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HILBERT-SPACE-VALUED STATES ON QUANTUM LOGICS

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Summary. We analyze finitely additive orthogonal states whose values lie in a real Hilbert space. We call them h-states. We first consider the important case of h-states on a standard Hilbert logic $L(H)$ of projectors in H—we describe the h-states $s\colon L(H_1) \to H_2$, where $\dim H_2 \leqslant \dim H_1 < \infty$. In particular, we show that, up to a unitary mapping, every h-state $s\colon L(H) \to H (3 \leqslant \dim H < \infty)$ has to be concentrated on a one-dimensional projection. We also study the h-states $s\colon L(H_1) \to H_2$ for the case of $\dim H_1 = \infty$. The results of the first part complement the papers [10] and [13]. In the second part we investigate h-states on general logics. Being motivated by the quantum axiomatics, the main question we ask here is as follows: Given a Hilbert space H with $\dim H < \infty$, what is the class of such logics L that, for any Boolean subalgebra B of L, every h-state $s\colon B \to H$ extends over L? We answer this question by finding a simple condition characterizing the class (Theorem 3.4). It turns out that the class is considerably large—it contains e.g. all concrete logics—but, on the other hand, it does not contain all finite logics (we construct a counterexample in the appendix).

Keywords: Hilbert-space-valued state, h-state

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1. PRELIMINARIES

Definition 1.1. A (quantum) logic is a set L endowed with a partial ordering, $\leqslant$, and a unary operation, $'$, such that

(i) $0, 1 \in L$,
(ii) $a \leqslant b \Rightarrow b' \leqslant a'$ for any $a, b \in L$,
(iii) $(a')' = a$ for any $a \in L$,
(iv) $a \vee b$ exists in L whenever $a, b \in L$ and $a \leqslant b'$,
(v) $a \vee a' = 1$ for any $a \in L$,
(vi) $b = a \vee (b \wedge a')$ whenever $a, b \in L$ and $a \leqslant b$.

In what follows, let us reserve the symbol L for logics. As a prototype of a logic we may take a Boolean algebra or the lattice $L(H)$ of all projections in a Hilbert space H. Generally, a logic need not be distributive and need not be a lattice. A systematic treatment of logics together with the physical interpretation can be found in [7].

Definition 1.2. Let L be a logic and let H be a real Hilbert space. A mapping $s\colon L \to H$ is called a Hilbert-space-valued orthogonal state (abbr. *an h-state*) if the following two conditions are satisfied:

(i) $\|s(1)\| = 1$ (here $\|\ \|$ stands for the Hilbert norm in H),

(ii) if $a, b \in L$ and $a \leqslant b'$ then $s(a)$ is orthogonal to $s(b)$ (in H) and, moreover, $s(a \vee b) = s(a) + s(b)$.

The h-states have been investigated in the articles [3], [10] and [13], in the two former cases in the σ-additive setup and stochastic vein, in the third case in the realm of universal algebras. The h-states can be viewed as generalized two-valued states as the following simple proposition asserts. In the proposition (and in all what follows) we denote by $\mathcal{S}_H(L)$ the set of all h-states $s\colon L \to H$.

Proposition 1.3. *Suppose that H is a Hilbert space. If $s \in \mathcal{S}_H(L)$ then*

(i) $s(0) = 0$,

(ii) the set $\mathcal{R}(s) = \{p \in H \mid p = s(k) \text{ for } k \in L\}$ is contained in the sphere $S_{\frac{1}{2}}(s(1))$, where $s(1)$ is its centre and $\frac{1}{2}$ is its radius,

(iii) if $H = R$ then $\mathcal{R}(s) = \{0, 1\}$,

(iv) the mapping $t_s\colon L \to \langle 0, 1\rangle$ defined by the formula $t_s(k) = \|s(k)\|^2$ is an ordinary state on L (i.e. t_s is a probability measure).

Proof (see also [13]).

(i) Since $s(1) = s(1 \vee 0) = s(1) + s(0)$, we see that $s(0) = 0$.

(ii) Suppose that $p = s(k)$, where $k \in L$. If $\langle .,. \rangle$ denotes the inner product in H, we have

$$\langle s(k),\ s(1) - s(k)\rangle = \langle s(k),\ s(k')\rangle = 0$$

and therefore

$$\begin{aligned}\left\| s(k) - \frac{s(1)}{2} \right\|^2 &= \frac{1}{4}\left\langle s(k) - \big(s(1) - s(k)\big),\ s(k) - \big(s(1) - s(k)\big)\right\rangle = \\ &= \frac{1}{4}\left(\|s(k)\|^2 + \|s(k')\|^2\right) = \frac{1}{4}.\end{aligned}$$

(iii) A simple corollary of the case (ii).

(iv) If $k_1, k_2 \in L$ and $k_1 \leqslant k_2'$ then

$$\begin{aligned}t_s(k_1 \vee k_2) &= \|s(k_1 \vee k_2)\|^2 = \|s(k_1) + s(k_2)\|^2 = \\ &= \|s(k_1)\|^2 + \|s(k_2)\|^2 = t_s(k_1) + t_s(k_2).\end{aligned}$$

Further, $t_s(1) = \|s(1)\|^2 = 1$. Thus, t_s is an ordinary state on L. □

In this section we characterize the h-states $s\colon L(H_1) \to H_2$, where $\dim H_2 \leqslant \dim H_1 < \infty$. (The case of $\dim H_2 > \dim H_1$ does not seem to allow a lucid characterization.) To avoid usual pathologies occurring in this area of problems, we restrict ourselves to the case of $\dim H_1 \geqslant 3$.

Let us first consider the case of $\dim H_1 = \dim H_2 < \infty$. Thus, we suppose that $H_1 = H_2 = H$ and $3 \leqslant \dim H < \infty$. In this case we have some natural h-states at our disposal—if $v \in H$ with $\|v\| = 1$ then the mapping $s_v : L(H) \to H$ given by the formula $s_v(P_M) = P_M v$, where P_M denotes the projection of H onto M, is obviously an h-state. Let us call it the ***h-state associated with*** $v \in H$. Our main result in this part asserts that $\mathcal{S}_H\big(L(H)\big)$ essentially contains only these h-states. This extends the research of R. Mayet carried on in [13]. (Prior to the next theorem, let us recall that a linear mapping $U\colon H \to H$ is called unitary if $\langle U(x), U(y)\rangle = \langle x, y\rangle$ for any $x, y \in H$.)

Theorem 2.1. *Let H be a Hilbert space such that $3 \leqslant \dim H < \infty$. If $s\colon L(H) \to H$ is an h-state then there exists a unit vector $v \in H$ and a unitary mapping $U : H \to H$ such that $s = U \circ s_v$, where s_v is the h-state associated with $v \in H$. (Thus, for $s\colon L(H) \to H$ there is a unit vector $v \in H$ such that $\big\|s(P_v)\big\| = 1$.)*

Proof. We shall need three lemmas.

Lemma 2.2. *Suppose that $\dim H = n$. Suppose further that we are given $n + 2$ vectors in H. Then there is a pair among the vectors, say vectors x, y ($x \neq y$), such that $\langle x, y\rangle \geqslant 0$.*

Proof of Lemma 2.2. We shall proceed by induction. The case of $n = 1$ is obvious. Suppose that the statement of Lemma 2.2 is valid for $n-1$ and suppose that we have $n+2$ vectors in H, say $x_1, x_2, \ldots, x_{n+1}, x_{n+2}$. Take a subspace M of H which contains the vectors $x_1, x_2, \ldots, x_{n-1}$. We may suppose that $\dim M = n - 1$. By the inductive assumption, there are pairs $\{y_1, z_1\} \subset \{x_1, \ldots, x_n, P_M x_n, P_M x_{n+1}\}$, $\{y_2, z_2\} \subset \{x_1, \ldots, x_{n-1}, P_M x_{n+1}, P_M x_{n+2}\}$ and $\{y_3, z_3\} \subset \{x_1, \ldots, x_{n-1}, P_M x_n, P_M x_{n+2}\}$ such that $\langle y_i, z_i\rangle \geqslant 0$ ($i = 1, 2, 3$). If $\{y_i, z_i\} \cap \{x_1, \ldots, x_{n-1}\} \neq \emptyset$ for at least one $i \in \{1, 2, 3\}$, then the proof is complete. Suppose on the contrary that $\langle P_M x_n, P_M x_{n+1}\rangle \geqslant 0$, $\langle P_M x_n, P_M x_{n+2}\rangle \geqslant 0$ and $\langle P_M x_{n+1}, P_M x_{n+2}\rangle \geqslant 0$. Let K denote the orthogonal complement of M in H. Then $\dim K = 1$ and we infer that at least one of the numbers $\langle P_K x_n, P_K x_{n+1}\rangle$, $\langle P_K x_n, P_K x_{n+2}\rangle$ and $\langle P_K x_n, P_K x_{n+1}\rangle$ is nonnegative. Suppose that $\langle P_K x_n, P_K x_{n+1}\rangle \geqslant 0$ (in the other cases we proceed similarly). Then $\langle x_n, x_{n+1}\rangle = \langle P_M x_n, P_M x_{n+1}\rangle + \langle P_K x_n, P_K x_{n+1}\rangle \geqslant 0$ and this completes the proof of Lemma 2.2. □

Before we formulate the following lemma, let us recall that the trace of an operator $A\colon H \to H$, denoted by $\operatorname{Tr} A$, is the number $\sum_{k=1}^{n} \langle Ae_k, e_k \rangle$, where $e_1, e_2, \ldots, e_n$ is an arbitrary orthonormal basis of H.

Lemma 2.3. *Let H be a Hilbert space such that $3 \leqslant \dim H < \infty$. Then for any h-state $s\colon L(H) \to H$ one can find a nonnegative operator $T_s\colon H \to H$ such that, for any $P_M, P_N \in L(H)$, it satisfies the equality*

$$\langle s(P_M), s(P_N) \rangle = \operatorname{Tr}(T_s P_M P_N) = \operatorname{Tr}(T_s P_N P_M).$$

Proof (a sketch, see also [10]). By Prop. 1.3, the mapping $t_s\colon L(H) \to \langle 0, 1 \rangle$ defined by setting $t_s(P_M) = \|s(P_M)\|^2$ is an ordinary state on $L(H)$. Thus, by the famous Gleason's theorem [5], we can find a nonnegative operator T_s such that $t_s(P_M) = \operatorname{Tr}(T_s P_M)$ $(P_M \in L(H))$. Let $S(H)$ denote the linear space of all symmetric operators on H and let us extend s to a linear mapping $\tilde{s}\colon S(H)$. This can be done in a straight-forward manner since $S(H)$ is generated by all projections (see [2]). Moreover, every $A \in S(H)$ can be expressed as a linear combination of mutually orthogonal projections [2], and therefore $\|\tilde{s}(A)\|^2 = \operatorname{Tr}(T_s A^2)$. In particular, if $P_M, P_N \in L(H)$ then $\|\tilde{s}(P_M + P_N)\|^2 = \operatorname{Tr}\bigl(T_s(P_M + P_N)^2\bigr) = 2\operatorname{Tr}(T_s P_M P_N) + \operatorname{Tr}(T_s P_M) + \operatorname{Tr}(T_s P_N)$. So we have $2\langle s(P_M), s(P_N)\rangle = \|s(P_M) + s(P_N)\|^2 - \|s(P_M)\|^2 - \|s(P_N)\|^2 = 2\operatorname{Tr}(T_s P_M P_N)$ and the proof of Lemma 2.3 is complete. □

Prior to the formulation of the following lemma, let us agree to denote by P_v $(v \in H)$ the projection on the linear span of the vector v.

Lemma 2.4. *Let $s\colon L(H) \to H$ $(3 \leqslant \dim H < \infty)$ be an h-state. Then there exists a vector $v \in H$ such that $\|s(P_v)\| = 1$. (A corollary: If we define $t_s\colon L(H) \to \langle 0, 1 \rangle$ by setting $t_s(P_M) = \|s(M)\|^2$, then t_s is a pure state on $L(H)$.)*

Proof of lemma 2.4. Let $T_s\colon H \to H$ be the operator from Lemma 2.3. We shall show that T_s has to be a projection on a one-dimensional subspace of H. Suppose on the contrary that e_1, e_2 are unit orthonormal eigenvectors corresponding to positive eigenvalues λ_1, λ_2. Suppose that $\dim H = n$ and take an orthogonal basis $v_1, v_2, \ldots, v_n$ in H. Obviously, we may take it such that $\|P_{v_i} e_j\| \neq 0$ $(i = 1, 2, \ldots, n,\ j = 1, 2)$. Then

$$\left\langle s(P_{v_i}), \frac{s(P_{e_j})}{\|s(P_{e_j})\|} \right\rangle = \frac{1}{\sqrt{\lambda_j}} \operatorname{Tr}(T_s P_{v_i} P_{e_j}) = \sqrt{\lambda_j} \|P_{v_i} e_j\|^2$$

($i = 1, 2, \ldots, n$, $j = 1, 2$). Moreover, we have

$$s(P_{v_i}) = \sum_{j=1}^{2} \sqrt{\lambda_j} \|P_{v_i} e_j\|^2 \frac{s(P_{e_j})}{\|s(P_{e_j})\|} + z_i, \quad (i = 1, 2, \ldots, n),$$

where z_i is orthogonal to both $s(P_{e_1})$, $s(P_{e_2})$. According to Lemma 2.2, there is a pair of indices i_1, i_2 such that $\langle z_{i_1}, z_{i_2} \rangle \geqslant 0$. Then we obtain $\langle s(P_{v_{i_1}}), s(P_{v_{i_2}}) \rangle = \sum_{j=1}^{2} \lambda_j \|P_{v_{i_1}} e_j\|^2 \cdot \|P_{v_{i_2}} e_j\|^2 + \langle z_{i_1}, z_{i_2} \rangle > 0$, which is a contradiction—the vectors v_{i_1}, v_{i_2} are orthogonal and so are the vectors $s(P_{v_{i_1}})$, $s(P_{v_{i_2}})$. It follows that T_s is a one-dimensional projection and the proof of Lemma 2.4 is complete. □

Let us now return to the proof of Theorem 2.1. Let $v \in H$ be a unit vector such that $T_s = P_v$ (Lemma 2.3, 2.4). Let s_v be the h-state associated with v. We shall first prove that $\langle s_v(P_M), s_v(P_N) \rangle = \langle s(P_M), s(P_N) \rangle$ for every $P_M, P_N \in L(H)$. Indeed, we have $\langle s_v(P_M), s_v(P_N) \rangle = \langle P_M v, P_N v \rangle = \langle P_N P_M v, v \rangle = \langle P_N P_M v, P_v v \rangle = \langle P_v P_N P_M v, v \rangle = \mathrm{Tr}(P_v P_N P_M) = \langle s(P_M), s(P_N) \rangle$. Put, as before, $\mathcal{R}(s_v) = \{s_v(P_M) \mid P_M \in L(H)\}$, and introduce a mapping $\tilde{\mathcal{U}} \colon \mathcal{R}(s_v) \to H$ by putting $\tilde{\mathcal{U}}(s_v(P_M)) = s(P_M)$ $(P_M \in L(H))$. Let us check that $\tilde{\mathcal{U}}$ is well defined and can be linearly extended over the linear span of $\mathcal{R}(s_v)$. Suppose that $\sum_{i=1}^{m} \alpha_i s_v(P_{M_i}) = s_v(M)$, where $\alpha_i \in R$ and $P_{M_i} \in L(H)$ $(i \leqslant m)$. We obtain

$$\begin{aligned}
\Big\| \sum_{i=1}^{m} \alpha_i s(P_{M_i}) - s(M) \Big\|^2 &= \Big\| \sum_{i=1}^{m} \alpha_i s(P_{M_i}) \Big\|^2 + \|s(P_M)\|^2 - 2 \Big\langle \sum_{i=1}^{m} \alpha_i s(P_{M_i}), s(P_M) \Big\rangle \\
&= \Big\| \sum_{i=1}^{m} \alpha_i s_v(P_{M_i}) \Big\|^2 + \|s_v(P_M)\|^2 - 2 \Big\langle \sum_{i=1}^{m} \alpha_i s_v(P_{M_i}), s_v(P_M) \Big\rangle \\
&= \Big\| \sum_{i=1}^{n} \alpha_i s_v(P_{M_i}) - s_v(P_M) \Big\|^2 = 0
\end{aligned}$$

The rest is easy—by applying the equality $\langle s_v(P_M), s_v(P_N) \rangle = \langle s(P_M), s(P_N) \rangle$ we extend $\tilde{\mathcal{U}}$ to a unitary mapping $\mathcal{U} \colon H \to H$ and the proof is complete. □

Let us now consider the case of the h-states $s \colon L(H_1) \to H_2$, where $\dim H_2 < \dim H_1$. The situation here is quite transparent.

Theorem 2.5. *Let H_1, H_2 be Hilbert spaces with $\dim H_2 < \dim H_1 < \infty$, $\dim H_1 \geqslant 3$. Then there is no h-state $s \colon L(H_1) \to H_2$.*

Proof (see also [13]). Suppose that $\dim H_1 = m$. Suppose further that $s \colon L(H_1) \to H_2$ is an h-state. Then $t_s \colon L(H_1) \to \langle 0, 1 \rangle$, $t_s(P_M) = \|s(P_M)\|^2$ is a state on $L(H_1)$. By Gleason's theorem [5] we can write $t_s(P_M) = \sum_{i=1}^{m} \alpha_i \|P_M(v_i)\|^2$, where $\alpha_1 > 0$ and $v_i \neq 0$, $v_i \in H$ $(i \leqslant m)$. Let $e_1, e_2, \ldots, e_m$ be an orthogonal basis of H_1 which is taken such that $\langle e_i, v_1 \rangle \neq 0$ $(i \leqslant m)$. Then

$$t_s(P_{e_i}) = \sum_{j=1}^{m} \alpha_j \left\| \langle e_i, v_j \rangle\, e_i \right\|^2 \geqslant \alpha_1 \left| \langle e_i, v_1 \rangle \right|^2 > 0.$$

This implies that $s(P_{e_i}) \neq 0$ $(i \leqslant m)$. Since the vectors $s(P_{e_i})$ have to be mutually orthogonal in H_2, we have derived a contradiction. The proof of Theorem 2.5 is complete. □

Obviously, the assumption of $\dim H_1 \geqslant 3$ in the last theorem is essential. For instance, if we define a mapping $s\colon L(R^2) \to R^1$ by putting $s(P_v) = 1$ or $s(P_v) = 0$ if $\arg v \in \left(0, \frac{\pi}{2}\right\rangle \cup \left(\pi, \frac{3\pi}{2}\right\rangle$ or $\arg v \in \left(\frac{\pi}{2}, \pi\right\rangle \cup \left(\frac{3\pi}{2}, 2\pi\right\rangle$, respectively, then s can be easily extended over $L(R^2)$ to an h-state. It should be also noted that a simple alternation of this example gives an h-state $t\colon L(R^2) \to R^2$ which does not allow the description of Theorem 2.2.

To conclude this section, let us ask if Theorems 2.1, 2.5 remain valid for $\dim H_1 = \infty$. As we shall see, Theorem 2.5 remains valid while Theorem 2.1 does not.

Proposition 2.6. *Let H_1, H_2 be Hilbert spaces such that $\dim H_1 = \infty$, $\dim H_2 < \infty$. Then there is no h-state $s\colon L(H_1) \to H_2$.*

Proof. Suppose that $\dim H_2 = n$ and write $H_1 = \bigoplus_{\alpha \in I} H_\alpha$, where each H_α is a unitary copy of an $(n+2)$-dimensional Hilbert space K. Define now a mapping $f\colon L(K) \to L(H_1)$ such that, for any $P_M \in L(K)$, $f(P_M) = \bigoplus_{\alpha \in I} P_{M_\alpha}$, where M_α is a unitary copy of M. One can easily check that $f(P_M \vee P_N) = f(P_M) \vee f(P_N)$ and $f(P_K) = P_{H_1}$. Moreover, if P_M is orthogonal to P_N then $f(P_M)$ is orthogonal to $f(P_N)$, too. If $s\colon L(H_1) \to H_2$ is an h-state, then so is $\tilde{s} = s \circ f\colon L(K) \to H_2$, which contradicts Theorem 2.5. The proof of Theorem 2.6 is complete. □

It should be noticed that the above proposition generalizes the analogous result for two-valued states (see [1]).

Proposition 2.7. *Let H be a Hilbert space with $\dim H = \infty$. Then for every natural number $n \in N$ there is an h-state $s\colon L(H) \to H$ such that $\left\| s(P_M) \right\| < 1$ whenever $\dim M = n$.*

Proof. Let $v_1, \ldots, v_{n+1}$ be such nonzero orthogonal vectors in H that $\sum_{i=1}^{n+1} \|v_i\|^2 = 1$. By identifying H with the direct sum $\bigoplus_{i=1}^{n+1} H_i$, where $H_i = H$ $(i \leqslant n+1)$, we can define an h-state $s\colon L(H) \to H$ be setting $s(P_N) = \bigoplus_{i=1}^{n+1} P_N v_i$. Then for the state t_s associated with s, $t_s(P_N) = \left\| s(P_N) \right\|^2$, we have $t_s(P_N) = \sum_{i=1}^{n+1} \|P_N v_i\|^2$.

Let M be a subspace of H with $\dim H = n$. We shall prove that $\|s(P_M)\| < 1$. Looking for a contradiction, let us suppose that $\|s(P_M)\| = 1$. Then for the orthogonal complement M' of M we have $t_s(P_{M'}) = 0$. Thus, $P_{M'}v_i = 0$ $(i \leqslant n+1)$. We finally obtain that $v_i \in M$ $(i = 1, 2, \ldots, n+1)$, which is a contradiction. The proof is complete. □

Remark 2.8. It should be noted that in the paper [12] the author studies formally similar questions to those we pursued in this section (esp. in Theorem 2.1). However, we have not been able to establish any concrete direct dependence of our results to his paper and vice versa. Moreover, he deals exclusively with Hilbert spaces over the complex numbers where he has the spectral theorem and its consequences at his disposal. We study only real Hilbert spaces where we cannot benefit from the spectral theorem and therefore have to derive our results by fully geometric reasonings.

3. The h-states on general logics. "Rich" logics and extensions of h-states

Let H be a Hilbert space with $\dim H < \infty$. As we have seen, there are logics having no h-states with values in H (Theorem 2.5). In fact, this "pathalogy" may occur even for finite logics (see the next section). Obviously, if we contemplate a potential application in quantum axiomatics we would be interested rather in logics having reasonably many h-states. This leads us to the following definition. (As usual, by a partition of unity we mean a set $\{\alpha_i \mid i \leqslant n\}$ of nonnegative numbers such that $\sum\limits_{i=1}^{n} \alpha_i = 1$.)

Definition 3.1. Let H be a Hilbert space and let $\dim H = n < \infty$. Let L be a logic. We say that L is H-rich if for any partition of unity $\{\alpha_i \mid i \leqslant n\}$ and any set $\{k_i \mid i \leqslant n\}$ of mutually orthogonal non-zero elements in L there is an h-state $s\colon L \to H$ such that $\|s(k_i)\|^2 = \alpha_i$ $(i \leqslant n)$.

Let us first see that for a given H the class of H-rich logics is considerably large.

Proposition 3.2. *The logic $L(H)$ is H-rich.*

Proof. If $\dim H = n$ and if we are given a partition of unity $\{\alpha_i \mid i \leqslant n\}$ and a set $\{P_{A_i} \mid i \leqslant n\}$. where P_{A_i} are orthogonal projections in H, then we can choose vectors $u_i \in A_i$ such that $\|u_i\| = \sqrt{\alpha_i}$. Then we put $s(P_M) = P_M\Big(\sum\limits_{i\leqslant n} u_i\Big)$. □

Prior to the following proposition, let us recall a natural and useful class of logics (see e.g. [7],[15], etc.). Let a logic be called *concrete* if it can be represented by a collection, say Δ, of a subsets of a set S. The relation $\leqslant$ in Δ means the inclusion relation and the operation $'$ means the formation of the complement (in S). Thus, (S, Δ) is a concrete logic if 1. $\emptyset \in \Delta$. 2. if $A \in \Delta$ then $A' \in \Delta$, and 3. if $A, B \in \Delta$ and $A \cap B = \emptyset$ then $A \cup B \in \Delta$. Obviously, every Boolean algebra can be viewed as a concrete logic (we take the set representation).

Proposition 3.3. *Let H be a Hilbert space with* $\dim H < \infty$. *Then every concrete logic is H-rich.*

Proof. Suppose that $\dim H = n$ and suppose that we are given a partition of unity $\{\alpha_i \mid i \leqslant n\}$ and a set $\{A_i \mid i \leqslant n\} \subset \Delta$, where A_i are mutually disjoint nonvoid subsets of S. Choose points $p_i \in A_i$ $(i \leqslant n)$ and take an orthonormal basis of H, $\{e_i \mid i \leqslant n\}$. For any $M \in \Delta$ write $C(M) = \{i \leqslant n \mid p_i \in M\}$ and define a mapping $s \colon \Delta \to H$ by putting $s(M) = \sum_{i \in C(M)} \sqrt{\alpha_i} e_i$. Then the mapping s is an h-state and moreover, $\|s(A_i)\| = \sqrt{\alpha_i}$. The proof of Prop. 3.3 is complete. □

We will now state the main result of this section. We shall prove that the H-rich logics are exactly those logics L for which the h-states of $\mathcal{S}_H(B)$, where B is an arbitrary Boolean subalgebra of L, admit extensions to h-states of $\mathcal{S}_H(L)$. This result can be viewed as a generalization of the classical result for Boolean algebras and two-valued states (see also the comments below). (The following Theorem 3.4 may also have certain hearing on the quantum axiomatics. We can understand the h-states as "generalized hidden variables" (compare with [4], [7], [13], etc.).) If we accept the hidden variables hypothesis for L then, for any Boolean subalgebra of L, the set $\mathcal{S}_H(L)$ should contain the set $\mathcal{S}_H(B)$. Thus, in this interpretation, Theorem 3.4 establishes the right class of logics for the hidden variables postulate.)

Theorem 3.4. *Let H be a Hilbert space with* $\dim H < \infty$ *and let L be a logic. Then the following two statements are equivalent:*

(i) The logic L is H-rich.

(ii) For any Boolean subalgebra B of L and any h-state $s \colon B \to H$ there is an h-state $\tilde{s} \colon L \to H$ such that $\tilde{s}(b) = s(b)$ for any $b \in B$.

Proof. (ii) $\Rightarrow$ (i). Suppose that $\dim H = n$ and suppose that we are given a partition of unity $\{\alpha_i \mid i \leqslant n\}$ and a set $\{k_i \mid i \leqslant n\}$ of mutually orthogonal non-zero elements in L. Obviously, the set $\{k_i \mid i \leqslant n\}$ generates a (finite) Boolean subalgebra B of L. By Prop. 3.3, there is an h-state $s \colon B \to H$ such that $\|s(k_i)\|^2 = \alpha_i$. Since by the condition (ii) the state s has an extension over L, we see that L is H-rich.

(i) $\Rightarrow$ (ii). Let us first observe that $\mathcal{S}_H(L)$ is a compact topological space when understood with the "pointwise" topology. Indeed, the set $\mathcal{S}_H(L)$ can be viewed as a subspace of the topological product E^L, where E is a unit ball in H (see Prop. 1.3). Since E is compact, E^L is compact as well (Tichonoff's theorem) and a straightforward verification gives that $\mathcal{S}_H(L)$ is closed in E^L.

Suppose now that we are given an h-state $s\colon B \to H$, where B is a Boolean subalgebra of L. Denote by $\mathcal{P}$ the set of all finite orthogonal coverings of B. The set $\mathcal{P}$ is naturally ordered by the refinement relation—we write $P \leqslant R$ $(P, R \in \mathcal{P})$ if for any $p \in P$ there exists na $r \in R$ such that $p \leqslant r$.

For any "partition" $P = \{p_1, p_2, \ldots, p_k\} \in \mathcal{P}$ put $\mathcal{S}_P = \{\bar{s} \in \mathcal{S}_H(L) \mid \bar{s}(p_i) = s(p_i)\}$. Let us see first that $\mathcal{S}_P \neq \emptyset$ for any $p \in \mathcal{P}$. Indeed, since $s(p_i)$ are orthogonal in H, we infer that at most $n(=\dim H)$ of them differ from 0. Thus, by our assumption, there is an h-state $t\colon L \to H$ such that $\|t(p_i)\| = \|s(p_i)\|$ $(i \leqslant k)$. Obviously, there is a unitary mapping $\mathcal{U}\colon H \to H$ such that $\mathcal{U}(t(p_i)) = s(p_i)$ $(i \leqslant k)$. Thus, if we put $\bar{t} = \mathcal{U} \circ t$ then $\bar{t} \in \mathcal{S}_P$.

We shall show now that the set $\{\mathcal{S}_P \mid P \in \mathcal{P}\}$ is a filter base in $\mathcal{S}_H(L)$ consisting of closed sets. The latter fact is obvious as $\mathcal{S}_H(L)$ is endowed with the pointwise topology. Let us show that for any $P_1, P_2 \in \mathcal{P}$ we have $\mathcal{S}_{P_1} \cap \mathcal{S}_{P_2} \neq \emptyset$. Put $Q = P_1 \cap P_2 = \{p \wedge q \mid p \in P_1, q \in P_2\}$. Then $Q \in \mathcal{P}$ and one sees easily that $\mathcal{S}_Q \subset \mathcal{S}_{P_1} \cap \mathcal{S}_{P_2}$. Thus, $\mathcal{S}_{P_1} \cap \mathcal{S}_{P_2} \neq \emptyset$.

Let us now complete the proof. Since $\mathcal{S}_H(L)$ is compact, we obtain $\bigcap\limits_{P \in \mathcal{P}} \mathcal{S}_P \neq \emptyset$. Choosen an h-state $\tilde{s}$ in $\bigcap\limits_{P \in \mathcal{P}} \mathcal{S}_P$. By the definition of $\mathcal{S}_P$ we have $\tilde{s}(b) = s(b)$ for any $b \in B$. Therefore $\tilde{s}$ extends s and the proof is complete. □

If we specialize the previous result to concrete logic, we obtain the following result. It should be noticed that its Boolean corollary generalizes the well-kown theorem on the extensions of two-valued states on Boolean algebras (see [9]).

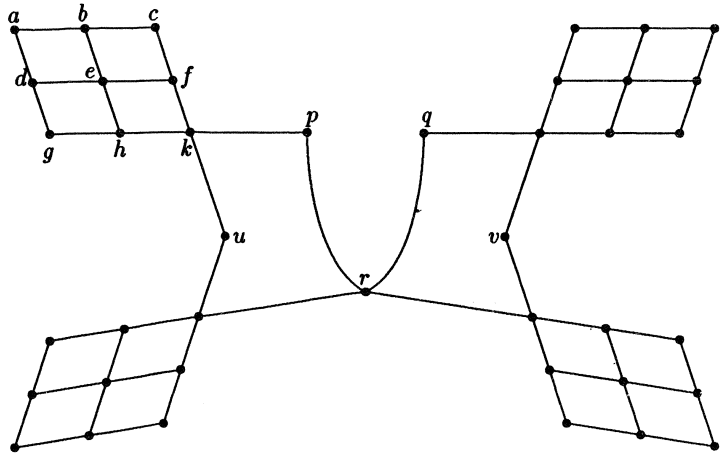
Theorem 3.5. *Let H be a Hilbert space with* $\dim H < \infty$. *Let L be a concrete logic and let B be a Boolean subalgebra of L. Then for every h-state $s\colon B \to H$ there exists an h-state $\tilde{s}\colon L \to H$ such that $\tilde{s}(b) = s(b)$ for any $b \in B$. A corollary: If B_1 is a Boolean subalgebra of a Boolean algebra B_2, then every h-state $s \in \mathcal{S}_H(B_1)$ admits an extension over B_2.*

Proof. It follows from Prop. 3.2 and Theorem 3.4. □

4. Appendix: A finite logic having no h-states

Here we construct a finite logic L such that $\mathcal{S}_H(L) = \emptyset$ for any Hilbert space H. We do so by utilizing the technique invented by R. Greechie ([6]). Since our example possesses ordinary states, which we leave to the reader to check, it actually presents a little novelty in this area of orthomodular curiosities. (Since the Greechie technique has become a "folklore" in orthomodular posets—see [6], [11], [14], [17], etc.—we will not repeat its ideas here. Let us only recall that the points in "Greechie diagram" represent atoms in L, and the sets of points connected by angleless lines represent the sets of atoms in L belonging to a Boolean subalgebra of L.)

E x a m p l e. The logic L determined by the Greechie diagram below has no h-states.



P r o o f. Let $s \in \mathcal{S}_H(L)$ for a Hilbert space H. We shall prove that $s(p) = s(u)$. Then by symmetry we obtain $s(p) = s(u) = s(r) = s(v) = s(q)$. Since p, q, r are pairwise orthogonal then so are the vectors $s(p)$, $s(q)$, $s(r)$ in H. Therefore $s(p) = s(q) = s(r) = 0$, which is absurd because p, q, r are all atoms of a Boolean subalgebra of L and therefore $\|s(p) + s(q) + s(r)\| = \|s(1)\| = 1$.

It remains to prove the equality $s(p) = s(u)$. As we see the Greechie diagram, $\big(s(a)+s(d)+s(g)\big) + \big(s(b)+s(e)+s(h)\big) = \big(s(a)+s(b)+s(c)\big) + \big(s(d)+s(e)+s(f)\big)$ (here the sums in brackets equal $s(1)$). We therefore obtain $s(g)+s(h) = s(e)+s(f)$ and this yields $s(p) = s(u)$. The proof is complete. $\square$

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