 2,$$

it is easy to derive the upper bound

$$\|\hat{u}_{nj}\|^2_{1,\Omega_\alpha^n} \leqslant (1 + C)\|u_{nj}\|^2_{1,\Omega_n}$$

with C independent of n. Lemma 1.1 yields that

$$\|u_n\|_{1,\Omega_n} \leqslant C_5, \tag{21}$$

so that

$$\|\hat{u}_n\|^2_{1,\Omega_\alpha^n} \leqslant (1 + C)C_5^2.$$

Since

$$\Omega_\alpha^n \supset \Omega_\alpha \equiv \Omega(v + \tfrac{1}{2}\alpha) \quad \forall n > n_0(\alpha),$$

it is readily seen that

$$\|\hat{u}_n\|_{1,\Omega_\alpha} \leqslant C_5(1 + C)^{1/2} \quad \forall n > n_0(\alpha). \tag{22}$$

There exists $\bar{u} \in [H^1(\Omega)]^2$ and a subsequence $\{u_{n_r}\}$ of $\{u_n\}$(which we denote for simplicity only by $\{u_r\}$) such that

$$\hat{u}_r \rightharpoonup \bar{u} \quad \text{(weakly) in} \quad [H^1(\Omega_\alpha)]^2. \tag{23}$$

Since $\hat{u}_r \in V(v + \frac{1}{2}\alpha)$ and $V(v + \frac{1}{2}\alpha)$ is weakly closed, we conclude that $\bar{u} \in V(\Omega_\alpha)$.

Let a test function $w \in V(v)$ be given. The symmetric extension $\hat{w}$ (with respect to $\Gamma(v)$) belongs to $V(v_r)$ for all r sufficiently large and we may write

$$a(v_r; u_r, \hat{w}) = \mathscr{F}(v_r; \hat{w}), \tag{24}$$

(we assume that r is such that $\Omega(v_r) \subset \Omega_\alpha$). Let us pass to the limit with $r \to \infty$. Then we have

$$|a(v_r; u_r, \hat{w}) - a(v; \bar{u}, w)| \leqslant |a(v_r; u_r, \hat{w}) - a(v; \hat{u}_r, \hat{w})| + |a(v; \hat{u}_r - \bar{u}, w)| \equiv K_1 + K_2,$$

$$K_1 \leqslant \int_{\Delta(\Omega_r, \Omega)} |c_{ijml} e_{ij}(\hat{u}_r) e_{ml}(\hat{w})| \, \mathrm{d}x \leqslant C_3 \|\hat{u}_r\|_{1,\Omega_\alpha} \|\hat{w}\|_{1,\Delta(\Omega_r,\Omega)} \to 0$$

where

$$\Delta(\Omega_r, \Omega) = (\Omega_r - \Omega) \cup (\Omega - \Omega_r), \tag{25}$$
$$\lim_{r \to \infty} \operatorname{meas} \Delta(\Omega_r, \Omega) = 0,$$

and the bound (22) have been used. Since also $K_2 \to 0$ by virtue of (23), we arrive at

$$\lim_{r \to \infty} a(v_r; u_r, \hat{w}) = a(v; \bar{u}, w). \tag{26}$$

It is easy to derive that

$$\lim_{r \to \infty} \mathscr{F}(v_r; \hat{w}) = \mathscr{F}(v; w). \tag{27}$$

In fact, using (25), we obtain that

$$|\mathscr{F}(v_r; \hat{w}) - \mathscr{F}(v; w)| = \left| \int_{\Omega_r} F_i \hat{w}_i \, \mathrm{d}x - \int_{\Omega} F_i w_i \, \mathrm{d}x \right| \leqslant \int_{\Delta(\Omega_r, \Omega)} |F_i w_i \, \mathrm{d}x| \to 0.$$

Combining (24), (26) and (27), we are led to the equation

$$a(v; \bar{u}, w) = \mathscr{F}(v; w).$$

By Lemma 1.1 the problem (2) is uniquely solvable in $\Omega(v)$, so that $\bar{u}|_\Omega = u(v)$ follows, since also $\bar{u}|_\Omega \in V(v)$.

It is readily seen that (23) yields

$$\hat{u}_r|_{G_m} = u_r|_{G_m} \rightharpoonup u(v)|_{G_m} \quad \text{(weakly) in} \quad V\left(v - \frac{1}{m}\right) \quad \forall m > \tfrac{1}{2}\alpha.$$

From the uniqueness of the solution $u(v)$ we conclude that the whole original sequence $\{u_n|_{G_m}\}_{n=1}^{\infty}$ converges to $u(v)|_{G_m}$. □

Proof of Proposition 2.1. Let us put $\sigma_n = \sigma(v_n)$, $\sigma = \sigma(v)$. It is readily seen that

$$\text{(28)} \qquad \sigma_n\big|_{G_m} \rightharpoonup \sigma\big|_{G_m} \quad \text{(weakly) in} \quad S(G_m) \quad \forall m > 1/\alpha$$

is a consequence of Lemma 2.1 and the relation (18).

Let us consider any $\varphi \in S(\Omega_\delta)$. Then we may write

$$\begin{aligned}
&\big|\langle\varphi, \tilde{\sigma}_n\rangle_{\Omega_\delta} - \langle\varphi, \tilde{\sigma}\rangle_{\Omega_\delta}\big| \leqslant \\
&\qquad \leqslant \big|\langle\varphi, \sigma_n\rangle_{G_m} - \langle\varphi, \sigma\rangle_{G_m} + \langle\varphi, \sigma_n\rangle_{\Omega_n - G_m} - \langle\varphi, \sigma\rangle_{\Omega - G_m}\big| \\
&\qquad \leqslant \big|\langle\varphi, \sigma_n\rangle_{G_m} - \langle\varphi, \sigma\rangle_{G_m}\big| + \big|\langle\varphi, \sigma_n\rangle_{\Omega_n - G_m}\big| + \big|\langle\varphi, \sigma\rangle_{\Omega - G_m}\big| \\
&\qquad \equiv K_1 + K_2 + K_3 \to 0
\end{aligned}$$

for $n \to \infty$. In fact, from (28) we conclude that

$$K_1 \to 0.$$

Moreover,

$$K_2 \leqslant \|\varphi\|_{0,\Omega_n - G_m}\|\sigma_n\|_{0,\Omega_n} \to 0 \quad \text{if} \quad n \to \infty, \quad m \to \infty, \quad n > n_1(m),$$

since

$$\|\sigma_n\|_{0,\Omega_n} \leqslant C \quad \forall n = 1, 2, \dots$$

follows from Lemma 1.1 (see(21)) and

$$\text{meas}(\Omega_n - G_m) \leqslant \frac{1}{m} + \|v_n - v\|_\infty \to 0.$$

By definition of G_m,

$$K_3 \to 0 \quad \text{for} \quad m \to \infty.$$

Combining the three limits, we obtain that

$$\text{(29)} \qquad \tilde{\sigma}_n \rightharpoonup \tilde{\sigma} \quad \text{(weakly) in} \quad S(\Omega_\delta).$$

Next substituting for the strain from the inverse strain-stress relation

$$e(u_n) = b\sigma_n,$$

we obtain

$$\text{(30)} \qquad \langle\tilde{\sigma}_n, b\tilde{\sigma}_n\rangle_{\Omega_\delta} = a(v_n; u_n, u_n) = \mathscr{F}(v_n; u_n).$$

Let us show that

(31) $$\lim_{n\to\infty} \mathscr{F}(v_n; u_n) = \mathscr{F}(v; u).$$

In fact, we have

$$\left|\int_{\Omega_n} F\cdot u_n \,\mathrm{d}x - \int_{\Omega} F\cdot u \,\mathrm{d}x\right| \leqslant \left|\int_{G_m} F\cdot (u_n - u)\,\mathrm{d}x\right|$$

$$+\left|\int_{\Omega_n - G_m} F\cdot u_n \mathrm{d}x\right| + \left|\int_{\Omega - G_m} F\cdot u\,\mathrm{d}x\right| \equiv$$

$$\equiv J_1 + J_2 + J_3,$$

$$J_1 \to 0 \quad \text{for} \quad n\to\infty,$$

by virtue of Lemma 2.1,

$$J_2 \leqslant \|F\|_{0,\Omega_n - G_m} \|u_n\|_{0,\Omega_n} \to 0$$

using (3) and

$$\operatorname{meas}(\Omega_n - G_m) \to 0 \quad \text{for} \quad n\to\infty, \quad m\to\infty, \quad n > n_1(m).$$

Finally,

$$J_3 \to 0 \quad \text{for} \quad m\to\infty.$$

The convergence

$$\int_{\Gamma_0} g\cdot u_n \mathrm{d}\Gamma \to \int_{\Gamma_0} g\cdot u\,\mathrm{d}\Gamma$$

follows from Lemma 2.1. Combining these results, we arrive at (31). Since

(32) $$\mathscr{F}(v; u) = a(v; u, u) = \langle\tilde{\sigma}(v), b\tilde{\sigma}(v)\rangle_{\Omega_\delta},$$

from (30), (31) and (32) we obtain that

(33) $$\langle\tilde{\sigma}_n, b\tilde{\sigma}_n\rangle_{\Omega_\delta} \to \langle\tilde{\sigma}(v), b\tilde{\sigma}(v)\rangle_{\Omega_\delta}.$$

The norm $\|\tau\|_{0,\Omega_\delta}$ is equivalent to the energy norm

$$\|\tau\|_{\Omega_\delta} = \langle\tau, b\tau\rangle_{\Omega_\delta}^{1/2}.$$

Then

$$\|\tilde{\sigma}_n - \tilde{\sigma}(v)\|_{0,\Omega_\delta} \leqslant C\|\tilde{\sigma}_n - \tilde{\sigma}(v)\|_{\Omega_\delta} \to 0$$

follows from the convergence of the norms (33) and the weak convergence (29). □

Lemma 2.2. *The set $\mathscr{E}_{\rm ad}$ is compact in $C^1([0,1])$.*

Proof. The set $\mathscr{U}_{\rm ad}$ is compact in $C^1([0,1])$—see [6],Lemma 2. Let $\mathscr{E}_{\rm ad} \neq \emptyset$ and consider a sequence $\{v_n\} \subset \mathscr{E}_{\rm ad}$. Then there exists a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, such that

$$(34) \qquad v_n \to v \quad \text{in} \quad C^1([0,1]), \quad v \in \mathscr{U}_{\rm ad}.$$

It is sufficient to verify that $\sigma(v) \in M(\Omega(v))$. Since

$$(35) \qquad \tilde{\sigma}(v_n) \in M(\Omega_\delta) \quad \forall n$$

and the set $M(\Omega_\delta)$ is closed by Lemma 1.3, Proposition 2.1 implies that

$$(36) \qquad \lim_{n\to\infty} \tilde{\sigma}(v_n) = \tilde{\sigma}(v) \in M(\Omega_\delta).$$

Consequently, $\sigma(v) \in M(\Omega(v))$. □

Remark 2.1. Note that the *strong* convergence of stress fields is superfluous here, as $M(\Omega_\delta)$ is weakly closed (see Lemma 1.3) and the weak convergence (29) is sufficient to prove Lemma 2.2.

Later on, however, we shall need the strong convergence to show the continuity of penalty functionals in Section 6.

Remark 2.2. We can see that $M(\Omega)$ is a convex cone with its vertex at the point $\varkappa \in S(\Omega)$, $\varkappa_{ij} = k\delta_{ij}$.

Theorem 2.1. *Let the set $\mathscr{E}_{\rm ad}$ be non-empty. Then there exists at least one solution of the Optimal Design Problem* (19).

Proof. By Lemma 2.2, the set $\mathscr{E}_{\rm ad}$ is compact in $C^1([0,1])$. Since the functional $j : C^1([0,1]) \to \mathbb{R}^1$ is continuous, it attains at least one minimum in the set $\mathscr{E}_{\rm ad}$. □

Remark 2.3. In case of a masonry dam and $k = 0$, we can satisfy the assumption $\mathscr{E}_{\rm ad} \neq \emptyset$, employing some results from [2].

3. Approximate solution

We shall propose an approximate solution of the Optimal Design Problem (19), which is based on piecewise linear approximations of the unknown boundary, finite element method for solving the elastostatic problem and penalty approach to satisfy the constraints on the stress field.

Let N be a positive integer, $h = 1/N$ and $\Delta_j = [(j-1)h, jh]$, $j = 1, \ldots, N$. We define

$$\mathscr{U}_{\mathrm{ad}}^h = \big\{v_h \in C([0,1]) \mid v_h\big|_{\Delta_j} \in P_1(\Delta_j),\ j = 1, \ldots, N,\ \alpha \leqslant v_h \leqslant \beta,$$
$$\left|\frac{\mathrm{d}v_h}{\mathrm{d}x_2}\right| \leqslant C_1,\ |\delta_h^2 v_h(jh)| \leqslant C_2,\ j = 1, \ldots, N-1\big\},$$

where $P_1(\Delta_j)$ is the space of linear polynomials defined on Δ_j,

$$\delta_h^2 v_h(jh) = h^{-2}\big[v_h\big((j+1)h\big) - 2v_h(jh) + v_h\big((j-1)h\big)\big]. \tag{37}$$

Let $\Omega_h = \Omega(v_h)$. The polygonal domain Ω_h will be divided into triangles as follows. Choose an initial function $v_h^0 \in P_1([0,1]) \cap \mathscr{U}_{\mathrm{ad}}^h$ and construct a regular system of uniform triangulations $\mathscr{T}_h(v_h^0)$. (For instance, in case of the dam we set— in accordance with the results of [2]—

$$v_h^0(x_2) = a - x_2/\sqrt{b}, \quad b = |F_2|/|\mathrm{d}g_1/\mathrm{d}x_2|,\ a = \mathrm{const.})$$

For a general $v_h \in \mathscr{U}_{\mathrm{ad}}^h$ we construct $\mathscr{T}_h(v_h)$ as a "distortion" of the initial triangulation $\mathscr{T}_h(v_h^0)$, preserving the number of nodes on any straight line segment

$$\{x \mid x_1 \in [0, v_h(jh)], x_2 = jh\}, \quad j = 0, \ldots, N, \tag{38}$$

and the uniform partition of these segments.

We shall employ the standard finite element space with linear polynomials on any triangle $K \in \mathscr{T}_h(v_h)$:

$$V_h(v_h) = \big\{w_h \in [C(\bar{\Omega}_h)]^2 \mid w_h\big|_K \in [P_1(K)]^2\ \forall K \in \mathscr{T}_h(v_h),\ w_h = 0 \text{ on } \Gamma_u(v_h)\big\}.$$

The finite element solution of the elastostatic problem (2) will be defined as a function $u_h(v_h) \in V_h(v_h)$ such that

$$a\big(v_h; u_h(v_h), w_h\big) = \mathscr{F}(v_h; w_h) \quad \forall w_h \in V_h(v_h). \tag{39}$$

Since $V_h(v_h) \subset V(v_h)$ and $v_h \in \mathscr{U}_{\mathrm{ad}}^0$, we easily prove that the problem (39) has a unique solution for any $v_h \in \mathscr{U}_{\mathrm{ad}}^h$, making use of (5), (6) and (7). Moreover, we obtain that

$$\|u_h(v_h)\|_{1,\Omega(v_h)} \leqslant C_5 \quad \forall v_h \in \mathscr{U}_{\mathrm{ad}}^h, \tag{40}$$

where C_5 is a constant independent of v_h.

Let us define an approximate stress field $\sigma^h(v_h)$ by means of the formulae

$$\sigma^h_{ij}(v_h) = c_{ijml} e_{ml}\big(u_h(v_h)\big), \quad i,j = 1,2.$$

Remark 3.1. The stress field $\sigma^h(v_h)$ is piecewise constant, i.e. $\sigma^h_{ij}(v_h) \in P_0(K)$ for all $K \in \mathscr{T}_h(v_h)$, provided the coefficients c_{ijml} are piecewise constant.

Next let us introduce the following functions

$$f_i(v,\sigma) = \int\limits_{\Omega(v)} (\sigma_{ii} - k)^+ \mathrm{d}x, \quad i = 1,2, \qquad \text{(no sum)}$$

$$f_3(v,\sigma) = \int\limits_{\Omega(v)} \big(\det(\sigma - \varkappa)\big)^- \mathrm{d}x, \quad (\varkappa_{ij} = k\delta_{ij})$$

where $(.)^+$ and $(.)^-$ denotes the positive and negative part, respectively.

A *Penalized Cost Functional* will be defined as follows

$$j_\varepsilon(v,\sigma) = j(v) + \frac{1}{\varepsilon}\sum_{i=1}^{3} f_i(v,\sigma), \quad \varepsilon > 0. \tag{41}$$

Finally, we introduce the *Approximate Optimal Design Problem*

$$v^\varepsilon_h = \operatorname*{argmin}_{v_h \in \mathscr{U}^h_{ad}} j_\varepsilon\big(v_h, \sigma^h(v_h)\big). \tag{42}$$

Theorem 3.1. *The approximate problem* (42) *has at least one solution* v^ε_h *for any fixed* $h = 1/N$ *and any real positive parameter* ε.

The proof is based on the two following lemmas.

Lemmma 3.1. *Let* $h = 1/N$ *be fixed and let* $v^n_h \in \mathscr{U}^h_{\mathrm{ad}}$,

$$\lim_{n\to\infty} v^n_h = v_h \quad \text{in} \quad C([0,1]).$$

Then

$$\lim_{n\to\infty} \tilde\sigma^h(v^n_h) = \tilde\sigma^h(v_h) \quad \text{in} \quad S(\Omega_\delta), \tag{43}$$

where $\tilde\sigma^h$ *denote extensions of* σ^h *by zero.*

Proof. For brevity, we set $v^n = v^n_h$, $v = v_h$, $\Omega_n = \Omega(v^n_h)$, $\Omega = \Omega(v_h)$. Let $\{w^n_r\}$, $r = 1, \ldots, d$, be basis functions of the subspace $V_h(v^n)$. (Note that d is independent of n due to the construction of $\mathcal{T}_h(v^n)$.) The solution $u_h(v^n)$ can be expressed as follows

$$u_h(v^n) = \sum_{r=1}^{d} U_r^n w_r^n$$

and the vector of coefficients U^n satisfies the linear system

$$K(v^n)U^n = b(v^n),$$

where the entries

$$K_{ij}(v^n) = a(v^n; w_i^n, w_j^n), \quad b_i(v^n) = \mathcal{F}(v^n; w_i^n)$$

depend continuously on v^n. (Cf. e.g. [10] for the details of the proof.) Passing to the limit with $n \to \infty$, we obtain

$$u_h(v) = \sum_{r=1}^{d} U_r w_r,$$

where

$$U = K^{-1}(v)b(v) = \lim_{n\to\infty} U^n \quad \text{in} \quad \mathbf{R}^d$$

and $\{w_r\}$ are basis functions of the subspace $V_h(v)$.

Let $\tilde{w}_r^n$ and $\tilde{w}_r$ be extensions of w_r^n and w_r by zero into the domains $\Omega_\delta - \Omega_n$ and $\Omega_\delta - \Omega$, respectively. Since we may write

$$\tilde{\sigma}_{ij}^h(v_h) = c_{ijml} e_{ml}\Big(\sum_{r=1}^{d} U_r^n \tilde{w}_r^n\Big) = \sum_{r=1}^{d} U_r^n c_{ijml} e_{ml}(\tilde{w}_r^n)$$

and since for the extensions

$$\lim_{n\to\infty} e_{ml}(\tilde{w}_r^n) = e_{ml}(\tilde{w}_r) \quad \text{in} \quad S(\Omega_\delta)$$

holds (cf. [10]), we obtain the assertion (43). □

Lemma 3.2. *Let the assumptions of Lemma 3.1. be fulfilled. Then*

$$\lim_{n\to\infty} f_i\big(v_h^n, \sigma^h(v_h^n)\big) = f_i\big(v_h, \sigma^h(v_h)\big), \quad i = 1,2,3.$$

P r o o f. First consider $i = 1,2$. We may write

$$\big|f_i(v^n, \sigma^h(v^n)) - f_i(v, \sigma^h(v))\big| = \Big|\int_{\Omega_\delta} \big[(\tilde{\sigma}_{ii}^h(v^n) - k)^+ - (\tilde{\sigma}_{ii}^h(v) - k)^+\big] \mathrm{d}x\Big| \leqslant$$

$$\leqslant \int_{\Omega_\delta} \big|\tilde{\sigma}_{ii}^h(v^n) - \tilde{\sigma}_{ii}^h(v)\big| \mathrm{d}x \leqslant C \big\|\tilde{\sigma}^h(v^n) - \tilde{\sigma}^h(v)\big\|_{0,\Omega_\delta} \to 0$$

by virtue of Lemma 3.1 and the inequality

$$|a^+ - b^+| \leqslant |a - b|.$$

Let us consider the case $i = 3$. Then we have

$$\begin{aligned}\left|f_3(v^n, \sigma^h(v^n)) - f_3(v, \sigma^h(v))\right| &= \left|\int_{\Omega_\delta} \left[(\det(\tilde{\sigma}^h(v^n) - \varkappa))^- (\det(\tilde{\sigma}^h(v) - \varkappa))^-\right] \mathrm{d}x\right| \\ &\leqslant \int_{\Omega_\delta} \left|\det(\tilde{\sigma}^h(v^n) - \varkappa) - \det(\tilde{\sigma}^h(v) - \varkappa)\right| \mathrm{d}x \to 0\end{aligned}$$

using again Lemmma 3.1 and the inequality

$$|a^- - b^-| \leqslant |a - b|.$$

□

Proof of Theorem 3.1. Making use of Lemma 3.2, we derive

$$\lim_{n\to\infty} j_\varepsilon(v^n, \sigma^h(v^n)) = j_\varepsilon(v, \sigma^h(v)) \quad \text{for} \quad v^n \to v \quad \text{in} \quad C([0,1]). \tag{44}$$

Let us introduce a vector $a \in \mathbb{R}^{N+1}$ with components

$$a_i = v_h(ih), \quad i = 0, \ldots, N,$$

and let

$$\mathscr{A} = \{a \in \mathbb{R}^{N+1} \mid \exists v_h \in \mathscr{U}_{\text{ad}}^h\colon a_i = v_h(ih),\ i = 0, \ldots, N\}.$$

By virtue of (44), the function

$$a \mapsto j_\varepsilon\left(v_h(a), \sigma^h\left(v_h(a)\right)\right) \tag{45}$$

is continuous in the set $\mathscr{A}$ which is clearly compact. Consequently, the minimum is attained in $\mathscr{A}$. □

The primal variational formulation (2) and the corresponding approximate solution defined by the system (39) is not the only possible way how to calculate the stress fields. Next we shall show another way, based on a well-known dual variational formulation, i.e., the principle of minimum complementary energy (Castigliano-Menabrea).

Let us define the inner product and the associated norm

$$(\sigma, \tau)_\Omega \equiv \int_\Omega b_{ijml}\sigma_{ij}\tau_{ml}\,\mathrm{d}x, \quad \|\sigma\|_\Omega \equiv (\sigma, \sigma)_\Omega^{\frac{1}{2}},$$

where b_{ijml} are coefficients of the inverse strain-stress relation

$$e_{ij} = b_{ijml}\sigma_{ml}, \quad i, j = 1, 2.$$

Let us introduce the set of equilibrium stress fields

$$E(v) = \{\sigma \in S(\Omega(v)) \mid \langle \sigma, e(w)\rangle_{\Omega(v)} = \mathscr{F}(v; w) \quad \forall w \in V(v)\}.$$

The actual stress field $\sigma(v)$ coincides with the minimizer of the complementary energy

$$\mathscr{S}(v; \tau) = \frac{1}{2}\,\|\tau\|_{\Omega(v)}^2$$

over the set $E(v)$, i.e.

$$\sigma(v) = \operatorname*{argmin}_{\tau \in E(v)} \mathscr{S}(v; \tau). \tag{46}$$

The problem (46) has a unique solution for any $v \in \mathscr{U}_{\mathrm{ad}}^0$ (see e.g. [9]-Chapter 7).

Let us apply the finite element method to the approximate solution of the problem (46). We introduce the following subsets

$$H_h(v_h) = \{\tau^h \in S(\Omega(v_h)) \mid \tau_{ij}^h\big|_K \in P_0(K) \quad \forall K \in \mathscr{T}_h(v_h), \quad i, j = 1, 2\},$$

$$E_h(v_h) = \{\tau^h \in H_h(v_h) \mid \langle \tau^h, e(w_h)\rangle_{\Omega(v_h)} = \mathscr{F}(v_h; w_h) \quad \forall w_h \in V_h(v_h)\}.$$

Note that $E_h(v_h)$ is an *external* approximation of the set $E(v_h)$, since $E_h(v_h) \not\subset E(v_h)$.

The Castigliano principle (46) leads us to the following finite-dimensional problem: Find $\sigma^h \in E_h(v_h)$ such that

$$(\sigma^h, \tau^h)_{\Omega_h} = 0 \quad \forall \tau^h \in E_h^0(v_h), \tag{47}$$

where $\Omega_h \equiv \Omega(v_h)$ and

$$E_h^0(v_h) = \{\tau^h \in H_h(v_h) \mid \langle \tau^h, e(w_h)\rangle_{\Omega_h} = 0 \quad \forall w_h \in V_h(v_h)\}.$$

Lemma 4.1. *The problem (47) has a unique solution $\sigma^h(v_h)$ for any $v_h \in \mathcal{U}_{\text{ad}}^h$ and any $h = \frac{1}{N}$.*

Proof. The set $E(v_h)$ is not empty, since $\sigma(v_h)$, defined by the formula (8), belongs to $E(v_h)$ (cf. (2) and Lemma 1.1).

Let us define the projection mapping

$$r_h \colon S(\Omega_h) \to H_h(v_h)$$

by means of the following relation

$$\langle \tau - r_h\tau, \omega^h\rangle_{\Omega_h} = 0 \quad \forall \omega^h \in H_h(v_h).$$

Then $r_h\sigma(v_h) \in E_h(v_h)$. In fact, for any $w_h \in V_h(v_h)$ we have $e(w_h) \in H_h(v_h)$ and $w_h \in V(v_h)$, so that

$$\langle r_h\sigma(v_h), e(w_h)\rangle_{\Omega_h} = \langle \sigma(v_h), e(w_h)\rangle_{\Omega_h} = \mathcal{F}(v_h; w_h).$$

Consequently, $E_h(v_h)$ is also non-empty. The problem (47) is equivalent with the following one

$$\sigma^h = \underset{\tau^h \in E_h(v_h)}{\operatorname{argmin}} \frac{1}{2}\|\tau^h\|_{\Omega_h}^2. \tag{48}$$

Since the set $E_h(v_h)$ is convex and closed in $S(\Omega_h)$, the unique solvability of the problem (47) follows. □

Let us consider the modified Approximate Optimal Design Problem (42), where $\sigma^h(v_h)$, however, is determined by the solution of the problem (47). Then Theorem 3.1 remains true, as well as Lemmas 3.1 and 3.2. In the proof, we have only to replace the proof of Lemma 3.1 by a new argument as follows.

Proof of Lemma 3.1. 1° Recall the abbreviated notations $v^n \equiv v_h^n$, $v \equiv v_h$, $\Omega_n \equiv \Omega(v_h^n)$. From (48) we obtain

$$\left(\sigma^h(v^n), \tau^h - \sigma^h(v^n)\right)_{\Omega_n} \geqslant 0 \quad \forall \tau^h \in E_h(v^n). \tag{49}$$

Substituting

$$\tau^h = r_h^n\sigma(v^n) \in E_h(v^n)$$

(cf. the proof of Lemma 4.1), we may write

$$(50)\qquad \|\sigma^h(v^h)\|^2_{\Omega_n} \leqslant \left(\sigma^h(v_h), r^n_h\sigma(v^n)\right)_{\Omega_n} \leqslant \|\sigma^h(v^n)\|_{\Omega_n}\| \, r^n_h\sigma(v^n)\|_{\Omega_n}.$$

Properties of the coefficients b_{ijml} are quite analogous to that of c_{ijml}, so that a positive constant b_0 exists such that

$$(51)\qquad b_0^{-1}\|\tau\|_{0,\Omega(v)} \leqslant \|\tau\|_{\Omega(v)} \leqslant b_0\|\tau\|_{0,\Omega(v)} \quad \forall v \in \mathscr{U}^0_{\text{ad}}\ \forall \tau \in S(\Omega(v)).$$

Consequently, we obtain

$$\|r^n_h\sigma(v^n)\|_{\Omega_n} \leqslant b_0\|r^n_h\sigma(v^n)\|_{0,\Omega_n} \leqslant b_0\|\sigma(v^n)\|_{0,\Omega_n} \leqslant b_0 C$$

using also (3) from Lemma 1.1, the formula (18) and the boundedness of c_{ijml}. Then (50) implies the estimate

$$(52)\qquad \|\tilde{\sigma}^h(v^n)\|_{0,\Omega_\delta} = \|\sigma^h(v^n)\|_{0,\Omega_n} \leqslant b_0\|\sigma^h(v^n)\|_{\Omega_n} \leqslant b_0^2 C \quad \forall n.$$

Therefore, $\sigma \in S(\Omega_\delta)$ and a subsequence (which will be denoted by the same symbol) exist such that

$$(53)\qquad \tilde{\sigma}^h(v^n) \rightharpoonup \sigma \quad \text{(weakly) in } S(\Omega_\delta).$$

2° We show that

$$(54)\qquad \sigma = 0 \quad \text{a.e. in } \Omega_\delta - \Omega(v).$$

Let $\|\sigma\|_{0,D} > 0$ for some measurable set $D \subset \Omega_\delta - \Omega(v)$. Introducing the characteristic function χ_D of the set D, we obtain for $n \to \infty$

$$\left\langle \tilde{\sigma}^h(v^n), \chi_D\sigma \right\rangle_{\Omega_\delta} \to \langle \sigma, \chi_D\sigma \rangle_{\Omega_\delta} = \|\sigma\|^2_{0,D} > 0.$$

On the other hand,

$$\left| \left\langle \tilde{\sigma}^h(v^n), \chi_D\sigma \right\rangle_{\Omega_\delta} \right| = \left| \left\langle \tilde{\sigma}^h(v^n), \sigma \right\rangle_{D\cap\Omega_n} \right| \leqslant \|\tilde{\sigma}^h(v^n)\|_{0,\Omega_\delta}\|\sigma\|_{0,D\cap\Omega_n} \to 0$$

by virtue of (52) and the fact that

$$\operatorname{meas}(D \cap \Omega_n) \to 0.$$

Thus we arrive at a contradiction and (54) holds.

3° Let us show that

$$\sigma\big|_{\Omega(v)} \in E_h(v). \tag{55}$$

We can write

$$\sigma^h(v^n) = \sum_{i=1}^{s} \hat{\sigma}_i(v^n)\vartheta^{(i)}(v^n), \tag{56}$$

where $\vartheta^{(i)}(v^n)$ are basis functions of the space $H_h(v^n)$, $s = \dim H_h(v^n)$ being independent of n, $\hat{\sigma}(v^n) \in \mathbf{R}^s$.

One can prove that positive constants n_0 and C_0 exist such that

$$\|\omega\|_{0,\Omega_n} \geqslant C_0 \|\hat{\omega}\|_{\mathbf{R}^s} \tag{57}$$

holds for all $n > n_0$ and $\omega \in H_h(v^n)$.

From (52) and (57) we conclude that a subsequence of $\{\hat{\sigma}(v_h)\}$ exists such that

$$\hat{\sigma}(v^n) \to \hat{\sigma} \text{ in } \mathbf{R}^s, \tag{58}$$

where $\hat{\sigma}$ is the vector of coefficients of σ in $\Omega(v)$. In addition to that, the convergence in (53) is even *strong*, since

$$\|\mathscr{F}^{(i)}(v^n) - \mathscr{F}^{(i)}(v)\|_{0,\Omega_\delta} \to 0 \quad \text{for } n \to \infty,\ i = 1, \ldots, s. \tag{59}$$

It is easy to realize that

$$\sigma \in E_h(v) \Leftrightarrow A(v)\hat{\sigma} = \mathscr{F}(v),$$

where the matrix $A(v)$ has the following entries

$$\left\langle \vartheta^{(i)}, e(w_r) \right\rangle_{\Omega(v)}$$

and w_r are basis functions of the space $V_h(v)$. The entries of $\mathscr{F}(v)$ are obvious. Since

$$\sigma^h(v^n) \in E_h(v^n) \Rightarrow A(v^n)\hat{\sigma}(v^n) = \mathscr{F}(v^n)$$

and $A(v^n)$, $\mathscr{F}(v^n)$ depend on v^n continuously, passing to the limit with $n \to \infty$ and using also (58), we arrive at

$$A(v)\hat{\sigma} = \mathscr{F}(v).$$

Consequently, (55) holds.

4° Let us consider an arbitrary test function $\tau \in E_h^0(v)$. Then we may write

$$\tau = \sum_{i=1}^{s} \hat{\tau}_i \vartheta^{(i)}(v), \quad A(v)\hat{\tau} = 0.$$

It is not difficult to realize that the $d \times s$ matrix $A(v)$ has a full rang $d = \dim V_h(v)$. (Note that

$$\dim V_h(v) = d < \frac{1}{2} \dim H_h(v) = \frac{s}{2}.)$$

Thus a suitable renumbering leads to the equation

$$A_1(v)\hat{\tau}^1 + A_2(v)\hat{\tau}^2 = 0,$$

where $A_1(v)$ is a nonsingular $d \times d$ matrix, so that

$$\hat{\tau}^1 \equiv \hat{\tau}^1(\hat{\tau}^2) = -A_1^{-1}(v)A_2(v)\hat{\tau}^2.$$

Obviously, defining

$$\hat{\tau}_n^1(\hat{\tau}^2) = -A_1^{-1}(v^n)A_2(v^n)\hat{\tau}^2,$$

we obtain that

(60) $$\tau(v^n) \equiv \sum_{i \leqslant d} \hat{\tau}_{ni}^1(\hat{\tau}^2)\vartheta_i(v^n) + \sum_{i>d} \hat{\tau}_i^2 \vartheta_i(v^n) \in E_h^0(v^n).$$

Introducing extensions of $\tau(v^n)$ and of τ as follows

$$\tilde{\tau}(v^n) = 0 \text{ in } \Omega_\delta - \Omega_n \text{ and } \tilde{\tau} = 0 \text{ in } \Omega_\delta - \Omega(v),$$

we obtain from (60) that

(61) $$\left(\tilde{\sigma}^h(v^n), \tilde{\tau}(v^n)\right)_{\Omega_\delta} = 0.$$

We show that

(62) $$\tilde{\tau}(v^n) \to \tilde{\tau} \text{ in } S(\Omega_\delta).$$

In fact, we have

$$\|\tilde{\tau}(v^n) - \tilde{\tau}\|_{0,\Omega_\delta}^2 \leqslant 2 \int_{\Omega_\delta} \left[\sum_{i=1}^{S} (\hat{\tau}_{ni} - \hat{\tau}_i)\tilde{\vartheta}_i(v^n)\right]^2 \mathrm{d}x + 2 \int_{\Omega_\delta} \left[\sum_{i=1}^{S} \hat{\tau}_i \left(\tilde{\vartheta}_i(v^n) - \tilde{\vartheta}_i(v)\right)\right]^2 \mathrm{d}x$$
$$\equiv J_{1n} + J_{2n}.$$

It is easy to find that

$$\lim_{n\to\infty} J_{2n} = 0$$

on the basis of (59). Denoting by $\|\cdot\|_*$ the standard spectral norm and using (60), we may write

$$J_{1n} = 2\int_{\Omega_\delta}\Big[\sum_{i\leqslant d}(\hat{\tau}^1_{ni}(\hat{\tau}^2) - \hat{\tau}^1_i(\hat{\tau}^2))\tilde{\vartheta}_i(v^n)\Big]^2 \,\mathrm{d}x \leqslant C\|\hat{\tau}^1_n(\hat{\tau}^2) - \hat{\tau}^1(\hat{\tau}^2)\|^2_{\mathbb{R}^d},$$

$$\|\hat{\tau}^1_n(\hat{\tau}^2) - \hat{\tau}^1(\hat{\tau}^2)\|_{\mathbb{R}^d} \leqslant \| - A_1^{-1}(v^n)A_2(v^n) + A_1^{-1}(v)A_2(v)\|_* \ \|\hat{\tau}^2\|_{\mathbb{R}^{s-d}} \to 0,$$

since the matrices depend continuously on the variable v^n. Consequently, we arrive at (62). Passing to the limit with $n \to \infty$ in (61) and making use of (53) and (62), we get

$$(\sigma, \tau)_{\Omega(v)} = 0.$$

As the solution of the problem (46) is unique (cf. Lemma 4.1) $\sigma = \sigma^h(v_h)$ follows and the whole sequence $\{\tilde{\sigma}^h(v_h^n)\}$ tends to $\tilde{\sigma}^h(v_h)$ in $S(\Omega_\delta)$. □

A comparison with the model of Giaquinta and Giusti

In the paper [5], Giaquinta and Giusti introduced a new mathematical model of masonry-like materials, which resembles that of so called perfect plasticity subject to Hencky's law. In the stress formulation the state problem is reduced to the minimization of the complementary energy $\mathscr{S}(v;\tau)$ over the set

$$\mathscr{K}(v) = M\big(\Omega(v)\big) \cap E(v)$$

(cf. [5]-Theorem 7.2). It is readily seen that $\mathscr{K}(v)$ is closed and convex. Consequently, the unique minimizer $\sigma^G(v)$ exists if and only if $\mathscr{K}(v) \neq \emptyset$.

Assume that $v \in \mathscr{E}_{\mathrm{ad}}$. Then $\sigma(v) \in \mathscr{K}(v)$ and $\sigma^G(v) = \sigma(v)$ follows from (46). Thus we obtain that

$$\mathscr{E}_{\mathrm{ad}} \subset \mathscr{P}_{\mathrm{ad}} \equiv \{v \in \mathscr{U}_{\mathrm{ad}} \mid \mathscr{K}(v) \neq \emptyset\}.$$

The optimal design problem could be now reformulated as follows:

(P) $$v_0^G = \operatorname*{argmin}_{v\in\mathscr{P}_{\mathrm{ad}}} j(v).$$

Since

$$\min_{v \in \mathscr{P}_{ad}} j(v) \leqslant \min_{v \in \mathscr{E}_{ad}} j(v),$$

we conclude that our setting (19) is "on the safe side".

The same conclusion follows from the fact that in our setting the compatibility of the stress field $\sigma(v)$ is guaranteed almost everywhere, whereas the stress $\sigma^G(v)$ can be incompatible in some subdomains of $\Omega(v)$ (cf. [5]-Introduction).

On the other hand, a mathematical analysis of the problem (P) would be much more difficult. It remains open, to the authors' knowledge, until now.

5. Gradient of the cost functional—adjoint problem

For an effective solution of the Approximate Optimal Design Problem (42) the gradient of the penalized cost functional with respect to the design variable is needed. To this end, the well-known method of an adjoint problem may be employed (cf. e.g. [8]).

5.1. Primal approach

Let us recall the proof of Theorem 3.1, where $a \in \mathbf{R}^{N+1}$ denoted the vector of nodal x_1-coordinates of the design function v_h. We may write

$$\sigma^h(v_h) = \sum_{r=1}^{d} U_r(a)\sigma^{(r)}(a), \tag{63}$$

where

$$\sigma_{ij}^{(r)}(a) = c_{ijml} e_{ml}\big(w_r(a)\big), \quad i,j = 1,2, \tag{64}$$

and $w_1(a)$, ..., $w_d(a)$ denote basis functions of $V_h(v_h)$.

The vector of coefficients $U(a) \in \mathbf{R}^d$ satisfies the following linear system

$$K(a)U(a) = b(a). \tag{65}$$

Substituting (63) into the formula for j_ε leads to the following definition

$$\mathscr{J}(a) \equiv J(a, U(a)) \equiv j_\varepsilon\big(v_h, \sigma^h(v_h)\big) = j_\varepsilon\big(v_h(a), \sum_{r=1}^{d} U_r(a)\sigma^{(r)}(a)\big).$$

Lemma 5.1. *Let $q \in \mathbf{R}^d$ be the solution of the so-called Adjoint Problem*

$$K(a)q = \nabla_U J(a, U). \tag{66}$$

Then

$$\nabla_a \mathscr{J}(a) = \nabla_a J(a,U)\big|_{U=U(a)} + \Big(\nabla_a b(a) - (\nabla_a K(a))U(a)\Big)^T q. \tag{67}$$

Proof. Obviously, we may write

$$\nabla_a \mathscr{J}(a) = \nabla_a J(a,U) + (\nabla_a U(a))^T \nabla_U J(a,U). \tag{68}$$

On the other hand, differentiating (65) we obtain

$$(\nabla_a K(a))U(a) + K(a)\nabla_a U(a) = \nabla_a b(a). \tag{69}$$

Using further the symmetry of $K(a)$, (66) and (69), we arrive at

$$\begin{aligned}(\nabla_a U(a))^T \nabla_U J(a,U) &= (\nabla_a U(a))^T K(a) q \\ &= \Big(\nabla_a b(a) - (\nabla_a K(a))U(a)\Big)^T q.\end{aligned} \tag{70}$$

To obtain (67), it suffices to substitute (70) into (68). □

A direct calculation leads now to the following lemma.

Lemma 5.2. *Components of* $\nabla_U J(a,U)$ *are*

$$\begin{aligned}\frac{\partial J(a,U)}{\partial U_r} &= \frac{1}{\varepsilon}\sum_{i=1}^{2} \int\limits_{\Omega(a)} H\big(\sigma_{ii}^h(a) - k\big)\sigma_{ii}^{(r)}(a)\,\mathrm{d}x \\ &- \frac{1}{\varepsilon}\int\limits_{\Omega(a)} H\big(-\det(\sigma^h(a) - \varkappa)\big)\big[\sigma_{11}^{(r)}(a)\big(\sigma_{22}^h(a) - k\big) + \big(\sigma_{11}^h(a) - k\big)\sigma_{22}^{(r)}(a) \\ &- 2\sigma_{12}^h(a)\sigma_{12}^{(r)}(a)\big]\,\mathrm{d}x, \quad r = 1,\dots,d,\end{aligned} \tag{71}$$

where $H(.)$ *is the Heaviside function,* $\sigma^h(a) \equiv \sigma^h(v_h)$ *and* $\sigma^{(r)}(a)$ *are defined by (63) and (64).*

Components of $\nabla_a J(a,U)$ *are given as follows*

$$\begin{aligned}\frac{\partial J(a,U)}{\partial a_j} &= \int\limits_0^1 \varphi_j(x_2) p\big(v_h(x_2), x_2\big)\,\mathrm{d}x_2 \\ &+ \frac{1}{\varepsilon}\sum_{i=1}^{2}\Big[\int_{\Omega(a)} H\big(\sigma_{ii}^h(a) - k\big)\sum_{r=1}^{d} U_r \frac{\partial \sigma_{ii}^{(r)}(a)}{\partial a_j}\,\mathrm{d}x \\ &+ \int_0^1 \varphi_j(x_2)\Big(\big(\sigma_{ii}^h(a) - k\big)\big|_{x_1=v_h(x_2)}\Big)^+ \mathrm{d}x_2\Big] - \frac{1}{\varepsilon}\Big[\int_{\Omega(a)} H\big(-\det(\sigma^h(a) - \varkappa)\big) \\ &\sum_{r=1}^{d} U_r \Big(\big(\sigma_{22}^h(a) - k\big)\frac{\partial \sigma_{11}^{(r)}}{\partial a_j} + \big(\sigma_{11}^h(a) - k\big)\frac{\partial \sigma_{22}^{(r)}}{\partial a_j} - 2\sigma_{12}^h(a)\frac{\partial \sigma_{12}^{(r)}}{\partial a_j}\Big)\,\mathrm{d}x - \\ &- \int_0^1 \varphi_j(x_2)\Big(\det(\sigma^h(a) - \varkappa)\big|_{x_1=v_h(x_2)}\Big)^- \mathrm{d}x_2\Big], \quad j = 0,\dots,N,\end{aligned} \tag{72}$$

where

$$v_h(x_2) = \sum_{i=1}^{N} a_i \varphi_i(x_2),$$

$\varphi_i\big|_{\nabla_j} \in P_1(\nabla_j)$ *and* $\varphi_i(jh) = \delta_{ij}$ *(Kronecker's delta).*

5.2. Dual approach

Let us recall that the dual problem (47) is equivalent with the constraint minimization problem (48). Applying the Lagrange multiplier method to the latter, we arrive at the following problem: Find a couple $\{\sigma^h, \lambda\}$ such that the Lagrangian

$$\mathscr{L}(\sigma^h, \lambda) \equiv \frac{1}{2}\|\sigma^h\|^2_{\Omega(v_h)} + \sum_{r=1}^{d} \lambda_r \big(\langle \sigma^h, e(w_r)\rangle_{\Omega(v_h)} - \mathscr{F}(v_h; w_r)\big) \tag{73}$$

attains a stationary value on $H_h(v_h) \times \mathbf{R}^d$. In accordance with the formula (56), we shall write

$$\sigma^h = \sum_{i=1}^{s} q_i \vartheta^{(i)}, \tag{74}$$

where $\vartheta^{(i)}$ are basis functions of the space $H_h(v_h)$.

The vanishing variations of the Lagrangian with respect to σ^h and λ yield

$$\begin{aligned} &(\sigma^h, \vartheta^{(i)})_{\Omega(v_h)} + \sum_{r=1}^{d} \lambda_r \left\langle \vartheta^{(i)}, e(w_r)\right\rangle_{\Omega(v_h)} = 0, \quad i = 1, \dots, s, \\ &\langle \sigma^h, e(w_r)\rangle_{\Omega(v_h)} = \mathscr{F}(v_h; w_r), \quad r = 1, \dots, d. \end{aligned} \tag{75}$$

Substituting (74), we arrive at the linear system

$$B \begin{pmatrix} q \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ G \end{pmatrix}, \tag{76}$$

where

$$B = \begin{pmatrix} B_{11}, & B_{12} \\ B_{12}^T, & 0 \end{pmatrix},$$

B_{11} is the $s \times s$ Gram matrix $(\vartheta^{(i)}, \vartheta^{(j)})_{\Omega(v_h)}$, B_{12} is $s \times d$ matrix $\langle \vartheta^{(i)}, e(w_r)\rangle_{\Omega(v_h)}$ and G is $d \times 1$ matrix $\mathscr{F}(v_h; w_r)$. Note that all matrices depend on the design vector parameter a.

We insert (74) into the formula for j_ε to obtain

$$\mathscr{Z}(a) \equiv Z\big(a, q(a)\big) \equiv j_\varepsilon\big(v_h(a), \sum_{i=1}^{s} q_i(a)\vartheta^{(i)}(a)\big).$$

Lemma 5.3. *Let* $\binom{\xi}{\mu}$, $\xi \in \mathbf{R}^s$, $\mu \in \mathbf{R}^d$, *be the solution of the Adjoint Problem*

$$B(a)\begin{pmatrix}\xi\\ \mu\end{pmatrix} = \begin{pmatrix}\nabla_q Z(a,q)\\ 0\end{pmatrix}. \tag{77}$$

Then

$$\nabla_a \mathscr{Z}(a) = \nabla_a Z(a,q)\big|_{q=q(a)} + \left(\begin{pmatrix}0\\ \nabla_a G(a)\end{pmatrix} - (\nabla_a B(a))\begin{pmatrix}q(a)\\ \lambda(a)\end{pmatrix}\right)^T \begin{pmatrix}\xi\\ \mu\end{pmatrix}. \tag{78}$$

Proof. Obviously, we have

$$\nabla_a \mathscr{Z}(a) = \nabla_a Z(a,q) + (\nabla_a q(a))^T \nabla_q Z(a,q). \tag{79}$$

Denoting

$$Q(a) \equiv \begin{pmatrix}q(a)\\ \lambda(a)\end{pmatrix}$$

and differentiating (76), we obtain

$$(\nabla_a B(a))Q(a) + B(a)\nabla_a Q(a) = \begin{pmatrix}0\\ \nabla_a G(a)\end{pmatrix}. \tag{80}$$

Using the symmetry of $B(a)$, (77) and (80), we find that

$$\begin{aligned}(\nabla_a q(a))^T \nabla_q Z(a,q) &= (\nabla_a Q(a))^T \begin{pmatrix}\nabla_q Z(a,q)\\ 0\end{pmatrix}\\ &= (\nabla_a Q(a))^T B(a)\begin{pmatrix}\xi\\ \mu\end{pmatrix}\\ &= \left(\begin{pmatrix}0\\ \nabla_a G(a)\end{pmatrix} - \nabla_a B(a)Q(a)\right)^T \begin{pmatrix}\xi\\ \mu\end{pmatrix}.\end{aligned} \tag{81}$$

To obtain (78), it remains to substitute (81) into (79). □

We may easily derive the following lemma.

Lemma 5.4. *Components of* $\nabla_q Z(a,q)$ *are*

$$\begin{aligned}\frac{\partial Z(a,q)}{\partial q_i} &= \frac{1}{\varepsilon}\sum_{j=1}^{2}\int_{\Omega(a)} H\big(\sigma_{jj}^h(a) - k\big)\vartheta_{jj}^{(i)}(a) - \frac{1}{\varepsilon}\int_{\Omega(a)} H\big(-\det(\sigma^h(a) - \varkappa)\big)\\ &\quad\times \big[\big(\sigma_{22}^h(a) - k\big)\vartheta_{11}^{(i)}(a) + \big(\sigma_{11}^h(a) - k\big)\vartheta_{22}^{(i)}(a) - 2\sigma_{12}^h(a)\vartheta_{12}^{(i)}(a)\big]\,\mathrm{d}x,\\ &\quad i = 1,\dots,s.\end{aligned} \tag{82}$$

Components of $\nabla_a Z(a,q)$ *are given as follows*

$$\frac{\partial Z(a,q)}{\partial a_j} = \int_0^1 \varphi_j(x_2) p(v_h(x_2), x_2)\,\mathrm{d}x_2 + \frac{1}{\varepsilon}\sum_{i=1}^{2}\Big[\int_{\Omega(a)} H\big(\sigma_{ii}^h(a) - k\big)\sum_{l=1}^{s} q_l \frac{\vartheta_{ii}^{(l)}(a)}{\partial a_j}$$

$$(83)\qquad + \int_0^1 \varphi_j(x_2)\Big(\big(\sigma_{ii}^h(a) - k\big)_{x_1 = v_h(x_2)}\Big)^+ \mathrm{d}x_2\Big] - \frac{1}{\varepsilon}\Big[\int_{\Omega(a)} H\Big(-\det\big(\sigma^h(a) - \varkappa\big)\Big)$$

$$\sum_{l=1}^{s} q_l \Big(\big(\sigma_{22}^h(a) - k\big)\frac{\partial \vartheta_{11}^{(l)}(a)}{\partial a_j} + \big(\sigma_{11}^h(a) - k\big)\frac{\partial \vartheta_{22}^{(l)}(a)}{\partial a_j} - 2\sigma_{12}^h(a)\frac{\partial \vartheta_{12}^{(l)}(a)}{\partial a_j}\Big)\,\mathrm{d}x$$

$$- \int_0^1 \varphi_j(x_2)\Big(\det\big(\sigma^h(a) - \varkappa\big)_{x_1 = v_h(x_2)}\Big)^- \mathrm{d}x_2\Big], \quad j = 0, \ldots, N.$$

Remark 5.1. For computation of $\nabla_a K(a)$, $\nabla_a b(a)$ in (67) and of $\nabla_a B(a)$, $\nabla_a G(a)$ in (78) we can use the isoparametric technique — see [4].

6. Convergence analysis

In the present chapter we investigate the distance between approximate and exact solutions. We divide the analysis by introducing an intermediate *Penalized Optimal Design Problem*, i.e., the following problem

$$(84)\qquad v_\varepsilon = \operatorname*{argmin}_{v \in \mathcal{U}_{ad}} j_\varepsilon(v, \sigma(v)),$$

where j_ε was defined in (41).

First of all we shall prove the following existence result.

Proposition 6.1. *There exists at least one solution of the problem* (84) *for any positive parameter* ε.

For the proof we shall need an auxiliary lemma.

Lemma 6.1. *The functions*

$$v \mapsto f_i\big(v, \sigma(v)\big), \quad i = 1, 2, 3,$$

are continuous in the subset $\mathcal{U}_{\mathrm{ad}}^0 \subset C([0,1])$.

Proof. Considering $i = 1, 2$, we may write for any sequence $v_n \to v$ in $C([0,1])$, $v_n \in \mathcal{U}^0_{\rm ad}$,

$$|f_i(v_n, \sigma(v_n)) - f_i(v, \sigma(v))| = \Big| \int_{\Omega_\delta} \Big[(\tilde\sigma_{ii}(v_n) - k)^+ - (\tilde\sigma_{ii}(v) - k)^+ \Big] \, {\rm d}x \Big|$$
$$\leqslant \int_{\Omega_\delta} |\tilde\sigma_{ii}(v_n) - \tilde\sigma_{ii}(v)| \, {\rm d}x$$
$$\leqslant ({\rm meas}\, \Omega_\delta)^{\frac12} \|\tilde\sigma(v_n) - \tilde\sigma(v)\|_{0,\Omega_\delta}.$$

The continuity of $f_i(v, \sigma(v))$ therefore follows from Proposition 2.1. Further, we may write

$$|f_3(v_n, \sigma(v_n)) - f_3(v, \sigma(v))| = \Big| \int_{\Omega_n} \big(\det(\sigma_n - \varkappa) \big)^- {\rm d}x - \int_\Omega \big(\det(\sigma - \varkappa) \big)^- {\rm d}x \Big|$$
$$= \Big| \int_{\Omega_\delta} \Big(\big(\det(\tilde\sigma_n - \varkappa) \big)^- - \big(\det(\tilde\sigma - \varkappa) \big)^- \Big) \, {\rm d}x \Big|$$
$$\leqslant \int_{\Omega_\delta} |\det(\tilde\sigma_n - \varkappa) - \det(\tilde\sigma - \varkappa)| \, {\rm d}x \to 0$$

again by virtue of Proposition 2.1. □

Proof of Proposition 6.1. The set $\mathcal{U}_{\rm ad}$ is compact in $C^1([0,1])$. The functionals $j(v), f_i(v, \sigma(v))$ are continuous in $C([0,1])$ by virtue of Lemma 6.1. Consequently, they are continuous in $C^1([0,1])$, as well and the existence of a minimizer of $j_\varepsilon(v, \sigma(v))$ follows. □

Theorem 6.1. *Let $\mathcal{E}_{\rm ad}$ be non-empty. Let $\{\varepsilon\}$, $\varepsilon \to 0^+$ be a sequence and $\{v_\varepsilon\}$ a sequence of solutions of the problem (84), $\{\sigma(v_\varepsilon)\}$ the sequence of corresponding stress fields. Then there exists a subsequence $\{\tilde\varepsilon\} \subset \{\varepsilon\}$ and an element $v^* \in \mathcal{U}_{\rm ad}$ such that*

$$v_{\tilde\varepsilon} \to v^* \quad \text{in } C^1([0,1]), \tag{85}$$
$$\tilde\sigma(v_{\tilde\varepsilon}) \to \tilde\sigma(v^*) \quad \text{in } S(\Omega_\delta), \tag{86}$$

where v^* is a solution of the Optimal Design Problem (19), *and $\tilde\sigma$ denote the extensions of the stress fields by zero.*

Proof. Since $\mathcal{U}_{\rm ad}$ is compact in $C^1([0,1])$, there exists a subsequence $\{v_{\tilde\varepsilon}\} \subset \{v_\varepsilon\}$ such that (85) holds with $v^* \in \mathcal{U}_{\rm ad}$. Proposition 2.1 implies (86). Let us show that the function v^* is a solution of the Optimal Design Problem (19).

From the definition of (84) it follows that

$$j(v_{\tilde{\varepsilon}}) + \frac{1}{\tilde{\varepsilon}} \sum_{i=1}^{3} f_i\big(v_{\tilde{\varepsilon}}, \sigma(v_{\tilde{\varepsilon}})\big) \leqslant j(v) + \frac{1}{\tilde{\varepsilon}} \sum_{i=1}^{3} f_i\big(v, \sigma(v)\big) \tag{87}$$

holds for any $v \in \mathcal{U}_{\text{ad}}$.

Taking now an element $v \in \mathcal{E}_{\text{ad}}$, we are led to

$$\tilde{\varepsilon} j(v_{\tilde{\varepsilon}}) + \sum_{i=1}^{3} f_i\big(v_{\tilde{\varepsilon}}, \sigma(v_{\tilde{\varepsilon}})\big) \leqslant \tilde{\varepsilon} j(v),$$

$$0 \leqslant \sum_{i=1}^{3} f_i\big(v_{\tilde{\varepsilon}}, \sigma(v_{\tilde{\varepsilon}})\big) \leqslant \tilde{\varepsilon} j(v).$$

Passing to the limit with $\tilde{\varepsilon} \to 0^+$ and using Lemma 6.1, we get

$$\sum_{i=1}^{3} f_i\big(v^*, \sigma(v^*)\big) = 0.$$

Consequently, $v^* \in \mathcal{E}_{\text{ad}}$ follows easily by definitions. Then (87) implies

$$j(v_{\tilde{\varepsilon}}) \leqslant j(v_{\tilde{\varepsilon}}) + \frac{1}{\tilde{\varepsilon}} \sum_{i=1}^{3} f_i\big(v_{\tilde{\varepsilon}}, \sigma(v_{\tilde{\varepsilon}})\big) \leqslant j(v) \quad \forall v \in \mathcal{E}_{\text{ad}}.$$

Passing to the limit with $\tilde{\varepsilon} \to 0^+$ and using (85), we deduce that

$$j(v^*) \leqslant j(v) \quad \forall v \in \mathcal{E}_{\text{ad}}.$$

□

To analyze the convergence of approximate solutions v_h^ε of the problems (42), we shall need the following assertion.

Proposition 6.2. *Let $\{v_h\}$, $h \to 0^+$, be a sequence of $v_h \in \mathcal{U}_{\text{ad}}$ such that*

$$v_h \to v \quad \text{in } C([0,1]). \tag{88}$$

Then

$$\tilde{\sigma}^h(v_h) \to \tilde{\sigma}(v) \quad \text{in } S(\Omega_\delta), \tag{89}$$

where $\tilde{\sigma}^h$ denote the extensions of σ^h by zero.

The proof will be given later. First we shall prove an auxiliary lemma.

Lemma 6.2. *Let the assumptions of Proposition 6.2 be fulfilled and let $u_h = u_h(v_h)$. Then*

$$u_h\big|_{G_m} \rightharpoonup u(v)\big|_{G_m} \quad \text{(weakly) in } [H^1(G_m)]^2 \quad \text{as} \quad h \to 0 \tag{90}$$

holds for any $m > \frac{1}{\alpha}$, (see Lemma 2.1 for the definition of G_m).

Proof. Let us denote

$$\Omega = \Omega(v), \quad \Omega_h = \Omega(v_h)$$

and define the extensions of u_h as follows:

$$\hat{u}_h(x) = u_h(x^*) \tag{91}$$

where

$$x^* = \big(2v_h(x_2) - x_1, x_2\big) \quad \text{for } x \in \Omega_\alpha^h - \Omega_h,\ \Omega_\alpha^h \equiv \Omega(v_h + \alpha).$$

Since the derivatives v_h' are bounded, we obtain (cf. the proof of Lemma 2.1)

$$\|\hat{u}_h\|_{1,\Omega_\alpha^h}^2 \leqslant (1 + C)\|u_h\|_{\Omega_h}^2$$

with C independent of h. Since $\Omega_\alpha^h \supset \Omega_\alpha \equiv \Omega(v + \frac{\alpha}{2})$ for all $h < h_0(\alpha)$,

$$\|\hat{u}_h\|_{1,\Omega_\alpha} \leqslant (1 + C)^{\frac{1}{2}} C_5 \quad \forall h < h_0(\alpha) \tag{92}$$

holds, by virtue of the upper bound

$$\|u_h\|_{1,\Omega_h} \leqslant C_5 \quad \forall h. \tag{93}$$

The latter estimate is a consequence of (5) and (7), since $v_h \in \mathscr{U}_{\mathrm{ad}}^0$ and $V_h(v_h) \subset V(v_h)$.

Then a subsequence (and we shall denote it by the same symbol) of $\{\hat{u}_h\}$ exists, such that

$$\hat{u}_h \rightharpoonup \bar{u} \text{ (weakly) in } [H^1(\Omega_\alpha)]^2 \tag{94}$$

for some $\bar{u} \in [H^1(\Omega_\alpha)]^2$. Since $\hat{u}_h \in V(v + \frac{\alpha}{2})$, we deduce that $\bar{u} \in V(v + \frac{\alpha}{2})$.

Let any $w \in V(v)$ be given. We construct an extension $\hat{w}$ of w by means of the symmetry (91) with respect to $\Gamma(v)$ and repeat the procedure j-times, where $j = \left[\frac{\delta}{\alpha}\right] + 1$ (i.e., with respect to $\Gamma(v + \alpha)$, $\Gamma(v + 2\alpha), \ldots$) if $j > 1$. Thus we obtain an extension $\hat{w} \in V(\delta)$.

There exists a sequence $\{w_\eta\}, \eta \to 0^+$, such that $w_\eta \in [C^\infty(\bar{\Omega}_\delta)]^2, w_\eta = 0$ in a neighbourhood of the x_1-axis and

$$w_\eta \to \hat{w} \text{ in } [H^1(\Omega_\delta)]^2. \tag{95}$$

Let us consider a family of triangulations $\{\mathcal{T}_h^*(v_h)\}$, generated as extensions of $\{\mathcal{T}_h(v_h)\}$ to the domain Ω_δ, which obey the same rules. By direct calculations, we can verify that the family

$$\{\mathcal{T}_h^*(v_h)\},\ h \to 0,\ v_h \in \mathcal{U}_{\mathrm{ad}}^h,$$

is *regular* (i.e., a positive constant ϑ_0 exists, independent of h and v_h, such that any interior angle in any $\mathcal{T}_h^*(v_h)$ is not less than ϑ_0).

Let $\pi_h w_\eta$ be the linear Lagrange interpolate of w_η over the triangulation $\mathcal{T}_h^*(v_h)$. Obviously, we have

$$\pi_h w_\eta\big|_{\Omega_h} \in V_h(v_h) \quad \forall h,$$

$$\|\pi_h w_\eta - w_\eta\|_{1,\Omega_\delta} \leqslant Ch\|w_\eta\|_{2,\Omega_\delta}. \tag{96}$$

We may write

$$a(v_h; u_h, \pi_h w_\eta) = \mathcal{F}(v_h; \pi_h w_\eta). \tag{97}$$

Let us pass to the limit with $h \to 0^+$. We have

$$\begin{aligned}|a(v_h; u_h, \pi_h w_\eta) - a(v; \bar{u}, w_\eta)| \leqslant{}& |a(v_h; u_h, \pi_h w_\eta - w_\eta)| \\ &+ |a(v_h; u_h, w_\eta) - a(v; \hat{u}_h, w_\eta)| \\ &+ |a(v; \hat{u}_h - \bar{u}, w_\eta)| \equiv K_1 + K_2 + K_3;\end{aligned}$$

$$K_1 \leqslant C_3\|u_h\|_{1,\Omega_h}\|\pi_h w_\eta - w_\eta\|_{1,\Omega_h} \leqslant C_3 C_5 Ch\|w_\eta\|_{2,\Omega_\delta} \to 0$$

by virtue of (6), (93) and (96);

$$K_2 \leqslant \int\limits_{\Delta(\Omega_h,\Omega)} |c_{ijml} e_{ij}(\hat{u}_h) e_{ml}(w_\eta)|\, \mathrm{d}x \leqslant C\|\hat{u}_h\|_{1,\Omega_\alpha}\|w_\eta\|_{1,\Delta(\Omega_h,\Omega)} \to 0$$

making use of (92) and

$$\operatorname{meas}\Delta(\Omega_h, \Omega) \to 0; \tag{98}$$

$K_3 \to 0$ follows from the weak convergence (94).

Consequently, we have

(99)
$$\lim_{h\to 0} a(v_h; u_h, \pi_h w_\eta) = a(v; \bar{u}, w_\eta).$$

It is easy to derive

(100)
$$\lim_{h\to 0} \mathscr{F}(v_h; \pi_h w_\eta) = \mathscr{F}(v; w_\eta)$$

In fact, we may write

$$\begin{aligned}|\mathscr{F}(v_h;\pi_h w_\eta) - \mathscr{F}(v; w_\eta)| &\leqslant \Big|\int_{\Omega_h} F^T(\pi_h w_\eta - w_\eta)\,\mathrm{d}x\Big| \\ &+ \Big|\int_{\Omega_h} F^T w_\eta\,\mathrm{d}x - \int_{\Omega} F^T w_\eta\,\mathrm{d}x\Big| + \Big|\int_{\Gamma_0} g^T(\pi_h w_\eta - w_\eta)d\Gamma\Big| \\ &\leqslant \|F\|_{0,\Omega_\delta}\|\pi_h w_\eta - w_\eta\|_{0,\Omega_\delta} \\ &+ \int_{\Delta(\Omega_h,\Omega)} |F^T w_\eta|\,\mathrm{d}x + \tilde{C}\|g\|_{0,\Gamma_0}\|\pi_h w_\eta - w_\eta\|_{1,\Omega_\delta} \to 0,\end{aligned}$$

using (96), (98) and the Trace Theorem on the space $[H^1(\Omega_\delta)]^2$.

Combining (97), (99) and (100), we arrive at

$$a(v; \bar{u}, w_\eta) = \mathscr{F}(v; w_\eta).$$

Passing to the limit with $\eta \to 0$ and using (95), we obtain

$$a(v; \bar{u}, w) = \mathscr{F}(v; w).$$

By virtue of the unique solvability of the problem (2), (see Lemma 1.1), $\bar{u}|_{\Omega} = u(v)$ follows and the whole sequence $\{\hat{u}_h|_{\Omega}\}$ tends to $u(v)$ weakly in $V(v)$. Finally, (90) is an immediate consequence of this result, since $G_m \subset \Omega$ for all $m > \frac{1}{\alpha}$ and $\hat{u}_h|_{G_m} \equiv u_h(v_h)|_{G_m}$ holds for all $h > h_0(m)$ such that $G_m \subset \Omega_h$. □

Proof of Proposition 6.2. Let us denote $\sigma = \sigma(v)$. From Lemma 6.2 and the definition of σ^h it follows that

(101)
$$\sigma^h|_{G_m} \rightharpoonup \sigma|_{G_m} \quad \text{(weakly) in } S(G_m) \quad \forall m > \frac{1}{\alpha}.$$

Let us show that

(102)
$$\tilde{\sigma}^h \rightharpoonup \tilde{\sigma} \text{ (weakly) in } S(\Omega_\delta) \text{ for } h \to 0.$$

To this end, let us consider any $\varphi \in S(\Omega_\delta)$. Then we have

$$\begin{aligned}\left|\langle\varphi,\tilde{\sigma}^h\rangle_{\Omega_\delta} - \langle\varphi,\tilde{\sigma}\rangle_{\Omega_\delta}\right| &\leqslant \left|\langle\varphi,\sigma^h\rangle_{G_m} - \langle\varphi,\sigma\rangle_{G_m} + \langle\varphi,\sigma^h\rangle_{\Omega_h-G_m} - \langle\varphi,\sigma\rangle_{\Omega-G_m}\right| \\ &\leqslant \left|\langle\varphi,\sigma^h-\sigma\rangle_{G_m}\right| + \left|\langle\varphi,\sigma^h\rangle_{\Omega_h-G_m}\right| + \left|\langle\varphi,\sigma\rangle_{\Omega-G_m}\right| \\ &\equiv K_1+K_2+K_3.\end{aligned}$$

Making use of (101), we conclude that

$$K_1 \to 0.$$

Furthermore,

$$K_2 \leqslant \|\varphi\|_{0,\Omega_h-G_m}\|\sigma^h\|_{0,\Omega_h} \to 0 \quad \text{if } h\to 0,\ m\to\infty,\ h<h_0(m),$$

since

$$\|\sigma^h\|_{0,\Omega_h} \leqslant C \quad \forall h$$

(cf. the definition of σ^h and (93)) and

$$\text{meas}(\Omega_h - G_m) \leqslant \frac{1}{m} + \|v_h - v\|_\infty \to 0.$$

By definition of G_m,

$$K_3 \to 0 \quad \text{for } m\to\infty.$$

Combining the three limits, we obtain (102).

Making use of the inverse relation

$$e(u_h) = b\sigma^h,$$

we may write

$$\langle\tilde{\sigma}^h, b\tilde{\sigma}^h\rangle_{\Omega_\delta} = a(v_h;u_h,u_h) = \mathscr{F}(v_h;u_h). \tag{103}$$

On the basis of Lemma 6.2 we can prove that

$$\lim_{h\to 0}\mathscr{F}(v_h;u_h) = \mathscr{F}(v;u) = a(v;u,u) = \langle\tilde{\sigma}(v), b\tilde{\sigma}(v)\rangle_{\Omega_\delta} \tag{104}$$

Indeed, the same arguing as that in proving (31) can be applied, if we replace Ω_n by Ω_h and u_n by u_h, respectively.

From (103) and (104) we conclude that

$$\lim_{h\to 0}\langle\tilde{\sigma}^h, b\tilde{\sigma}^h\rangle_{\Omega_\delta} = \langle\tilde{\sigma}(v), b\tilde{\sigma}(v)\rangle_{\Omega_\delta}. \tag{105}$$

The equivalence of the energy norm with the standard norm of $S(\Omega_\delta)$, the weak convergence of $\{\tilde{\sigma}^h\}$ and (105) yield that

$$\|\tilde{\sigma}^h - \tilde{\sigma}(v)\|_{0,\Omega_\delta} \to 0 \text{ for } h\to 0.$$

□

Using Proposition 6.2 and arguing like in the proof of Lemma 6.1, we obtain the following assertion.

Proposition 6.3. *Let the assumptions of Proposition 6.2 be satisfied. Then*

$$\lim_{h\to 0} j_\varepsilon\big(v_h, \sigma^h(v_h)\big) = j_\varepsilon\big(v, \sigma(v)\big).$$

Theorem 6.2. *Let* $\{v_h^\varepsilon\}$, $h \to 0^+$, *be a sequence of solutions of the Approximate Optimal Design Problems* (42). *Then there exists a subsequence* $\{v_{\tilde{h}}^\varepsilon\} \subset \{v_h^\varepsilon\}$ *and an element* v^ε *such that*

$$v_{\tilde{h}}^\varepsilon \to v^\varepsilon \quad \text{in } C([0,1]), \tag{106}$$

$$\tilde{\sigma}^h(v_{\tilde{h}}^\varepsilon) \to \tilde{\sigma}(v^\varepsilon) \quad \text{in } S(\Omega_\delta) \tag{107}$$

as $\tilde{h} \to 0$, *and* v^ε *is a solution of the Penalized Optimal Design Problem* (74).

Proof. Since $\mathscr{U}_{\rm ad}^h \subset \mathscr{U}_{\rm ad}^0$ for any h and $\mathscr{U}_{\rm ad}^0$ is compact in $C([0,1])$, there exists a subsequence $\{v_{\tilde{h}}^\varepsilon\}$ such that (106) holds with $v^\varepsilon \in \mathscr{U}_{\rm ad}^0$. By a slight modification of Lemma 3.2 in the paper [8], we obtain that $v^\varepsilon \in \mathscr{U}_{\rm ad}$.

Let us consider an arbitrary $v \in \mathscr{U}_{\rm ad}$. From (a modification of) Lemma 3.1 of [8] it follows that a sequence $\{v_{\tilde{h}}\}$ exists such that $v_{\tilde{h}} \in \mathscr{U}_{\rm ad}^{\tilde{h}}$ and

$$v_{\tilde{h}} \to v \text{ in } C([0,1]).$$

By definition, we may write

$$j_\varepsilon\big(v_{\tilde{h}}^\varepsilon, \sigma^{\tilde{h}}(v_{\tilde{h}}^\varepsilon)\big) \leqslant j_\varepsilon\big(v_{\tilde{h}}, \sigma^{\tilde{h}}(v_{\tilde{h}})\big).$$

Let us pass to the limit with $\tilde{h} \to 0$ and apply Proposition 6.3 to both sides of the inequality to obtain

$$j_\varepsilon\big(v^\varepsilon, \sigma(v^\varepsilon)\big) \leqslant j_\varepsilon\big(v, \sigma(v)\big).$$

Consequently, v^ε is a solution of the problem (84). The convergence (107) of stress fields follows from Proposition 6.2. □

Finally, let us discuss the *dual approach* to approximate solutions $\sigma^h(v_h)$ of the elastostatic problem, i.e., let $\sigma^h(v_h)$ be determined by (47).

Assuming that the coefficients are *piecewise constant*, i.e.

$$b_{ijml} (\text{or } c_{ijml}) \in P_0(K) \quad \forall i,j,m,l = 1,2 \tag{108}$$

and for all $K \in \mathscr{T}_h(v_h)$, $h \to 0^+$, $v_h \in \mathscr{U}_{\rm ad}^h$, we shall prove that *Proposition 6.2 remains true.*

Proof of Proposition 6.2.

1° We may decompose $\sigma^h \equiv \sigma^h(v_h)$ as follows

$$\sigma^h = r_h \sigma(v_h) + \omega^h,$$

where r_h is the projection mapping from the proof of Lemma 4.1. Recall that

$$r_h \sigma(v_h) \in E_h(v_h) \quad \text{and} \quad \omega^h \in E_h^0(v_h).$$

Therefore, we get by (47) that

$$0 = (\sigma^h, \omega^h)_{\Omega_h} = (r_h \sigma(v_h), \omega^h)_{\Omega_h} + \|\omega^h\|_{\Omega_h}^2.$$

Consequently,

$$\|\omega^h\|_{\Omega_h}^2 \leqslant \|r_h \sigma(v_h)\|_{\Omega_h} \|\omega^h\|_{\Omega_h},$$
$$\|\omega^h\|_{\Omega_h} \leqslant C \|r_h \sigma(v_h)\|_{0,\Omega_h} \leqslant C \|\sigma(v_h)\|_{0,\Omega_h} \leqslant \tilde{C} C_5$$

by virtue of Lemma 1.1. Then

$$\|\tilde{\sigma}^h\|_{0,\Omega_\delta} \leqslant \|r_h \sigma(v_h)\|_{0,\Omega_h} + \|\omega^h\|_{0,\Omega_h} \leqslant C$$

and a subsequence of $\{\tilde{\sigma}^h\}$ exists such that

$$\tilde{\sigma}^h \rightharpoonup \sigma \quad \text{(weakly) in } S(\Omega_\delta) \tag{109}$$

for some $\sigma \in S(\Omega_\delta)$.

Next we shall prove that

$$\sigma = 0 \quad \text{a.e. in } \Omega_\delta - \Omega(v). \tag{110}$$

Let a subset $D \subset \Omega_\delta - \Omega(v)$ exist such that $\operatorname{meas} D > 0$,

$$\|\sigma\|_{0,D} > 0.$$

Denote by χ_D the characteristic function of the set D. From (109) we deduce that

$$\langle \tilde{\sigma}^h, \chi_D \sigma \rangle_{\Omega_\delta} \to \langle \sigma, \chi_D \sigma \rangle_{\Omega_\delta} = \|\sigma\|_{0,D}^2 > 0.$$

On the other hand,

$$\left| \langle \tilde{\sigma}^h, \chi_D \sigma \rangle_{\Omega_\delta} \right| \leqslant \|\tilde{\sigma}^h\|_{0,\Omega_\delta} \|\sigma\|_{0,D \cap \Omega_h} \to 0 \quad \text{for } h \to 0,$$

since $\tilde{\sigma}^h$ are bounded in $S(\Omega_\delta)$ and

$$\text{meas}(D \cap \Omega_h) \to 0.$$

Thus we arrive at a contradiction and (110) is verified.

Moreover, we can show that

$$\sigma\big|_{\Omega(v)} \in E(v). \tag{111}$$

Indeed, let us consider any $w \in V(v)$, its extension $\hat{w} \in V(\delta)$ and a sequence $\{w_\eta\}, \eta \to 0^+$, such that (95) holds. Using the Lagrange linear interpolate $\pi_h w_\eta \in V_h(v_h)$, we may write

$$\langle \tilde{\sigma}^h, e(\pi_h w_\eta) \rangle_{\Omega_\delta} = \mathscr{F}(v_h; \pi_h w_\eta).$$

Making use of (96), we obtain that

$$\|e(\pi_h w_\eta) - e(w_\eta)\|_{0,\Omega_\delta} \leqslant Ch \|w_\eta\|_{2,\Omega_\delta}.$$

Passing to the limit with $h \to 0$, and using also (109), (100), we arrive at

$$\langle \sigma, e(w_\eta) \rangle_{\Omega_\delta} = \mathscr{F}(v; w_\eta).$$

Passing to the limit with $\eta \to 0$ and making use of (95), (110), we are led to (111).

2° Next we show that

$$(\sigma, \tau)_{\Omega(v)} = 0 \quad \forall \tau \in E^0(v), \tag{112}$$

where

$$E^0(v) = \{\tau \in S(\Omega(v)) |\, \langle \tau, e(w) \rangle_{\Omega(v)} = 0 \quad \forall w \in V(v)\}.$$

Let an arbitrary $\tau \in E^0(v)$ be given. Let us define the extension $\tilde{\tau}$ by zero for all $x_1 > v(x_2)$, $0 < x_2 < 1$, and introduce a modified function τ^λ by the formulas

$$\tau^\lambda_{ij}(x_1, x_2) = a_{ij} \tilde{\tau}_{ij}(t x_1, x_2), \quad \text{(no sum)},$$

where

$$t = 1 + \lambda, \quad \lambda > 0,$$
$$a_{11} = t^{-1},\ a_{22} = t,\ a_{12} = 1.$$

Let us employ new variables

$$y_1 = t x_1, \quad y_2 = x_2$$

and consider any function $w \in V(v_h)$. If we define a new function $w^* = (w_1^*, w_2^*)$ by the formulae

$$w_1^*(y) = t^{-1} w_1\left(\frac{y_1}{t}, y_2\right),$$
$$w_2^*(y) = w_2\left(\frac{y_1}{t}, y_2\right),$$

then $w^*|_{\Omega(v)} \in V(v)$ for all $h < h_0(\lambda)$. We may write

$$\langle \tau^\lambda, e(w)\rangle_{\Omega_h} = \int_{\Omega(\frac{v}{t})} a_{ij}\tilde{\tau}_{ij}(tx_1, x_2) e_{ij}(w)\,\mathrm{d}x = \int_{\Omega(v)} \tau_{ij}(y) e_{ij}(w^*(y))\,\mathrm{d}y = 0,$$

since

$$\Omega\left(\frac{v}{t}\right) \subset \Omega_h \text{ and } \tau \in E^0(v),$$
$$e_{11}(w(x)) = t^2 e_{11}(w^*(y)), \quad e_{22}(w(x)) = e_{22}(w^*(y)),$$
$$e_{12}(w(x)) = t e_{12}(w^*(y)).$$

Consequently,

$$\tau^\lambda|_{\Omega_h} \in E^0(v_h) \quad \forall h < h_0(\lambda).$$

It is readily seen that

$$r_h \tau^\lambda \in E_h^0(v_h),$$

since

$$\langle r_h \tau^\lambda, e(w_h)\rangle_{\Omega_h} = \langle \tau^\lambda, e(w_h)\rangle_{\Omega_h} = 0$$

holds for any $w_h \in V_h(v_h) \subset V(v_h)$.

By the definition (47)

(113) $$(\sigma^h, r_h\tau^\lambda)_{\Omega_h} = (\tilde{\sigma}_h, r_h\tau^\lambda)_{\Omega_\delta} = 0$$

(where r_h applies to the extended triangulation $\mathscr{T}_h^*(v_h)$). It is well-known that

$$\lim_{h\to\infty} \|r_h\tau^\lambda - \tau^\lambda\|_{0,\Omega_\delta} = 0.$$

Passing to the limit with $h \to 0$ in (113) and using also (109), (110), we arrive at

(114) $$(\sigma, \tau^\lambda)_{\Omega(v)} = 0.$$

Next we show that

(115) $$\lim_{\lambda\to 0} \|\tau^\lambda - \tau\|_{0,\Omega(v)} = 0.$$

Indeed, since $y = (tx_1, x_2)$ and $\tau_{ij}^{\lambda}(x) = a_{ij}\tilde{\tau}_{ij}(y)$, we may write

$$\|\tau^{\lambda}(x) - \tau(x)\| \leqslant \|\tau^{\lambda}(x) - \tilde{\tau}(y)\| + \|\tilde{\tau}(y) - \tau(x)\| \leqslant \lambda\|\tilde{\tau}(y)\| + \|\tilde{\tau}(y) - \tau(x)\|$$

so that

$$\begin{aligned}\|\tau^{\lambda} - \tau\|_{0,\Omega(v)}^2 &= \int_{\Omega(v)} \|\tau^{\lambda}(x) - \tau(x)\|^2 \mathrm{d}x \\ &\leqslant 2\lambda^2 \int_{\Omega(\frac{v}{t})} \|\tau(y)\|^2 \mathrm{d}x + 2\int_{\Omega(v)} \|\tilde{\tau}(y) - \tau(x)\|^2 \mathrm{d}x \equiv J_1 + J_2.\end{aligned}$$

It is obvious that

$$J_1 \leqslant \frac{2}{t}\lambda^2 \int_{\Omega(v)} \|\tau(y)\|^2 dy \to 0 \quad \text{as } \lambda \to 0.$$

We can prove that J_2 tends to zero, as well, applying the following argument. There exists a sequence $\{\tau^n\}_{n=1}^{\infty}$ such that

$$\tau^n \in \left[C_0^{\infty}(\Omega(v))\right]^4 \cap S(\Omega(v)), \quad \tau^n \to \tau \text{ in } S(\Omega(v)). \tag{116}$$

We extend τ^n by zero outside of $\Omega(v)$. Then

$$\|\tau^n(y) - \tau^n(x)\|_{0,\Omega(v)} \leqslant \beta^{\frac{1}{2}}\lambda\|\tau^n\|_{C^1(\bar{\Omega}(v))} \tag{117}$$

follows from the mean value theorem. Moreover, we find that

$$\int_{\Omega(v)} \|\tilde{\tau}(y) - \tau^n(y)\|^2 \mathrm{d}x \leqslant \|\tau - \tau^n\|_{0,\Omega(v)}^2 \to 0 \quad \text{as } n \to \infty. \tag{118}$$

Finally, we may write

$$\begin{aligned}\|\tilde{\tau}(y) - \tau(x)\|_{0,\Omega(v)} &\leqslant \|\tilde{\tau}(y) - \tau^n(y)\|_{0,\Omega(v)} \\ &\quad + \|\tau^n(y) - \tau^n(x)\|_{0,\Omega(v)} + \|\tau^n - \tau\|_{0,\Omega(v)}\end{aligned}$$

and using (118), (117), (116), we obtain $J_2 \to 0$ as $\lambda \to 0$. Consequently, (115) has been proven. Passing to the limit with $\lambda \to 0$ in (114), we arrive at (112). From the unique solvability of the problem (46) it follows that

$$\sigma\big|_{\Omega(v)} = \sigma(v)$$

and the whole sequence $\{\tilde{\sigma}^h(v_h)\}$ tends to $\tilde{\sigma}(v)$ weakly in $S(\Omega_\delta)$.

3° It remains to verify the strong convergence. Making use of (47), we can write

$$(\tilde{\sigma}^h, \tilde{\sigma}^h)_{\Omega_\delta} = \left(\sigma^h, r_h\sigma(v_h) + \omega^h\right)_{\Omega_h} = \left(\sigma^h, r_h\sigma(v_h)\right)_{\Omega_h} \tag{119}$$

since $\omega^h \in E_h^0(v_h)$.

On the other hand, we have

$$\left(\sigma^h, r_h\sigma(v_h)\right)_{\Omega_h} = \left\langle b\sigma^h, r_h\sigma(v_h)\right\rangle_{\Omega_h} = \left\langle b\sigma^h, \sigma(v_h)\right\rangle_{\Omega_h} = (\tilde{\sigma}^h, \tilde{\sigma}(v_h))_{\Omega_\delta}, \tag{120}$$

since $b\sigma^h \in H_h(v_h)$ follows by the assumption (108). Using Proposition 2.1 and the weak convergence of $\tilde{\sigma}^h$ (see (109)), we obtain that

$$\left(\tilde{\sigma}^h, \tilde{\sigma}(v_h)\right)_{\Omega_\delta} \to \left(\tilde{\sigma}(v), \tilde{\sigma}(v)\right)_{\Omega_\delta} \text{ as } h \to 0. \tag{121}$$

Combining (119), (120) and (121), we arrive at

$$(\tilde{\sigma}^h, \tilde{\sigma}^h)_{\Omega_\delta} \to (\tilde{\sigma}(v), \tilde{\sigma}(v))_{\Omega_\delta} \quad \text{as } h \to 0. \tag{122}$$

The equivalence of the norms, the weak convergence and (122) imply that

$$\|\tilde{\sigma}^h - \tilde{\sigma}(v)\|_{0,\Omega_\delta} \leqslant C\|\tilde{\sigma}^h - \tilde{\sigma}(v)\|_{\Omega_\delta} \to 0.$$

□

It is easy to see that Proposition 6.3 and Theorem 6.2 remain true, provided the coefficients of the stress-strain relations are piecewise constant (108).

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MINIMALIZACE VÁHY PRUŽNÝCH TĚLES, KTERÁ NEVZDORUJÍ VĚTŠÍM TAHOVÝM NAPĚTÍM
I. OBLASTI S JEDNOU ZAKŘIVENOU STRANOU

IVAN HLAVÁČEK, MICHAL KŘÍŽEK

Uvažuje se optimalizace tvaru rovinného pružného tělesa za předpokladu, že materiál nesnese větší tahová napětí. Jde o zobecnění problému zděné přehrady, zatížené vlastní váhou a hydrostatickým tlakem. Je dokázána existence optimální oblasti. Na základě metody penalizace a konečných prvků se navrhují přibližná řešení a studuje jejich konvergence.

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