Applications of Mathematics

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Applications of Mathematics, Vol. 37 (1992), No. 1, 40–50

Persistent URL: http://dml.cz/dmlcz/104490

SUM OF OBSERVABLES IN FUZZY QUANTUM SPACES

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(Received June 11, 1990)

Summary. We introduce the sum of observables in fuzzy quantum spaces which generalize the Kolmogorov probability space using the ideas of fuzzy set theory.

Keywords: Fuzzy quantum space, observable, sum of observables.

AMS classification: primary 81B10; secondary 03E72.

1. INTRODUCTION

The main notion of the Kolmogorov classical model of probability theory [5] is a σ-algebra of subsets of a set. This model has been very useful, however, it does not describe situation in quantum mechanical measurements. There are many axiomatic models of quantum mechanics, and today there is a widespread model of quantum logics, see for example [12]. Two of the most important examples of non-Boolean quantum logic models are the system of all closed subspaces of a Hilbert space [6] and the quantum probability spaces introduced by Suppes [11].

The Kolmogorov probability model may be uniquely represented by a system of characteristic functions of subsets of a set X from the given σ-algebra $\mathcal{S}$, which have values in the closed interval [0,1]. When a quantum mechanical event a, say, is described vaguely, then by a fuzzy set a, that is a fuzzy event a, we shall understand a real-valued function $a\colon X \to [0,1]$ which describes the quantum machanical event a: this is a basic idea of Zadeh's theory [13].

The intersection $\cap$ and the union $\cup$ of fuzzy sets $\{a_i\}$, the complement $\perp$ of a fuzzy set a are defined as

$$\bigcap_i a_i := \inf_i a_i,$$
$$\bigcup_i a_i := \sup_i a_i,$$
$$a^{\perp} := 1 - a.$$

If f and g are two $\mathscr{S}$-measurable functions, then the measurability of the sum $f + g$ may be proved using the following simple relation

$$\{x \in X\colon (f+g)(x) < t\} = \bigcup_{r \in Q} \{x \in X : f(x) < r\} \cap \{x \in X : g(x) < t - r\}, \tag{1.1}$$

where Q is the set of all rationals.
Using this fact in the present note, we will define the sum of any pair of F-observables of a fuzzy quantum space.

2. Fuzzy quantum spaces

Definition 2.1. A fuzzy quantum space is a couple (X, M), where X is a nonempy set and $M \subset [0,1]^X$ satisfies the following conditions:
(i) if $1(x) = 1$ for any $x \in X$, then $1 \in M$;
(ii) if $a \in M$, then $a^{\perp} := 1 - a \in M$;
(iii) if $\frac{1}{2}(x) = \frac{1}{2}$ for any $x \in X$, then $\frac{1}{2} \notin M$;
(iv) $\bigcup_{n=1}^{\infty} a_n := \sup_n a_n \in M$ for any $\{a_n\}_{n=1}^{\infty} \subset M$.

In the fuzzy sets theory the system M is called a soft σ-algebra [7].

This structure has been suggested by Riečan [9] as an alternative axiomatic model for quantum mechanics. More general structure assuming that M is closed with respect to the union of any sequence of mutually orthogonal fuzzy sets has been proposed by Pykacz [8] and studied by Dvurečenskij and Chovanec [1]. Some fuzzy sets ideas have been studied also by Guz [4], but his approach is different from ours.

The analogue of a random variable is an F-quantum observable: An f-observable on a fuzzy quantum space(X, M) is a mapping $x\colon B(R^1) \to M$ with the following properties:
(i) $x(E^c) = 1 - x(E)$ for every $E \in B(R^1)$;
(ii) if $\{E_n\}_{n=1}^{\infty} \subset B(R^1)$, then $x(\bigcup_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} x(E_n)$, where $B(R^1)$ is the Borel σ-algebra of the real line R^1 andE^c denotes the complement of E in R^1.

In particular, for $a \in M$, the mapping $x_a\colon B(R^1) \to M$ defined by

$$x_a(E) = \begin{cases} a \cap a^\perp & 0,1 \notin E \\ a^\perp & 0 \in E, 1 \notin E \\ a & 0 \notin E, 1 \in E \\ a \cup a^\perp & 0,1 \in E \end{cases} \qquad (E \in B(R^1))$$

is an F-observable of (X, M) called the indicator of the fuzzy set $a \in M$.

If $f\colon R^1 \to R^1$ is a Borel measurable function and x is an F-observable, then $f \circ x\colon E \to x(f^{-1}(E))$, $E \in B(R^1)$, is an F-observable, too. In particular, if $\alpha \in R^1$, then $\alpha x\colon E \to x(\{t \in R^1\colon \alpha t \in E\})$ for any $E \in B(R^1)$.

Let (X, M) be a fuzzy quantum space. The set M may be regarded as a partually ordered set in which we define $a \leqslant b$ iff $a(x) \leqslant b(x)$ for any $x \in X$. Using the complementation $\perp\, : a \to a^\perp = 1 - a$ for any fuzzy set $a \in M$, we see that $\perp$ satisfies two conditions

(i) $(a^\perp)^\perp = a$ for any $a \in M$;

(ii) if $a \leqslant b$, then $b^\perp \leqslant a^\perp$. It is evident that $a \cup a^\perp = 1$ iff a is a crisp set. Hence M is a distributive σ-lattice with the complementation $\perp$, for which de Morgan laws

$$\left(\bigcup_i a_i\right)^\perp = \bigcap_i a_i^\perp, \tag{2.1}$$

$$\left(\bigcap_i a_i\right)^\perp = \bigcup_i a_i^\perp \tag{2.2}$$

hold whenever $\{a_i\} \subset M$.

A nonempty subset $A \subset M$ is called a Boolean algebra (σ-algebra) of a fuzzy quantum space (X, M) if

(i) there are minimal and maximal elements 0_A and 1_A from A such that for any $a \in A$, $0_A \leqslant a \leqslant 1_A$ and $a \cup a^\perp = 1_A$ (we recall that 0_A and 1_A are not crisp sets, in general);

(ii) A is boolean algebra (σ-algebra).

It is clear that $0_A \neq 1_A$. For example, if a is a fuzzy set from M, then $A_a = \{a \cap a^\perp, a^\perp, a, a \cup a^\perp\}$ is a Boolean algebra with the minimal and maximal elements $0_{A_a} = a \cap a^\perp$ and $1_{A_a} = a \cup a^\perp$, respectively.

In particular, if x is an f-observable of (X, M), then the range $R(x) = \{x(E)\colon E \in B(R^1)\}$ is a Boolean σ-algebra of (X, M) with the minimal and maximal elements $0_{R(x)} = x(\emptyset)$ and $1_{R(x)} = x(R^1)$.

In accordance with the theory of quantum logics, we say that two elements $a, b \in M$ are:

(i) orthogonal if $a \leqslant 1 - b$, and we write $a \perp b$;
(ii) compatible if $a = a \cap b \cup a \cap b^{\perp}, b = b \cap a \cup b \cap a^{\perp}$, and we write $a \leftrightarrow b$;
(iii) strongly compatible if $a \leftrightarrow b \leftrightarrow a^{\perp} \leftrightarrow b^{\perp} \leftrightarrow a$, and we write $a \overset{s}{\leftrightarrow} b$. Two observables x and y are compatible if $x(E) \leftrightarrow y(F)$ for any $E, F \in B(R^1)$.

The following result has been proved by Dvurečenskij and Riečan [2]:

Theorem 2.2. *Let $\{a_t : t \in T\}$ be a system of fuzzy sets from M. The following assertions are equivalent:*

(i) $\{a_t : t \in T\}$ is a system of mutually strongly compatible elements;

(ii) $a_s \cup a_s^{\perp} = a_t \cup a_t^{\perp}$ for any $s, t \in T$;

(iii) there is a Boolean σ-algebra of M containing all $\{a_t : t \in T\}$.

Now we characterize F-observables of a fuzzy quantum space (X, M).

Theorem 2.3. *Let x be an F-observable of a fuzzy quantum space (X, M) and let $B_x(t) = x\big((-\infty, t)\big)$, $t \in R^1$. Then the system $\{B_x(t) : t \in R^1\}$ fulfils the following conditions:*

(i) $B_x(s) \leqslant B_x(t)$ *if* $s < t$;

(ii) $\bigcup_t B_x(t) = a$;

(iii) $\bigcap_t B_x(t) = a^{\perp}$;

(iv) $\bigcup_{t<s} B_x(t) = B_x(s)$;

(v) $B_x(t) \cup B_x^{\perp}(t) = a$, *where* $a = X(R^1)$ *and* $a^{\perp} = x(\emptyset)$.

Conversely, if a system $\{B(t) : t \in R^1\}$ of fuzzy sets of a fuzzy quantum space (X, M) fulfils the conditions (i)–(v) for some $a \in M$, then there is a unique F-observable x such that $B_x(t) = B(t)$ for any t, and $x(R^1) = a$.

Proof. (i) is trivial.

(ii) Let $a = x(R^1)$, then $x((-\infty, t)) \leqslant a$. For every integer n we have

$$x\big((-\infty, n)\big) \leqslant a \text{ and } x(R^1) = x\Big(\bigcup_{n=1}^{\infty}(-\infty, n)\Big) = \bigcup_{n=1}^{\infty} x((-\infty, n)).$$

Similarly we prove (iii).

(iv) the condition (i) implies $B_x(t) \leqslant B_x(s)$ for every $t \leqslant s$, so that

$$B_x(s) = \bigcup_{n=1}^{\infty} B_x\left(s - \frac{1}{n}\right).$$

(v) It may be proved as follows: $B_x(t) \cup B_x^{\perp}(t) = x((-\infty,t) \cup (-\infty,t)^c) = x(R^1)$.

For the fuzzy quantum space (X, M) let now a system $\{B(t) : t \in R^1\}$ satisfying (i)–(v) be given. Due to (v), the system $\{B(t) : T \in R^1\}$ consists of mutually strongly compatible elements of M so that, according to Theorem 2.2, there is a minimal Boolean σ-algebra $\mathscr{A}$ of M containing all $B(t)$'s. By the Loomis-Sikorski theorem [10], there is a measurable space $(\Omega, \mathscr{S})$ and a homomorphism h from $\mathscr{S}$ onto $\mathscr{A}$. Let r_1, r_2, r_3, ... be any distinct enumeration of the rationals. We claim to construct, by induction, sets A_1, A_2, ... from $\mathscr{S}$ such that

(a) $h(A_i) = B(r_i)$;

(b) $A_i \subset A_j$ if $r_i < r_j$;

(c) $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

We note that if $A \subset B$, $A \in \mathscr{S}$ and if there is a $c \in \mathscr{A}$ such that $h(A) \leqslant c \leqslant h(B)$, then there is a $C \in \mathscr{S}$ such that $A \subset C \subset B$, $h(C) = c$. Indeed, since h maps $\mathscr{S}$ onto $\mathscr{A}$, there is a $C_1 \in \mathscr{S}$ such that $h(C_1) = c$. If we define $C = (C_1 \cap B) \cup A$ then C has the given property. Let A_1 be any set in $\mathscr{S}$ such that $h(A_1) = B(r_1)$ Suppose A_1, A_2, ..., $A_n \in \mathscr{S}$ have been constructed so that (a) and (b) hold. We shall construct A_{n+1} as follows. Let $(i_1, \ldots, i_n)$ be the permutation of $(1, \ldots, n)$ such that $r_{i_1} < \ldots < r_{i_n}$. Then only one of the following conditions holds:

(2.3)
(i) $r_{n+1} < r_{i_1}$;
(ii) $r_{n+1} > r_{i_n}$;
(iii) there is a unique $k = 1, \ldots, n-1$ such that $r_{i_k} < r_{n+1} < r_{i_{k+1}}$,

and by the above observation we can select A_{n+1} such that $h(A_{n+1}) = B(r_{n+1})$ and

(i) $A_{n+1} \subseteq A_i$;

(ii) $A_{n+1} \supseteq A_i$;

(iii) $A_{i_k} \subseteq A_{n+1} \subseteq A_{i_{k+1}}$,

according to (2.3). Then the system $\{A_1, \ldots, A_{n+1}\}$ fulfils (a) and (b). Thus, by induction, it follows that there is a sequence $\{A_j\}$ of sets in $\mathscr{S}$ with the properties (a) and (b). As

$$h\Big(\bigcap_{j=1}^{\infty} A_j\Big) = \bigcap_{j=1}^{\infty} h(A_j) = \bigcap_{j=1}^{\infty} B(r_j) = 0_{\mathscr{A}},$$

we may replacing A_j by $A_j - \bigcap_i A_i$ if necessary, assume that $\bigcap_j A_j = \emptyset$. We define an $\mathscr{S}$-measurable function f as follows:

$$f(\omega) = \begin{cases} 0 & \text{if } \omega \notin \bigcup_{j=1}^{\infty} A_j \\ \inf\{r_j : \omega \in A_j\} & \text{if } \omega \in \bigcup_{j=1}^{\infty} A_j. \end{cases}$$

The function f is everywhere well-defined and finite. Moreover,

$$f^{-1}((-\infty, r_k)) = \begin{cases} \bigcup\limits_{r_j < r_k} A_j \text{ if } r_k \leqslant 0 \\ \bigcup\limits_{r_j < r_k} A_j \cup (\Omega - \bigcup\limits_i A_i) \quad \text{if} \quad r_k > 0, \end{cases}$$

hence f is $\mathscr{S}$-measurabale and $h(f^-1((-\infty, r_k))) = B(r_k)$. If define an observable by $x(E) = h(f^{-1}(E))$, $E \in B(R^1)$, then $x((-\infty, t)) = B(t)$ for every $t \in R^1$. The equality $x_1((-\infty, t)) = x_2((-\infty, t))$ for every $t \in R^1$ implies $x_1 = x_2$, hence, the uniqueness of x is shown and the proof is complete. □

3. Existence of a sum

In accordance with (1.1), we define the sum of two observables as follows.

Definition 3.1. Let x and y be two F-observables of a fuzzy quantum space (X, M). If the system $\{B_{x+y}(t) : t \in R^1\}$,

$$B_{x+y}(t) = \bigcup_{r \in Q} (B_x(r) \cap B_y(t - r)), \; t \in R^1, \tag{3.1}$$

where Q is the set of all rationals, determines an F-observable z of (X, M), then we call it the sum of x and y, and we write $z = x + y$.

It is clear that if the sum exists, then it is unique. For the proof of Theorem 3.3 the followings lemma is useful.

Lemma 3.2. *Let S be a countable set in R^1. For observables x and y let us denote*

$$B^S_{x+y}(t) = \bigcup_{s \in S} (B_x(s) \cap B_y(t - s)), \tag{3.2}$$

then

$$B^S_{x+y}(t) = B_{x+y}(t) \quad \text{for every } t \in R^1.$$

Proof. We can show that if $t_n \uparrow t$, $t_n \in S$, then $B^S_{x+y}(t) = \bigcup\limits_n B^S_{x+y}(t_n)$. Indeed,

$$\bigcup_n B^S_{x+y}(t_n) = \bigcup_n \bigcup_{s \in S} (B_x(s) \cap B_y(t_n - s)) =$$
$$\bigcup_{s \in S} (B_x(s) \cap \bigcup_n B_y(t_n - s)) = \bigcup_{s \in S} (B_x(s) \cap B_y(t - s)).$$

Let now n be any integer, then for each $s \in S$ there is $r = r(s) \in Q$ such that we have $s < r < s + \frac{1}{n}$. Therefore, $B_x(s) \cap B_y(t - n^{-1} - s) \leqslant B_x(r) \cap B_y(t - r)$ and $B^S_{x+y}(t - n^{-1}) \leqslant B_{x+y}(t)$, $B^S_{x+y}(t) = \bigcup\limits_n B^S_{x+y}(t - n^{-1}) \leqslant B_{x+y}(t)$. Similarly we show that $B_{x+y}(t) \leqslant B^S_{x+y}(t)$. □

Theorem 3.3. *For every two F-observables x and y of a fuzzy quantum space (X, M) their sum exists.*

Proof. We show that the system $\{B_{x+y}(t)\colon t \in R^1\}$ fulfils the conditions of Theorem 2.3. The proof of (i), (ii) and (iv) is simple, due to the σ-continuity of M, that is, if $a_1 \leqslant a_2 \leqslant \ldots \in M$, then for any $b \in M$, $b \cap (\bigcup_i a_i) = \bigcup_i b \cap a_i$.

Calculate:

$$
\begin{aligned}
a &= B_{x+y}(t) \cup B^{\perp}_{x+y}(t) = \\
&= \bigcup_{r\in Q} (B_x(r) \cap B_y(t-r)) \cup \bigcap_{s\in Q} (B^{\perp}_x(s) \cup B^{\perp}_y(t-s)) = \\
&= \bigcap_s \bigcup_r (B_x(r) \cap B_y(t-r)) \cup (B^{\perp}_x(s) \cup B^{\perp}_y(t-s)) = \\
&= \bigcap_s \bigcup_r ((B_x(r) \cup B^{\perp}_x(s) \cup B^{\perp}_y(t-s)) \cap (B_y(t-r) \cup B^{\perp}_x(s) \cup B^{\perp}_y(t-s))).
\end{aligned}
$$

Since $B_x(r) \cup B^{\perp}_x(r) = x(R^1)$ and $B_x(r) \cup B^{\perp}_x(s) = x(R^1)$ for $s \leqslant r$, and

$$
\begin{aligned}
a = \bigcap_s (\bigcup_{r\geqslant s} (B_x(r) \cup B^{\perp}_x(s) \cup B^{\perp}_y(t-r)) \cap (B_y(t-r) \cup B^{\perp}_x(s) \cup B^{\perp}_y(t-s)) \cup \\
\cup \bigcup_{r<s} (B_x(r) \cup B^{\perp}_x(s) \cup B^{\perp}_y(t-s)) \cap (B_y(t-r) \cup B^{\perp}_x(s) \cup B^{\perp}_y(t-s))) = \\
= \bigcap_s ((x(R^1) \cup B^{\perp}_y \perp (t-s)) \cap (\bigcup_{r\geqslant s} B_y(t-r) \cup \\
\cup B^{\perp}_x(s) \cup B_y \perp (t-s)) \cap (\bigcup_{r<s} B_x(r) \cup B^{\perp}_x(s) \cup B^{\perp}_y(t-s)) \cap (y(R^1) \cup B^{\perp}_x(s))),
\end{aligned}
$$

we conclude that

$$
r \downarrow s \text{ implies } t-r \uparrow t-s \text{ and } \bigcup_{r\geqslant s} B_y(t-r) = B_y(t-s).
$$

Then

$$
\begin{aligned}
a = \bigcap_s \left((x(R^1) \cup B^{\perp}_y(t-s)) \cap \left(\bigcup_{r\geqslant s} B_y(t-r) \cup B^{\perp}_x(s) \cup B^{\perp}_y(t-s) \right) \cap \right. \\
\cap \left(\bigcup_{r<s} (B_x(r) \cup B^{\perp}_x(s) \cup B^{\perp}_y(t-s)) \cap (y(R^1) \cup B^{\perp}_x(s))) \right) = \\
= \bigcap_s \left((x(R^1) \cup B^{\perp}_y(t-s)) \cap (y(R^1) \cup B^{\perp}_x(s)) \cap x(R^1) \cup B^{\perp}_y(t-s) \right) \cap \\
\cap (y(R^1) \cup B^{\perp}_x(s)) = \\
= \bigcup_s \left((x(R^1) \cup B^{\perp}_y(t-s) \cap (y(R^1) \cup B^{\perp}_x(s))) = \right. \\
= (x(R^1) \cup \bigcap B_y(t-s)) \cap (y(R^1) \cup \bigcap_s B_x(s)) \Big).
\end{aligned}
$$

In other words, we have proved

$$a = (x(R^1) \cup y(\emptyset)) \cap (y(R^1) \cup x(\emptyset)) = x(R^1) \cap y(R^1),$$

which means the strong compatibility of $\{B_{x+y}(t) : t \in R^1\}$, too. To prove

$$\bigcap_{r \in R^1} B_{x+y}(t) = x(\emptyset) \cup y(\emptyset) = a^{\perp}, \tag{3.3}$$

we take into account that, by virtue of the property (v) of Theorem 2.3, $\{B_{x+y}(t) : t \in R^1\}$ is a system of mutually strongly compatible elements of a fuzzy quantum space (X, M). By Lemma 3.2, it suffices to prove (3.3) for $t \in T$, where T is a countable dense subset of R^1. By Theorem 2.2, there exists a Boolean σ-algebra $A \subset M$ containing all $B_{x+y}(t)$ for any $t \in R^1$. Every Boolean σ-algebra A of M is σ-distributive, that is, if T and S are countable sets, then

$$\bigcup_{t \in T} \bigcap_{s \in S} a_{ts} = \bigcap_{\mathscr{C} \in S^T} \bigcup_{t \in T} a_{t\mathscr{C}(t)}$$

for any two indexed sequences $\{a_{ts} : t \in T,\ s \in S\} \subset M$.

In particular, by Sikorski [10] a Boolean σ-algebra A is σ-distributive iff for any $a \in A$, $a \neq 0_A$, and any sequence $\{a_n\} \subset A$ there exists $\{e(n)\}_{n=1}^{\infty} \in \{0, 1\}$ such that $a \cap \bigcap_{n=1}^{\infty} a_n^{e(n)} \neq 0_A$, where $a_n^0 = a_n^{\perp}$, $a_n^1 = a_n$, which is easily verifiable in our case. Then

$$\bigcap_{t \in T} \bigcup_{r \in Q} (B_x(r) \cap B_y(t - r)) = \bigcup_{\mathscr{C} \in Q^T} \bigcap_{t \in T} (B_x(\mathscr{C}(t)) \cap B_y(t - \mathscr{C}(t))). \tag{3.4}$$

it is clear that

$$\bigcap_t B_{x+y}(t) \geqslant x(\emptyset) \cup y(\emptyset) = x(R^1) \cap y(\emptyset) \cup x(\emptyset) \cap y(R^1). \tag{3.5}$$

Let $\mathscr{C} \in Q^T$, then

$$\bigcap_{t \in T} (B_x(\mathscr{C}(t)) \cap B_y(t - \mathscr{C}(t))) = \bigcap_{t \in T} (B_x(\mathscr{C}(t)) \cap \bigcap_{t \in T} B_y(t - \mathscr{C}(t))).$$

There are two possible cases:

(a) $\inf_{t \in T} \mathscr{C}(t) = k > -\infty$, then $\bigcap_{t \in T} B_x(\mathscr{C}(t)) \cap \bigcap_{t \in T} B_y(t - \mathscr{C}(t)) =$

$$= \bigcap_{t \in T} \left(B_x(\mathscr{C}(t)) \cap \bigcap_{t \in T} B_y(t - \mathscr{C}(t)) \right) =$$

$$= B_x(k) \cap \bigcap_{t \in T} (B_y(t - \mathscr{C}(t))) =$$

$$= B_x(k) \cap y(\emptyset) \leqslant x(R^1) \cap y(\emptyset) \leqslant x(\emptyset) \cup y(\emptyset).$$

(b) $\inf_{t \in T} \mathscr{C}(t) = -\infty$, then $\bigcap_{t \in T} B_x(\mathscr{C}(t) \cap \bigcap_{t \in T} B_y(t - \mathscr{C}(t))) =$

$$= x(\emptyset) \cap y(R^1) \leqslant x(\emptyset) \cup y(\emptyset).$$

For every $\mathscr{C} \in Q^T$, we have

$$\bigcap_{t\in T} \big(B_x(\mathscr{C}(t)) \cap B_y(t - \mathscr{C}(t))\big) \leqslant x(\emptyset) \cup y(\emptyset),$$

and taking into account (3.4) and (3.5), the following inequalities hold:

$$x(\emptyset) \cup y(\emptyset) \leqslant \bigcap_t B_{x+y}(t) \leqslant x(\emptyset) \cup y(\emptyset).$$

□

4. Properties of the sum

In the present part, we establish some of the basic properties of the sum. We recall that if $x \leftrightarrow y$, then, according to Dvurečenskij and Riečan [3], there exists an F-observable z and two Borel measurable functions f and g such that $x = f \circ z$, $y = g \circ z$.

Theorem 4.1.
(i) $x + y = y + x$ *for any two F-observables x and y;*
(ii) $(x + y) + z = x + (y + z)$ *for any three F-observables x, y and z;*
(iii) if $x \leftrightarrow y$, *then* $x + y = (f + g) \circ z$ *provided* $x = f \circ z$, $y = g \circ z$;
(iv) Let $u \in R^1$ *and put*

$$I_u(E) = \begin{cases} 1 & u \in E \\ 0 & u \notin E, \end{cases} \qquad (E \in B(R^1))$$

then $x + I_u = f_u \circ x$, *where* $f_u(t) = t + u$;
(v) $\alpha(x + y) = \alpha x + \alpha y$ *for any* $\alpha \in R^1$ *and all F-observables x and y.*

Proof. (i) Let $t \in R^1$ and denote $S_t = \{t - r : r \in Q\}$. Then S_t is dense in R^1 and using Lemma 3.2, we have

$$B_{x+y}(t) = \bigcup_{r\in Q} (B_x(r) \cap B_y(t - r)) = \bigcup_{s\in S_t} (B_y(s) \cap B_x(t - s)) = B^S_{y+x}(t) = B_{y+x}(t).$$

$$\begin{aligned}
\text{(ii) } B_{(x+y)+z}(t) &= \bigcup_{r\in Q} (B_{x+y}(r) \cap B_z(t - r)) = \\
&= \bigcup_{r\in Q} \Big(\bigcup_{s\in Q} (B_x(s) \cap B_y(r - s)) \cap B_z(t - r)\Big) = \\
&= \bigcup_{s\in Q} B_x(s) \cap (\bigcup_{r\in Q} B_y(r - s) \cap B_z(t - r)) = \\
&= \bigcup_{s\in Q} B_x(s) \cap \Big(\bigcup_{r\in Q} B_y(r - s) \cap B_z(t - s - (r - s))\Big).
\end{aligned}$$

We denote $C_r = \{r - s\colon s \in Q\}$, then C_r is a countable dense set in R^1. Hence, by Lemma 3.2, we have

$$\bigcup_{s\in Q} B_x(s) \cap \Big(\bigcup_{r\in Q} B_y(r-s) \cap B_z((t-s)-(r-s))\Big) =$$
$$= \bigcup_{s\in Q} \Big(B_x(s) \cap B_{y+z}(r-s)\Big) = B_{x+(y+z)}(t).$$

(iii) Calculate:

$$\begin{aligned} B_{x+y}(t) &= \bigcup_{r\in Q} (x((-\infty, r)) \cap y((-\infty, t-r))) = \\ &= \bigcup_{r\in Q} (z(f^{-1}((-\infty, r))) \cap z(g^{-1}((-\infty, t-r)))) = \\ &\bigcup_{r\in Q} (z(E) \cap z(F)) = \bigcup_{r\in Q} z(E\cap F) = \\ &= z\Big(\bigcup_{r\in Q} (f^{-1}((-\infty, r)) \cap g^{-1}((-\infty, t-r)))\Big) = \\ &= z((f+g)^{-1}((-\infty, t))) = (f+g)\circ z((-\infty, t)) = \\ &= B_{(f+g)\circ z}(t). \end{aligned}$$

(iv) Since $B_{I_u}(t-r) = 0$ if $t - u < r$ and $B_{I_u}(t-r) = 1$ otherwise, we have

$$B_{x+I_u}(t) = \bigcup_{r\geqslant t-u} (0 \cap B_x(r)) \cup \bigcup_{r\leqslant t-u} (1 \cap B_x(r)) =$$
$$= \bigcup_{r\leqslant t-u} B_x(r) = B_x(t-u) = f_u \circ z((-\infty, t)).$$

(v) is evident. □

Remark 4.2. If M consist of crisp subsets, that is, M is a σ-algebra of subsets of M (more precisely, M is a set of all characteristic functions of sets from the given σ-algebra), then the sum of F-observables coincides with the pointwise defined sum. Indeed, in this case for x and y there are unique mappings $u, v\colon X \to R^1$ such that $x(E) = u^{-1}(E)$ and $y(F) = v^{-1}(F), E, F \in B(R^1)$, and $(x+y)(E) = (u+v)^{-1}(E)$ for any $E \in B(R^1)$ (see the proof of Theorem 3.3).

Remark 4.3. If $\mathcal{O}(M)$ is the set of all F-observables of a fuzzy quantum space (X, M), then $\mathcal{O}(M)$ is a real vector space with respect to the sum.

Remark 4.4. We define the subtraction of F-observables x and y as $x - y = x + (-y)$, where $(-y)(E) = y(\{t\colon -t \in E\})$, $E \in B(R^1)$.

References

[1] *A. Dvurečenskij and F. Chovanec*: Fuzzy quantum spaces and compatibility, Inter. J. Theor. Phys. *27* (1988), 1069–1082.

[2] *A. Dvurečenskij and B. Riečan*: On joint observables for F-quantum spaces, Busefal *35* (1988), 10–14.

[3] *A. Dvurečenskij and B. Riečan*: On joint distribution of observables for F-quantum spaces, Fuzzy Sets and Systems *39* (1991), 67–73.

[4] *W. Guz*: Fuzzy σ-algebras of physics, Inter. J. Theor. Phys. *24* (1985), 481–493.

[5] *A. N. Kolmogorov*: Grundebegriffe der Wahrscheinlichkeitsrechnung, Berlin, 1933.

[6] *J. von Neumann*: Mathematische Grundlagen der Quantenmechanik, Springer, Berlin, 1932.

[7] *K. Piasecki*: Probability of fuzzy events defined as denumerable additivity measure, Fuzzy Sets and Systems *17* (1985), 271–284.

[8] *J. Pykacz*: Quantum logics and soft fuzzy probability spaces, Busefal *32* (1987), 150–157.

[9] *B. Riečan*: A new approach to some basic notions of statistical quantum theory, Busefal *35* (1987), 4–6.

[10] *R. Sikorski*: Boolean algebras, Springer-Verlag, 1964.

[11] *P. Suppes*: The probability argument for a nonclassical logic of quantum mechanics, Phil. Sc. *33* (1966), 14–21.

[12] *V. S. Varadarjan*: Geometry of quantum theory, Van Nostrand, New Jersey, 1968.

[13] *L. A. Zadeh*: Probability measures on fuzzy events, J. Math. Anal. Appl. *23* (1968), 421–427.

Súhrn

SÚČET POZOROVATEĽNÝCH VO FUZZY KVANTOVÝCH PRIESTOROCH

ANATOLIJ DVUREČENSKIJ, ANNA TIRPÁKOVÁ

V práci je zavedený súčet pozorovateľných vo fuzzy kvantových priestoroch, ktoré zovšeobecňujú Kolmogorov pravdepodobnostný priestor, použijúc idey teórie fuzzy množín.

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