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THE LOCALLY BEST ESTIMATORS OF THE FIRST AND SECOND ORDER PARAMETERS IN EPOCH REGRESSION MODELS

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Summary. In the linear epoch regression model

$$E(Y^{(j,j)}) = E\begin{pmatrix} Y_1^{(1)} \\ \vdots \\ Y_j^{(j)} \end{pmatrix} = \begin{pmatrix} X_{11}, & X_{21}, & 0, & \dots, & 0 \\ X_{12}, & 0, & X_{22}, & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ X_{1j}, & 0, & 0, & \dots, & X_{2j} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_{21} \\ \vdots \\ \beta_{2j} \end{pmatrix},$$

$$\operatorname{var}(Y^{(j,j)}) = \begin{pmatrix} \sum_{s_1}^{p_1} \vartheta_{1s_1} H_{1s_1}, & \dots, & 0 \\ \dots & \dots & \dots \\ 0, & \dots, & \sum_{s_j}^{p_j} \vartheta_{js_j} H_{js_j} \end{pmatrix}$$

the locally best linear unbiased estimators of the first order parameters and the locally minimum variance quadratic unbiased and invariant estimators of an unbiasedly and invariantly estimable linear function of the second order parameters in the jth epoch and after the jth epoch are derived.The algorithms mentioned utilize the special block structure of the model and the sparseness of the covariance matrix of the observation vector.

Keywords: regression linear model, epoch model

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1. Motivation of the problem

The research of stability of a region using geodetical methods consists in carrying out and processing measurements at points of a suitable geodetic network repeated in certain time intervals—the so called epoch measurements. The network consists of a group of stable points and a group of non-stable, moving points. The problem is to determine the coordinates of both these groups of points and simultaneously the parameters of the covariance matrix of the observation vector within each epoch of measurement and after certain epoch or, finally, after all epochs of measurement in order to compare them and to discover the laws of the recent movements of the territory. The aim is to get a reliable prediction of these movements necessary, e.g., for a responsible decision on building large structures (nuclear power stations, dams etc).

From the point of view of testing hypotheses on existence of significant movements of the region against their non-existence statistically rigorous estimates both of the first and the second order parameters are of a great importance.

Essential advantage of all algorithms obtained mainly of those giving estimators after the jth epoch is that they operate with the input observation vector, its covariance matrix and the design matrix whose dimensions correspond to a single epoch only (realize that the dimension of the input observation vector and consequently the dimensions of all the above mentioned matrices after the jth epoch are, roughly speaking, j multiples of the corresponding dimension within the individual epochs). This is a consequence of the block structure of the linear regression model describing the epoch measurements and of the simultaneous sparseness of the covariance matrix of the observation vector [2]. This fact represents an essential reduction of numerical calculations.

The epistemological importance of such research cannot be neglected.

2. Formulation of the problem

Let the results of j epochs of the measurement be at our disposal.

In the simplest case the state of the measurement after the jth epoch is described by the model

$$(1)\qquad E\begin{pmatrix} Y^{(1)} \\ \vdots \\ Y^{(j)} \end{pmatrix} = \begin{pmatrix} X_1, & X_2, & 0, & \dots, & 0 \\ X_1, & 0, & X_2, & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ X_1, & 0, & 0, & \dots, & X_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_{21} \\ \vdots \\ \beta_{2j} \end{pmatrix},$$

$$\operatorname{var}\begin{pmatrix} Y^{(1)} \\ \vdots \\ Y^{(j)} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^p \vartheta_i H_i, & \dots, & 0 \\ \dots & \dots & \dots \\ 0, & \dots, & \sum_{i=1}^p \vartheta_i H_i \end{pmatrix},$$

written frequently in its brief form

$$E[Y^{(j,j)}] = (1_{(j)} \otimes X_1, I_{(j,j)} \otimes X_2) \begin{pmatrix} \beta_1 \\ \beta_{21} \\ \vdots \\ \beta_{2j} \end{pmatrix}, \quad \operatorname{var}(Y^{(j,j)}) = \sum_{i=1}^p \vartheta_i (I \otimes H_i),$$

where $Y^{(i)}$ is an n dimensional observation vector, X_1 an $(n \times k)$ dimensional first part and X_2 an $(n \times l)$ dimensional second part of the design matrix, β_1 is a k dimensional vector of unknown coordinates of the group of stable points, β_{2i} is an l dimensional vector of unknown coordinates of the group of the non-stable points, $i = 1, \ldots, j$, $\vartheta' = (\vartheta_1, \ldots, \vartheta_p)$ is a vector of the second order parameters, $H_1, \ldots, H_p$ are known $(n \times n)$ symmetric matrices, $(Y^{(j,j)})' = (Y^{(1)}, \ldots, Y^{(j)})$ is a jn dimensional observation vector after the jth epoch, $(1^{(j)})' = (1_1, \ldots, 1_j)$, $I_{(j,j)}$ is a $j \times j$ unit matrix and $\otimes$ denotes the Kronecker multiplication.

This is the case when in all epochs of measurement the same linear regression model

$$E(Y) = (X_1, X_2)\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}, \ \operatorname{var}(Y) = \sum_{s=1}^{p} \vartheta_s H_s = \Sigma$$

$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \in \mathcal{R}^{k+l}$, $\vartheta \in \underline{\vartheta} \subset \mathcal{R}^p$ (the definition domain $\underline{\vartheta}$ of the second order parameters is assumed to include a non-empty topological interior) is realized.

In order to distinguish various epochs of measurement, a simple upper index indicating the number of the epoch will be used to denote the values concerning this epoch. A double upper index denotes the values after the epoch.

In what follows a more general epoch linear regression model is dealt with, namely the model of the form

$$E(Y^{(j,j)}) = E\begin{pmatrix} Y^{(1)}_{1(n_1)} \\ \vdots \\ Y^{(j)}_{j(n_j)} \end{pmatrix} = \begin{pmatrix} X_{11}, & X_{21}, & 0, & \ldots, & 0 \\ X_{12}, & 0, & X_{22}, & \ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ X_{1j}, & 0, & 0, & \ldots, & X_{2j} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_{21} \\ \vdots \\ \beta_{2j} \end{pmatrix},$$

(2)
$$\Sigma^{(j,j)} = \operatorname{var}(Y^{(j,j)}) = \begin{pmatrix} \sum_{s_1=1}^{p_1} \vartheta_{1s_1} H_{1s_1}, & \ldots, & 0 \\ \ldots & \ldots & \ldots \\ 0, & \ldots, & \sum_{s_j}^{p_j} \vartheta_{js_j} H_{js_j} \end{pmatrix} =$$
$$= \sum_{i=1}^{j} \sum_{s_i=1}^{p_i} \vartheta_{is_i} e_{i(j)} e'_{i(j)} \otimes H_{is_i},$$

where $Y_i^{(i)}$ is an n_i dimensional observation vector of the ith epoch, $Y^{(j,j)}$ is a $(\sum_{i=1}^{j} n_i)$ dimensional observation vector after the jth epoch of measurement, X_{1i} is an $(n_i \times k)$ dimensional first part and X_{2i} an $(n_i \times l_i)$ dimensional second part of the design matrix, β_1 is a k dimensional vector of coordinates of the group of stable points (of this group is unchangable, it is not allowed to add new points or to omit some of them in any epoch of measurement), β_{2i} is an l_i dimensional vector of the unknown coordinates of the group of non-stable points observed in the ith epoch of measurement (this group may be changed in different epochs of the measurement), ϑ_{is_i} are unknown parameters of the second order, H_{is_i} is an $(n_i \times n_i)$

known symmetric matrix, $i = 1, \ldots, j$, $s_i = 1, \ldots, p_i$, $e_{i(j)}$ is the ith unit vector of $\mathcal{R}^j$.

In this case in the ith epoch of measurement the linear regression model

$$E(Y_i) = (X_{1i}, X_{2i}) \begin{pmatrix} \beta_1 \\ \beta_{2i} \end{pmatrix}, \ \text{var}(Y_i) = \sum_{s_i=1}^{p_i} \vartheta_{is_i} H_{is_i} = \Sigma_i$$

$\begin{pmatrix} \beta_1 \\ \beta_{2i} \end{pmatrix} \in \mathcal{R}^{k+l_i}$, $\vartheta_i \in \underline{\vartheta}_i \subset \mathcal{R}^{p_i}$, is realized and it may differ from the models realized in other epochs.

The problem is to determine $\vartheta_0 - LBLUE$s (locally best linear unbiased estimators) of the first order parameters and a $\vartheta_0 - LMVQUIE$ (locally minimum variance quadratic unbiased invariant estimator) of unbiasedly and invariantly estimable functions of the second order parameters

(i) within the jth epoch of the measurement,

(ii) after the jth epoch of the measurement.

In accordance with the agreement concerning the symbols used for distinguishing the values *in* the jth epoch and *after* the jth epoch, $\hat{\beta}_1^{(j)}$, $\hat{\beta}_{2j}^{(j)}$, $\widehat{g_j'\vartheta_j}^{(j)}$ are the estimators *in* the jth epoch and $\hat{\beta}_1^{(j,j)}$, $\hat{\beta}_{21}^{(j,j)}$, $\hat{\beta}_{22}^{(j,j)}, \ldots, \hat{\beta}_{2j}^{(j,j)}$, $\widehat{g_1'\vartheta_1}^{(j,j)}, \ldots, \widehat{g_j'\vartheta_j}^{(j,j)}$ the estimators after the jth epoch of measurement.

3. Review of basic assertions

Lemma 1. *In the regular linear regression model*

$$(3) \qquad E(Y) = X\beta, \ \text{var}(Y) = \sum_{s=1}^{p} \vartheta_s V_s, \ R(X) = m < n, \ R[\text{var}(Y)] = n$$

(Y is an n dimensional observation vector, X an $(n \times m)$ dimensional design matrix, β an m dimensional vector of unknown first order parameters, ϑ a p dimensional vector of unknown second order parameters, V_s, $s = 1, \ldots, p$, are known $(n \times n)$ dimensional symmetric matrices), the matrices V_s, $s = 1, \ldots, p$, being supposed to be linearly independent, a $\vartheta_0 - LBLUE$ (a ϑ_0-locally best linear unbiased estimator, where ϑ_0 is an approximate value of the unknown second order vector parameter, $\Sigma_0 = \sum_{s=1}^{p} \vartheta_{s0} V_s$) of the parameter β is

$$\hat{\beta}(Y|\Sigma_0) = (X'\Sigma_0^{-1}X)^{-1}X'\Sigma_0^{-1}Y, \ \text{var}[\hat{\beta}(Y|\Sigma_0)|\Sigma_0] = (X'\Sigma_0^{-1}X)^{-1}.$$

Proof. Cf. [1], Theorem 5.1.2, p. 148, see also [1], p. 203. □

Let M_X denote a projection matrix $I - X(X'X)^- X'$ on the orthogonal (in the Euclidean norm) complement of the column space of the matrix X; $(X'X)^-$ means a generalized inverse (g-inverse) of the matrix $X'X$, i.e. $X'X(X'X)^- X'X = X'X$; A^+ is a special kind of a g-inverse of the matrix A fulfilling in addition to the relation $AA^+A = A$ the following relations: $A^+AA^+ = A^+$, $A^+A = (A^+A)'$ and $AA^+ = (AA^+)'$ (the Moore-Penrose inverse A^+ of the matrix A is determined uniquely; for details see [1], pp. 10–19).

Let W be an $n \times n$ positive definite matrix and X an $n \times k$ matrix, then

$$(M_X W M_X)^+ = W^{-1} - W^{-1}X(X'W^{-1}X)^- X'W^{-1} = W^{-1}M_X^{W^{-1}},$$

where $M_X^{W^{-1}} = I - P_X^{W^{-1}}$, $P_X^{W^{-1}}$ being the projection matrix in the norm $\|u\| = \sqrt{u'W^{-1}u}$, $u \in \mathcal{R}^n$, on the column space of the matrix X; in this relation the term $W^{-1}X(X'W^{-1}X)^- X'W^{-1}$ is invariant with respect to the choice of the g inverse.

In what follows the notation $(M_X W M_X)^+$ is preferred to the notation $W^{-1}M_X^{W^{-1}}$.

Lemma 2. *A ϑ_0−LMVQUIE (a ϑ_0-linear minimum variance quadratic unbiased and invariant estimator) of a linear function $g'\vartheta$ of the second order parameters $\vartheta \in \underline{\vartheta} \subset \mathcal{R}^p$ ($\underline{\vartheta}$ is supposed to contain a non-empty topological interior) in the model (3) exists iff $g \in \mathcal{M}(C_W^{(I)})$, where $C_W^{(I)}$ is the I-criterion matrix corresponding to the model (3). Its (s,t)th element reads*

$$\{C_W^{(I)}\}_{s,t} = \mathrm{Tr}[(M_X W M_X)^+ V_s (M_X W M_X)^+ V_t], \tag{4}$$

where W is an arbitrary $(n \times n)$ positive definite matrix and $(M_X W M_X)^+ = W^{-1} - W^{-1}X(X'W^{-1}X)^{-1}X'W^{-1}$.

P r o o f. Cf. [1], Theorem 5.6.4, p. 188. □

Lemma 3. *A $\vartheta_0 -$ LMVQUIE of $g'\vartheta$, $\vartheta \in \underline{\vartheta} \subset \mathcal{R}^p$, $g \in \mathcal{M}(C_W^{(I)})$ is*

$$\widehat{g'\vartheta}(Y|\Sigma_0) = \sum_{s=1}^{p} \kappa_s Y'(M_X \Sigma_0 M_X)^+ V_s (M_X \Sigma_0 M_X)^+ Y, \tag{5}$$

where $\kappa = (\kappa_1, \dots, \kappa_p)'$ is any solution of the system of equations

$$C_{\Sigma_0}^{(I)} \kappa = g,$$

and $C_{\Sigma_0}^{(I)}$ is the I-criterion matrix for $W = \Sigma_0$.

P r o o f. Cf. [1], Theorem 5.6.5, p. 190 and Remark 6.6.1, p. 261. □

Lemma 4. *In the block matrix*

$$\begin{pmatrix} C_{11}, & C_{12} \\ C_{21}, & C_{22} \end{pmatrix}$$

let the blocks C_{11} *and* C_{22} *be regular square matrices. Then*

$$\begin{pmatrix} C_{11}, & C_{12} \\ C_{21}, & C_{22} \end{pmatrix}^{-1} = \begin{pmatrix} C^{11}, & C^{12} \\ C^{21}, & C^{22} \end{pmatrix},$$

where

$$C^{11} = (C_{11} - C_{12}C_{22}^{-1}C_{21})^{-1} = C_{11}^{-1} + C_{11}^{-1}C_{12}(C_{22} - C_{21}C_{11}^{-1}C_{12})^{-1}C_{21}C_{11}^{-1}$$
$$C^{12} = -(C_{11} - C_{12}C_{22}^{-1}C_{21})^{-1}C_{12}C_{22}^{-1} = -C_{11}^{-1}C_{12}(C_{22} - C_{21}C_{11}^{-1}C_{12})^{-1}$$
$$C^{21} = -C_{22}^{-1}C_{21}(C_{11} - C_{12}C_{22}^{-1}C_{21})^{-1} = -(C_{22} - C_{21}C_{11}^{-1}C_{12})^{-1}C_{21}C_{11}^{-1}$$
$$(6)\, C^{22} = C_{22}^{-1} + C_{22}^{-1}C_{21}(C_{11} - C_{12}C_{22}^{-1}C_{21})^{-1}C_{12}C_{22}^{-1} = (C_{22} - C_{21}C_{11}^{-1}C_{12})^{-1}.$$

P r o o f. The assertion can be proved directly. □

4. $\vartheta_{j0} - LBLUE$s AND $\vartheta_{j0} - LMVQUIE$ OF THE FIRST AND SECOND ORDER PARAMETERS, RESPECTIVELY, IN THE jTH EPOCH

Theorem 1. *Within the model of the jth epoch*

$$E(Y_j^{(j)}) = X^{(j)} \begin{pmatrix} \beta_1 \\ \beta_{2j} \end{pmatrix}, \ \Sigma_j^{(j)} = \mathrm{var}(Y_j^{(j)}) = \sum_{s_j=1}^{p_j} \vartheta_{js_j} H_{js_j} = \Sigma_j, \tag{7}$$

where $X^{(j)} = (X_{1j}, X_{2j})$ *and* $V_{js_j}^{(j)} = H_{js_j}$, $s_j = 1, \ldots, p_j$, *in which the regularity conditions, i.e.* $R(X_{1j}X_{2j}) = k + l_j$, $R(X_{1j}) = k$, $R(X_{2j}) = l_j$, $R[\mathrm{var}(Y_j^{(j)})] = n_j$, *are fulfilled and the* $(n_j \times n_j)$ *symmetric matrices* H_{js_j}, $s_j = 1, \ldots, p_j$, *are linearly independent, the* $\vartheta_{j0} - LBLUE$*s of the first order parameters* β_1 *and* β_2 *are*

$$\hat{\beta}_1^{(j)}(Y_j^{(j)}|\Sigma_{j0}) = N_{\Sigma_{j0}}^{-1} X_{1j}'(M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^{+} Y_j^{(j)}, \tag{8}$$

$$\hat{\beta}_{2j}^{(j)}(Y_j^{(j)}|\Sigma_{j0}) = (X_{2j}'\Sigma_{j0}^{-1}X_{2j})^{-1}X_{2j}'\Sigma_{j0}^{-1}[I - X_{1j}N_{\Sigma_{j0}}^{-1}X_{1j}'(M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^{+}]Y_j^{(j)} \equiv$$
$$\equiv (X_{2j}'\Sigma_{j0}^{-1}X_{2j})^{-1}X_{2j}'\Sigma_{j0}^{-1}[Y_j^{(j)} - X_{1j}\hat{\beta}_1^{(j)}(Y_j^{(j)}|\Sigma_{j0})], \tag{9}$$

where $N_{\Sigma_{j0}} = X_{1j}'(M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^{+}X_{1j}$ *and* $(M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^{+} = \Sigma_{j0}^{-1} - \Sigma_{j0}^{-1}X_{2j} \times (X_{2j}'\Sigma_{j0}^{-1}X_{2j})^{-1}X_{2j}'\Sigma_{j0}^{-1}$ *when simultaneously*

$$\mathrm{var}[\hat{\beta}_1^{(j)}(Y_j^{(j)}|\Sigma_{j0})|\Sigma_{j0}] = N_{\Sigma_{j0}}^{-1},$$

$$\mathrm{var}[\hat{\beta}_{2j}^{(j)}(Y_j^{(j)}|\Sigma_{j0})|\Sigma_{j0}] =$$
$$(X_{2j}'\Sigma_{j0}^{-1}X_{2j})^{-1} + (X_{2j}'\Sigma_{j0}^{-1}X_{2j})^{-1}X_{2j}'\Sigma_{j0}^{-1}X_{1j}N_{\Sigma_{j0}}^{-1}X_{1j}'\Sigma_{j0}^{-1}X_{2j}(X_{2j}'\Sigma_{j0}^{-1}X_{2j})^{-1},$$

$$\operatorname{cov}[\hat\beta_1^{(j)}(Y_j^{(j)}|\Sigma_{j0}),\ \hat\beta_{2j}^{(j)}(Y_j^{(j)}|\Sigma_{j0})|\Sigma_{j0}] = -N_{\Sigma_{j0}}^{-1}X_{1j}'\Sigma_{j0}^{-1}X_{2j}(X_{2j}'\Sigma_{j0}^{-1}X_{2j})^{-1}.$$

P r o o f. The assertion follows immediately from Lemmas 1 and 4 applied to the model (7) of the measurement in the jth epoch. Realize that

$$((X^{(j)})'W_j^{-1}X^{(j)})^{-1} = \begin{pmatrix} X_{1j}'W_j^{-1}X_{1j}, & X_{1j}'W_j^{-1}X_{2j} \\ X_{2j}'W_j^{-1}X_{1j}, & X_{2j}'W_j^{-1}X_{2j} \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} N_{W_j}^{-1}, & -N_{W_j}^{-1}X_{1j}'W_j^{-1}X_{2j}(X_{2j}'W_j^{-1}X_{2j})^{-1} \\ -(X_{2j}'W_j^{-1}X_{2j})^{-1}X_{2j}'W_j^{-1}X_{1j}N_{W_j}^{-1}, & (X_{2j}'W_j^{-1}X_{2j})^{-1} + (X_{2j}^{-1}W_j^{-1}X_{2j})^{-1}X_{2j}'W_j^{-1}X_{1j}N_{W_j}^{-1}X_{1j}'W_j^{-1}X_{2j}(X_{2j}'W_j^{-1}X_{2j})^{-1} \end{pmatrix}.$$

□

R e m a r k 1. The equivalent forms of the $\vartheta_{j0} - LBLUE$s of the first order parameters β_1 and β_{2j} in the jth epoch of the measurement are

$$\begin{aligned}
\hat\beta_1^{(j)}(Y_j^{(j)}|\Sigma_{j0}) &= (X_{1j}'\Sigma_{j0}^{-1}X_{1j})^{-1}X_{1j}'\Sigma_{j0}^{-1}\{I - X_{2j}[X_{2j}'\Sigma_{j0}^{-1}X_{2j} - \\
&\quad - X_{2j}'\Sigma_{j0}^{-1}X_{1j}(X_{1j}'\Sigma_{j0}^{-1}X_{1j})^{-1}X_{1j}'\Sigma_{j0}^{-1}X_{2j}]^{-1}\times \\
&\quad X_{2j}'[\Sigma_{j0}^{-1} - \Sigma_{j0}^{-1}X_{1j}(X_{1j}'\Sigma_{j0}^{-1}X_{1j})^{-1}X_{1j}'\Sigma_{j0}^{-1}]\}Y_j^{(j)} \equiv \\
&\equiv (X_{1j}'\Sigma_{j0}X_{1j})^{-1}X_{1j}'\Sigma_{j0}^{-1}[Y_j^{(j)} - X_{2j}\hat\beta_{2j}^{(j)}(Y_j^{(j)}|\Sigma_{j0})], \\
\hat\beta_{2j}^{(j)}(Y^{(j)}|\Sigma^{j0}) &= [X_{2j}'\Sigma_{j0}^{-1}X_{2j} - X_{2j}'\Sigma_{j0}^{-1}X_{1j}(X_{1j}'\Sigma_{j0}^{-1}X_{1j})^{-1}X_{1j}'\Sigma_{j0}^{-1}X_{2j}]^{-1}\times \\
&\quad X_{2j}'[\Sigma_{j0}^{-1} - \Sigma_{j0}^{-1}X_{1j}(X_{1j}'\Sigma_{j0}^{-1}X_{1j})^{-1}X_{1j}'\Sigma_{j0}^{-1}]Y_j^{(j)},
\end{aligned}$$

when simultaneously

$$\begin{aligned}
\operatorname{var}[\hat\beta_j^{(j)}(Y_j^{(j)}|\Sigma_{j0})|\Sigma_{j0}] &= (X_{1j}'\Sigma_{j0}^{-1}X_{1j})^{-1} + (X_{1j}\Sigma_{j0}^{-1}X_{1j})^{-1}X_{1j}'\Sigma_{j0}^{-1}X_{2j} \\
[X_{2j}'\Sigma_{j0}^{-1}X_{2j} - X_{2j}'\Sigma_{j0}^{-1}X_{1j}&(X_{1j}\Sigma_{j0}^{-1}X_{1j})^{-1}X_{1j}'\Sigma_{j0}^{-1}X_{2j}]^{-1}X_{2j}'\Sigma_{j0}^{-1}X_{1j}(X_{1j}'\Sigma_{j0}^{-1}X_{1j})^{-1}, \\
\operatorname{var}[\hat\beta_{2j}^{(j)}(Y_j^{(j)}|\Sigma_{j0})|\Sigma_{j0}] &= [X_{2j}'\Sigma_{j0}^{-1}X_{2j} - X_{2j}\Sigma_{j0}^{-1}X_{1j}(X_{1j}'\Sigma_{j0}^{-1}X_{1j})^{-1}X_{1j}'\Sigma_{j0}^{-1}X_{2j}]^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{cov}[\hat\beta_1^{(j)}(Y^{(j)}|\Sigma_{j0}),\ &\hat\beta_{2j}^{(j)}(Y^{(j)}|\Sigma_{j0})|\Sigma_{j0}] = \\
&= -(X_{1j}'\Sigma_{j0}^{-1}X_{1j})^{-1}X_{1j}'\Sigma_{j0}^{-1}X_{2j}\times \\
&[X_{2j}'\Sigma_{j0}^{-1}X_{2j} - X_{2j}'\Sigma_{j0}^{-1}X_{1j}(X_{1j}'\Sigma_{j0}^{-1}X_{1j})^{-1}X_{1j}'\Sigma_{j0}^{-1}X_{2j}]^{-1}.
\end{aligned}$$

The equivalent forms of $\vartheta_{j0} - LBLUE$s and their variances at the point ϑ_{j0} are good tools for checking their correctness and numerical stability.

Theorem 2. *A $\vartheta_{j0} - LMVQUIE$ of a linear function $g_j'\vartheta_j$ of the second order parameter $\vartheta_j \in \underline{\vartheta}_j \subset \mathcal{R}^j$, $g_j \in \mathcal{M}(C_{W_j}^{(I)})$, $C_{W_j}^{(I)}$ being a $(p_j \times p_j)$ dimensional I-criterion matrix of the regular model (7) describing the jth epoch of measurement is*

$$\begin{aligned}\widehat{g_j'\vartheta_j^{(j)}}(Y_j^{(j)}|\Sigma_{j0}) = &\sum_{s_j=1}^{p_j} \kappa_{js_j}^{(j)}(Y_j^{(j)})'[I - X_{1j}N_{\Sigma_{j0}}^{-1}X_{1j}'(M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^+]'\times \\ &(M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^+ H_{js_j}(M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^+\times \\ &[I - X_{1j}N_{\Sigma_{j0}}^{-1}X_{1j}'(M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^+]Y_j^{(j)} \equiv \\ \equiv &\sum_{s_j=1}^{p_j} \kappa_{s_j}^{(j)}[Y_j^{(j)} - X_{1j}\hat{\beta}_1^{(j)}(Y_j^{(j)}|\Sigma_{j0})]'(M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^+\times \\ &H_{js_j}(M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^+[Y_j^{(j)} - X_{1j}\hat{\beta}_1^{(j)}(Y_j^{(j)}|\Sigma_{j0})]. \end{aligned} \tag{10}$$

Here

$$\begin{aligned}\{C_{W_j}^{(I)}\}_{s_j,t_j} = \mathrm{Tr}\{&[I - (M_{X_{2j}}W_jM_{X_{2j}})^+X_{1j}N_{W_j}^{-1}X_{1j}'](M_{X_{2j}}W_jM_{X_{2j}})^+H_{js_j}\times \\ &(M_{X_{2j}}W_jM_{X_{2j}})^+[I - X_{1j}N_{W_j}^{-1}X_{1j}'(M_{X_{2j}}W_jM_{X_{2j}})^+]H_{jt_j}\},\end{aligned} \tag{11}$$

W_j is an arbitrary $(n_j \times n_j)$ positive definite matrix and the p_jth dimensional vector $\kappa^{(j)}$ of the unknown Lagrange coefficients is any solution of the system of equations

$$C_{\Sigma_{j0}}^{(I)}\kappa_j^{(j)} = g_j,$$

where $C_{\Sigma_{j0}}^{(I)}$ is the I-criterion matrix for $W_j = \Sigma_{j0}$ and $\hat{\beta}_1^{(j)}(Y_j^{(j)}|\Sigma_{j0})$ is given by (8).

Proof. The assertion is a direct consequence of Lemmas 2, 3 and 4 applied to the model (7) corresponding to the state of measurement in the jth epoch. The crucial point of the proof consists in the fact that

$$\begin{aligned}(M_{X^{(j)}}W_jM_{X^{(j)}})^+ = &(M_{X_{2j}}W_jM_{X_{2j}})^+ \\ &- (M_{X_{2j}}W_jM_{X_{2j}})^+X_{1j}N_{W_j}^{-1}X_{1j}'(M_{X_{2j}}W_jM_{X_{2j}})^+.\end{aligned}$$

□

5. $(\vartheta_{10}, \dots, \vartheta_{j0}) - LBLUE$s and $(\vartheta_{10}, \dots, \vartheta_{j0}) - LMVQUIE$ of the first and second order parameters, respectively, after the jth epoch

Theorem 3. *Within the model (2) of the measurement after the jth epoch in which the regularity conditions, i.e.* $R(X^{(j,j)}) = k + \sum_{i=1}^{j} l_i$, $R(X_{1i}) = k$, $R(X_{2i}) = l_i$, $X^{(j,j)}$ *denoting the matrix*

$$X^{(j,j)} = \begin{pmatrix} X_{11}, & X_{21}, & 0, & \dots, & 0 \\ X_{12}, & 0, & X_{22}, & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ X_{1j}, & 0, & 0, & \dots, & X_{2j} \end{pmatrix},$$

$R[\mathrm{var}(Y^{(j,j)})] = \sum_{i=1}^{j} n_i$, $R[\mathrm{var}(Y_i^{(i)})] = n_i$, $i = 1, \dots, j$, *are fulfilled and the* $(n_i \times n_i)$ *symmetric matrices* H_{is_i}, $s_i = 1, \dots, p_i$, $i = 1, \dots, j$, *are linearly independent, the* $(\vartheta_{10}, \dots, \vartheta_{j0}) - LBLUEs$ *of the first order parameters* $\beta_1, \beta_{21}, \dots, \beta_{2j}$ *are*

$$\hat{\beta}_1^{(j,j)}(Y^{(j,j)}|\Sigma_0^{(j,j)}) = N_{\Sigma_0^{(j,j)}}^{-1} \sum_{i=1}^{j} X_{1i}(M_{X_{2i}}\Sigma_{i0}M_{X_{2i}})^{+}Y_i^{(i)}, \tag{12}$$

$$\begin{aligned} \hat{\beta}_{2k}^{(j,j)}(Y^{(j,j)}|\Sigma_0^{(j,j)} &= (X'_{2k}\Sigma_{k0}^{-1}X_{2k})^{-1}X'_{2k}\Sigma_{k0}^{-1} \times \\ &\times [Y_k^{(k)} - X_{1k}N_{\Sigma_0^{(j,j)}}^{-1}\sum_{i=1}^{j} X'_{1i}(M_{X_{2i}}\Sigma_{i0}M_{X_{2i}})^{+}Y^{(i)}] \equiv \\ &\equiv (X'_{2k}\Sigma_{k0}^{-1}X_{2k})^{-1}X'_{2k}\Sigma_{k0}^{-1}[Y_k^{(k)} - X_{1k}\hat{\beta}_1^{(j,j)}(Y^{(j,j)}|\Sigma_0^{(j,j)})], \end{aligned} \tag{13}$$

$k = 1, \dots, j$, *where* $N_{\Sigma_0^{(j,j)}} = \sum_{i=1}^{j} X'_{1i}(M_{X_{2i}}\Sigma_{i0}M_{X_{2i}})^{+}X_{1i}$.

Simultaneously

$$\mathrm{var}[\hat{\beta}_1^{(j,j)}(Y^{(j,j)}|\Sigma_0^{(j,j)})|\Sigma_0^{(j,j)}] = N_{\Sigma_0^{(j,j)}},$$

$$\begin{aligned} \mathrm{var}[\hat{\beta}_{2k}^{(j,j)}(Y^{(j,j)}|\Sigma_0^{(j,j)}|\Sigma_0^{(j,j)}] &= (X'_{2k}\Sigma_{k0}^{-1}X_{2k})^{-1} + \\ &+ (X'_{2k}\Sigma_{k0}^{-1}X_{2k})^{-1}X'_{2k}\Sigma_{k0}^{-1}X_{1k}N_{\Sigma_0^{(j,j)}}^{-1}X'_{1k}\Sigma_{k0}^{-1}X_{2k}(X'_{2k}\Sigma_{k0}^{-1}X_{2k})^{-1}, \end{aligned}$$

$k = 1, \dots, j$,

$$\begin{aligned} \mathrm{cov}[\hat{\beta}_1^{(j,j)}(Y^{(j,j)}|\Sigma_0^{(j,j)}), \hat{\beta}_{2k}^{(j,j)}(Y^{(j,j)}|\Sigma_0^{(j,j)})|\Sigma_0^{(j,j)}] &= \\ = -N_{\Sigma_0^{(j,j)}}^{-1}X'_{1k}\Sigma_{k0}^{-1}X_{2k}(X'_{2k}\Sigma_{k0}^{-1}X_{2k})^{-1}, \end{aligned}$$

$k = 1, \dots, j$,

$$\begin{aligned} \mathrm{cov}[\hat{\beta}_{2k}^{(j,j)}(Y^{(j,j)}|\Sigma_0^{(j,j)}), \ \hat{\beta}_{2l}^{(j,j)}(Y^{(j,j)}|\Sigma_0^{(j,j)})|\Sigma_0^{(j,j)}] &= \\ = -(X'_{2k}\Sigma_{k0}^{-1}X_{2k})^{-1}X_{2k}\Sigma_{k0}^{-1}X_{1k}N_{\Sigma_0^{(j,j)}}^{-1}X'_{1l}\Sigma_{l0}^{-1}X_{2l}(X'_{2l}\Sigma_{l0}^{-1}X_{2l})^{-1}, \end{aligned}$$

$k, l = 1, \dots, j,\ k \neq l.$

Proof. The assertion is a direct application of Lemma 1 to the model (2). It suffices to realize that

$$[(X^{(j,j)})'(W^{(j,j)})^{-1}X^{(j,j)}]^{-1} =$$

$$= \begin{pmatrix} \sum_{i=1}^{j} X'_{1i}W_i^{-1}X_{1i}, & X'_{11}W_1^{-1}X_{21}, & X'_{12}W_2^{-1}X_{22}, & \dots, & X'_{1j}W_j^{-1}X_{2J} \\ X'_{21}W_1^{-1}X_{11}, & X_{21}, W_1^{-1}X_{21}, & 0, & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ X'_{2j}W_j^{-1}X_{1j}, & 0, & 0, & \dots, & X'_{2j}W_j^{-1}X_{2j} \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} N^{-1}_{W(j,j)}, & -N^{-1}_{W(j,j)}F'_1, & -N^{-1}_{W(j,j)}F'_2, & \dots, & -N^{-1}_{W(j,j)}F'_j \\ -F_1N^{-1}_{W(j,j)}, & (X'_{21}W_1^{-1}X_{21})^{-1} + F_1N^{-1}_{W(j,j)}F'_1, & F_1N^{-1}_{W(j,j)}F'_2, & \dots, & F_1N^{-1}_{W(j,j)}F'_j \\ \dots & \dots & \dots & \dots & \dots \\ -F_jN^{-1}_{W(j,j)}, & F_jN^{-1}_{W(j,j)}F'_1, & F_jN^{-1}_{W(j,j)}F'_2, & \dots, & (X'_{2j}W_j^{-1}X_{2j})^{-1} + F_jN^{-1}_{W(j,j)}F'_j \end{pmatrix},$$

where $N_{W(j,j)} = \sum_{i=1}^{j} X_{1i}(M_{X_{2i}}W_iM_{X_{2i}})^+X_{1i}$ and $F_i = (X'_{2i}W_i^{-1}X_{2i})^{-1}X'_{2i}W_i^{-1}X_{1i}$, $i = 1, \dots, j$. Lemma 4 was applied here (only the indicated division of the matrix $[X^{(j,j)}(W^{(j,j)})^{-1}X^{(j,j)}]$ into blocks C_{11}, C_{12}, C_{21} and C_{22} is allowed for the application of the first part of relations (6); in the other case it is not easy to invert the matrices C_{22} and $(C_{11} - C_{12}C_{22}^{-1}C_{21})$). □

Theorem 4. *A $(\vartheta_{10}, \dots, \vartheta_{j0})$-LMVQUIE of a linear function $(g'_1, \dots, g'_j)\begin{pmatrix}\vartheta_1\\ \vdots \\ \vartheta_j\end{pmatrix}$ of the second order parameter $\begin{pmatrix}\vartheta_1\\ \vdots \\ \vartheta_j\end{pmatrix} \in \underline{\vartheta}_1 \times \dots \times \underline{\vartheta}_j \subset \mathcal{R}^{\sum_{i=1}^{j} p_i}$, $(g'_1, \dots, g'_j)' \in \mathcal{M}(C^{(I)}_{W(j,j)})$, where $C^{(I)}_{W(j,j)}$ is a $[(\sum_{i=1}^{j} p_i) \times (\sum_{i=1}^{j} p_i)]$ dimensional I-criterion matrix of the regular model (2) describing the measurement after the jth epoch is*

$$\sum_{k=1}^{j} \widehat{g'_k\vartheta_k}^{(j,j)}(Y^{(j,j)}|\Sigma_0^{(j,j)}) = \sum_{k=1}^{j}\sum_{s_k=1}^{p_k} \lambda^{(j,j)}_{ks_k}\{Y_k^{(k)} - X_{1k}N^{-1}_{\Sigma_0^{(j,j)}} \times$$

$$\times [\sum_{i=1}^{j} X'_{1i}(M_{X_{2i}}\Sigma_{i0}M_{X_{2i}})^+Y_i^{(i)}]\}' \times$$

$$\times (M_{X_{2k}}\Sigma_{k0}M_{X_{2k}})^+H_{ks_k}(M_{X_{2k}}\Sigma_{k0}M_{X_{2k}})^+ \times$$

$$\times \{Y_k^{(k)} - X_{1k}N^{-1}_{\Sigma_0^{(j,j)}}[\sum_{i=1}^{j} X_{1i}(M_{X_{2i}}\Sigma_{i0}M_{X_{2i}})^+Y_i^{(i)}]\} =$$

(14)
$$= \sum_{k=1}^{j} \sum_{s_k=1}^{p_k} \lambda_{ks_k}^{(j,j)} [Y_k^{(k)} - X_{1k}\hat{\beta}_1^{(j,j)}(Y^{(j,j)}|\Sigma_0^{(j,j)})]' \times$$
$$\times (M_{X_{2k}} \Sigma_{k0} M_{X_{2k}})^+ H_{ks_k} (M_{X_{2k}} \Sigma_{k0} M_{X_{2k}})^+ \times$$
$$\times [Y_k^{(k)} - X_{1k}\hat{\beta}_1^{(j,j)}(Y^{(j,j)}|\Sigma_0^{(j,j)})];$$

here

$$\{C_{W^{(j,j)}}^{(I)}\}_{ik,\ s_i t_k} = \mathrm{Tr}[(M_{X_{2k}} W_k M_{X_{2k}})^+ X_{1k} N_{W^{(j,j)}}^{-1} X_{1i}' (M_{X_{2i}} W_i M_{X_{2i}})^+ \times H_{is_i} (M_{X_{2i}} W_i M_{X_{2i}})^+ X_{1i} N_{W^{(j,j)}}^{-1} X_{1k}' (M_{X_{2k}} W_k M_{X_{2k}})^+ H_{kt_k}]$$

$i, k = 1, \ldots, j$, $s_i = 1, \ldots, p_i$, $t_k = 1, \ldots, p_k$, $i \neq k$ (a non-diagonal block of the I-criterion matrix) and

$$\{C_{W^{(j,j)}}^{(I)}\}_{kk,\ s_k t_k} = Tr\{[I - (M_{X_{2k}} W_k M_{X_{2k}})^+ X_{1k} N_{W^{(j,j)}}^{-1} X_{1k}'](M_{X_{2k}} W_k M_{X_{2k}})^+ \times H_{ks_k} (M_{X_{2k}} W_k M_{X_{2k}})^+ [I - X_{1k} N_{W^{(j,j)}}^{-1} X_{1k}' (M_{X_{2k}} W_k M_{X_{2k}})^+] H_{kt_k}\},$$

$k = 1, \ldots, j$, $s_k, t_k = 1, \ldots, p_k$ (the diagonal block of the I-criterion matrix; this consists of $j \times j$ blocks, the (i,k)th being $(p_i \times p_i)$ dimensional), $W^{(j,j)} = \sum_{i=1}^{j} e_{i(j)} e_{i(j)}' \otimes W_i$, where W_i, $i = 1, \ldots, j$, are arbitrary $(n_i \times n_i)$ positive definite matrices; the $\sum_{i=1}^{j} p_i$ dimensional vector $(\lambda^{(j,j)})' = (\lambda_{1(p_1)}^{(j,j)}, \ldots, \lambda_{j(p_j)}^{(j,j)})$, $(\lambda_i^{(j,j)})' = (\lambda_{1i}^{(j,j)}, \ldots, \lambda_{ip_i}^{(j,j)})$ of the unknown Lagrange coefficients is any solution of the system of equations

$$C_{\Sigma_0^{(j,j)}}^{(I)} \lambda^{(j,j)} = g, \quad g' = (g_1, \ldots, g_j),$$

where $C_{\Sigma_0^{(j,j)}}^{(I)}$ is the I-criterion matrix for $W^{(j,j)} = \Sigma_0^{(j,j)} = \sum_{i=1}^{j} e_{i(j)} e_{i(j)}' \otimes \Sigma_{0i}$, and $\hat{\beta}_1^{(j,j)}(Y^{(j,j)}|\Sigma_0^{(j,j)})$ is given by (12).

P r o o f. The assertion follows from Lemmas 3 and 4. Its crucial points consist in the following facts

(i)

$$(M_{X(j,j)}\Sigma_0^{(j,j)}M_{X(j,j)})^+ =$$

$$= \begin{pmatrix} (M_{X_{21}}\Sigma_{10}M_{X_{21}})^+[I - X_{11}N^{-1}_{\Sigma_0^{(j,j)}}X'_{11}(M_{X_{21}}\Sigma_{10}M_{X_{21}})^+], \\ -(M_{X_{22}}\Sigma_{20}M_{X_{22}})^+X_{12}N^{-1}_{\Sigma_0^{(j,j)}}X'_{11}(M_{X_{21}}\Sigma_{10}M_{X_{21}})^+, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ -(M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^+X_{1j}N^{-1}_{\Sigma_0^{(j,j)}}X'_{11}(M_{X_{21}}\Sigma_{10}M_{X_{21}})^+, \end{pmatrix.}$$

$$\begin{matrix} -(M_{X_{21}}\Sigma_{10}M_{X_{21}})^+X_{11}N^{-1}_{\Sigma_0^{(j,j)}}X'_{12}(M_{X_{22}}\Sigma_{20}M_{X_{22}})^+, \\ (M_{X_{22}}\Sigma_{20}M_{X_{22}})^+[I - X_{12}N^{-1}_{\Sigma_0^{(j,j)}}X'_{12}(M_{X_{22}}\Sigma_{20}M_{X_{22}})^+], \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ -(M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^+X_{1j}N^{-1}_{\Sigma_0^{(j,j)}}X'_{12}(M_{X_{22}}\Sigma_{20}M_{X_{22}})^+, \end{matrix}$$

$$\left.\begin{matrix} \dots, & -(M_{X_{21}}\Sigma_{10}M_{X_{21}})^+X_{11}N^{-1}_{\Sigma_0^{(j,j)}}X'_{1j}(M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^+ \\ \dots, & -(M_{X_{22}}\Sigma_{20}M_{X_{22}})^+X_{12}N^{-1}_{\Sigma_0^{(j,j)}}X'_{1j}(M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^+ \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ \dots, & (M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^+[I - X_{1j}N^{-1}_{\Sigma_0^{(j,j)}}X'_{1j}(M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^+] \end{matrix}\right),$$

(ii)

$$(M_{X(j,j)}W^{(j,j)}M_{X(j,j)})^+V_{is_i}^{(j,j)} = (M_{X(j,j)}W^{(j,j)}M_{X(j,j)})^+(e_{i(j)}e'_{i(j)}\otimes H_{is_i}) =$$

$$= \begin{pmatrix} 0_{1(n_1)}, & \dots, & 0_{i-1}, & -(M_{X_{21}}\Sigma_{10}M_{X_{21}})^+X_{11}N^{-1}_{\Sigma_0^{(j,j)}}X'_{1i}(M_{X_{2i}}\Sigma_{i0}M_{X_{2i}})^+H_{is_i}, \\ \cdots & \cdots & \cdots & \cdots\cdots\cdots\cdots \\ 0_{1(n_i)}, & \dots, & 0_{i-1}, & (M_{X_{2i}}\Sigma_{i0}M_{X_{2i}})^+[I - X_{1i}N^{-1}_{\Sigma_0^{(j,j)}}X'_{1i}(M_{X_{2i}}\Sigma_{i0}M_{X_{2i}})^+H_{is_i}], \\ \cdots & \cdots & \cdots & \cdots\cdots\cdots\cdots \\ 0_{1(n_j)}, & \dots, & 0_{i-1}, & -(M_{X_{2j}}\Sigma_{j0}M_{X_{2j}})^+X_{1j}N^{-1}_{\Sigma_0^{(j,j)}}X'_{1i}(M_{X_{2i}}\Sigma_{i0}M_{X_{2i}})^+H_{is_i}, \end{pmatrix.}$$

$$\left.\begin{matrix} 0_{i+1}, & \dots, & 0_j \\ \cdots & \cdots & \cdots \\ 0_{i+1}, & \dots, & 0_j \\ \cdots & \cdots & \cdots \\ 0_{i+1}, & \dots, & 0_j \end{matrix}\right).$$

□

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