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SHAPE OPTIMIZATION OF ELASTO-PLASTIC AXISYMMETRIC BODIES

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Summary. A minimization of a cost functional with respect to a part of a boundary is considered for an elasto-plastic axisymmetric body obeying Hencky's law. The principle of Haar-Kármán and piecewise linear stress approximations are used to solve the state problem. A convergence result and the existence of an optimal boundary is proved.

Keywords: domain optimization, control of variational inequalities, Hencky's law of elasto-plasticity.

AMS Subject class: 65K10, 65N30, 73E99.

INTRODUCTION

In the present paper we solve the following optimal design problem. Given body forces, surface loads and material characteristics of an elasto-plastic axisymmetric body, find the shape of its meridian section such that a cost functional is minimized. The cost functional is defined by an integral of the yield function and either zero displacements or zero surface tractions are prescribed on the unknown part of the boundary.

One of the simplest mathematical models describing the elasto-plastic behaviour of solid bodies is given by the constituent law of Hencky (see e.g. [1]). The classical boundary value problems can be formulated in terms of a variational inequality by means of stresses, i.e. by Haar-Kármán principle.

Extending some ideas of Falk and Mercier [2], we introduce piecewise linear approximations of the stress field and of the unknown boundary to define approximate optimal design problems. The main result of the paper is a convergence analysis of the approximate solutions and the existence proofs for both the approximate and the original optimization problems.

The paper represents a continuation of some previous author's results in the field of shape optimization [3], [4], [5], [7].

1. Formulations of the optimization problems

Let us recall some basic relations of the elasto-plastic bodies obeying the law of Hencky.

Let a bounded elasto-plastic body occupy an axisymmetric domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary, which is generated by rotation of a two-dimensional domain D about the $x_3 \equiv z$ – axis. If all the data of the problem are axisymmetric, we pass to the cylindrical coordinate system (r, z, ϑ). The physical components of the displacement-vector $\boldsymbol{u}$ are denoted by $u_r \equiv u$, $u_z \equiv w$ $(u_\vartheta \equiv 0)$.

The space of displacement functions with finite energy is then

$$\mathscr{H}(D) = \{\mathbf{v} \equiv (u, w) \in (W_r^{1,2}(D) \cap L_{1/r}^2(D)) \times W_r^{1,2}(D)\}\,,$$

$$\|\mathbf{v}\|_{\mathscr{H}(D)} = \left(\int_D \left[u^2/r^2 + \left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + w^2 + \right.\right.$$

$$\left.\left. + \left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] r\,\mathrm{d}r\,\mathrm{d}z\right)^{1/2}.$$

Henceforth $W_r^{k,2}(D)$, $k = 1, 2, \ldots$, denotes the weighted Sobolev space with the norm

$$\|u\|_{k,r,D} = \left(\int_D \sum_{|\alpha| \leq k} |D^\alpha u|^2\, r\,\mathrm{d}r\,\mathrm{d}z\right)^{1/2};$$

$L_{r^n}^2(D)$ is the space of functions with the following norm

$$\|u\|_{0,r^n,D} = \left(\int_D u^2 r^n\,\mathrm{d}r\,\mathrm{d}z\right)^{1/2},$$

where n is an integer; $L_r^2(\Gamma)$, where $\Gamma \subset \partial D \dot{-} \Gamma_0$, Γ_0 being the intersection of ∂D with the z-axis, is defined analogously on the boundary of D.

We define $\mathbb{R}_\sigma$ as the space of symmetric 3×3 matrices such that $\sigma_{13} = \sigma_{23} = 0$ and identify indices r, z, ϑ with 1, 2, 3. Let the repeated index imply summation over the range 1, 2, 3. Then let

$$\|\sigma\| = (\sigma_{ij}\sigma_{ij})^{1/2}\,, \quad \sigma \in \mathbb{R}_\sigma\,.$$

Assume that a yield function $f\colon \mathbb{R}_\sigma \to \mathbb{R}$ is given, which is convex, Lipschitz and

(1) $$f(\lambda\sigma) = |\lambda|\, f(\sigma)$$

holds for all $\lambda \in \mathbb{R}$, $\sigma \in \mathbb{R}_\sigma$.

Example 1. These conditions are fulfilled by the von Mises function

$$f(\sigma) = \text{const}\, \|\sigma^d\|\,, \quad \text{const} > 0\,,$$

where

$$\sigma_{ij}^d = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\sigma_{mm}\,, \quad (\delta_{ij} = \text{Kronecker's delta})$$

is the stress-deviator. Note, that

$$f(\sigma) = \text{const}\left(\|\sigma\|^2 - \tfrac{1}{3}(\sigma_{mm})^2\right)^{1/2}. \qquad \square$$

We introduce the following space

$$S(D) = \{\tau\colon D \to \mathbb{R}_\sigma \mid \tau_{ij} \in L^2_r(D);\ i, j = 1, 2, 3\}$$

and the scalar product

$$\langle \sigma, \mathbf{e} \rangle_D = \int_D \sigma_{ij} e_{ij} r \,\mathrm{d}r \,\mathrm{d}z\,, \quad \sigma, \mathbf{e} \in S(D)\,,$$
$$\|\sigma\|_{0,r,D} = \langle \sigma, \sigma \rangle_D^{1/2}\,.$$

Moreover, we shall use also the energy scalar product

$$(\sigma, \tau)_D = \langle b\sigma, \tau \rangle_D$$

and the associated norm

$$\|\sigma\|_D = (\sigma, \sigma)_D^{1/2}\,,$$

where $b\colon S(D) \to S(D)$ is the isomorphism defined by the generalized Hooke's law, i.e.,

$$\mathbf{e} = b\sigma \Leftrightarrow e_{ij} = b_{ijkm}\sigma_{km}\,.$$

We assume that positive constants b_0, b_1 exist such that

$$b_0 \|\sigma\|^2_{0,r,D} \leqq \|\sigma\|^2_D \leqq b_1 \|\sigma\|^2_{0,r,D} \quad \forall \sigma \in S(D) \tag{2}$$

and

$$\langle b\sigma, \tau \rangle_D = \langle \sigma, b\tau \rangle_D \quad \forall \sigma, \tau \in S(D)\,.$$

Assume that

$$\partial D \dot{-} \Gamma_0 = \Gamma_u \cup \Gamma_g\,, \quad \Gamma_u \cap \Gamma_g = \emptyset\,.$$

Let a body force-vector $\mathbf{F} \equiv (F_r, F_z) \in [L^2_r(D)]^2$ and a surface traction-vector $\mathbf{g} \equiv (g_r, g_z) \in [L^2_r(\Gamma_g)]^2$ be given.

We define the set of plastically admissible stress fields

$$P(D) = \{\tau \in S(D) \mid f(\tau) \leqq 1 \text{ a.e. in } D\}$$

and the set of statically admissible (equilibriated) stress fields

$$E(D) = \{\tau \in S(D) \mid \langle \tau, \mathbf{e}(\mathbf{v}) \rangle_D = \mathscr{F}_D(\mathbf{v}) \quad \forall \mathbf{v} \in V(D)\}\,,$$

where

$$\mathbf{e}(\mathbf{v}) = \begin{bmatrix} \partial u/\partial r, & \frac{1}{2}(\partial u/\partial z + \partial w/\partial r), & 0 \\ & \partial w/\partial z & 0 \\ \text{sym.} & & u/r \end{bmatrix}$$

$V(D) = \{\mathbf{v} \in \mathscr{H}(D) \mid \mathbf{v} = 0 \text{ on } \Gamma_u\}$ (in the sense of traces-see e.g. [4] – Lemma 1),

$$\mathscr{F}_D(\mathbf{v}) = \int_D (F_r u + F_z w)\, r \,\mathrm{d}r \,\mathrm{d}z + \int_{\Gamma_g} (g_r u + g_z w)\, r \,\mathrm{d}s\,.$$

The *Haar-Kármán principle* says that the actual stress field minimizes the complementary energy

$$\tfrac{1}{2}\|\tau\|^2_D$$

over the set $P(D) \cap E(D)$. (For the derivation of the principle – see [1] or [2]). The principle is equivalent to the following variational inequality: find $\sigma \in P(D) \cap \cap E(D)$ such that

$$(3) \qquad (\sigma, \tau - \sigma)_D \geqq 0 \quad \forall \tau \in P(D) \cap E(D) .$$

Passing to the shape optimization problem, we introduce the following set of admissible design variables

$$U_{ad} = \{\alpha \in C^{(1),1}([0, 1]) | \alpha_{min} \leqq \alpha(z) \leqq \alpha_{max}, |d\alpha/dz| \leqq C_1 ,$$
$$|d^2\alpha/dz^2| \leqq C_2 \text{ a.e.} , \int_0^1 \alpha^2(z) \, dz = C_3\} ,$$

where α_{min}, α_{max}, C_1, C_2 and C_3 are given positive constants. (Here $C^{(1),1}$ denotes the space of differentiable functions, the derivatives of which are Lipschitz in the interval $[0, 1]$).

We shall consider a class of domains $D(\alpha)$, where $\alpha \in U_{ad}$ and

$$D(\alpha) = \{(r, z) | \, 0 < z < 1, \, 0 < r < \alpha(z)\} .$$

For any $\alpha \in U_{ad}$, the graph $\Gamma(\alpha)$ of the function α will concide either with the part Γ_u (Case I) or with Γ_g (Case II).

The function α has to be determined from the following *Optimal Design Problem*:

$$(4) \qquad \alpha = \arg \min_{\beta \in U_{ad}} \mathscr{J}(\beta, \sigma(\beta)) ,$$

where

$$\mathscr{J}(\beta, \sigma) = \int_{D(\beta)} f^2(\sigma) \, r \, dr \, dz$$

and $\sigma(\beta)$ is the solution of the variational inequality (3) on $D = D(\beta)$.

In what follows, we shall present some conditions, sufficient for unique solvability of (3) for all $\alpha \in U_{ad}$.

Assume that

$$b_{ijkm} \in L^\infty(\hat{D})$$

are given and (2) holds for all $\sigma \in S(\hat{D})$, where

$$\hat{D} = (0, \delta) \times (0, 1) , \quad \delta > \alpha_{max} .$$

Case I. Assume that forces

$$\mathbf{F} \in [L_r^2(\hat{D})]^2 \quad \text{and} \quad \mathbf{g} \in [L_r^2(\partial\hat{D} \dot{-} \Gamma_0 \dot{-} \Gamma_\delta)]^2$$

are given, where

$$\Gamma_0 = \{(r, z) | \, r = 0, \, z \in (0, 1)\} , \quad \Gamma_\delta = \{(r, z) | \, r = \delta, \, z \in (0, 1)\} .$$

$(A1)$ Moreover, assume that a tensor field $\sigma^0 \in E(\hat{D})$ and constants $C_L > 0$, $\varepsilon > 0$ exist such that

$$(A2) \qquad \|\sigma^0(r, z) - \sigma^0(t, z)\| \leqq C_L|r - t|$$

holds for almost all $z \in (0, 1)$ and all $r, t \in [0, \delta]$,

$$(A3) \qquad (1 + \varepsilon)\, \sigma^0 \in P(\hat{D})\,.$$

Case II. Assume that $\boldsymbol{F} \in [L_r^2(\hat{D})]^2$, $\boldsymbol{g} \equiv 0$,

$(A4)$ a tensor field $\sigma^0 \in S(\hat{D})$ exists such that

$$\sigma^0|_{D(\alpha)} \in E(D(\alpha)) \quad \forall \alpha \in U_{ad}^0\,,$$

where

$$U_{ad}^0 = \{\alpha \in C^{(0),1}([0, 1]) | \alpha_{min} \leqq \alpha(z) \leqq \alpha_{max}\}\,,$$

$(A2)$ and $(A3)$ hold.

Example 2. (for Case II). Let us consider

$$\boldsymbol{g} \equiv 0 \quad \text{and} \quad \boldsymbol{F} = (F_r, 0)\,,$$

$$F_r = \omega^2 r \varrho(r, z)\,,$$

(centrifugal forces only), where ω = const. and the density ϱ satisfies the following conditions

$$\varrho \in L^\infty(\hat{D})\,, \quad |\varrho(r, z) - \varrho(t, z)| \leqq C_\varrho|r - t|$$

for a.a. $z \in (0, 1)$ and all $r, t \in [0, \delta]$.

We define $\sigma_{ij}^0 = 0$ with the exception of

$$\sigma_{33}^0 = rF_r\,.$$

It is then easy to verify $(A2)$, $(A3)$, $(A4)$ for sufficiently small norm $\|F_r\|_{L^\infty(\hat{D})}$, using also the property (1) of the yield function.

Proposition 1.1. *The Haar-Kármán principle has a unique solution for any* $D(\alpha)$, $\alpha \in U_{ad}$.

Proof. The set $E(D(\alpha)) \cap P(D(\alpha))$ is non-empty, as follows from the assumptions. (Note that (1) implies $f(0) = 0$ so that $0 \in P(D(\alpha))$ and $\sigma^0|_{D(\alpha)} \in P(D(\alpha))$ follows from $(A3)$, since $P(D(\alpha))$ is convex.)

In Case I we use the fact that any restriction $\sigma^0|_{D(\alpha)} \in E(D(\alpha))$. Indeed, an extension $\boldsymbol{v}^\sim$ of $\boldsymbol{v} \in V(D(\alpha))$ by zero belongs to $V(\hat{D})$ and

$$\langle \sigma^0, \mathbf{e}(\boldsymbol{v}) \rangle_{D(\alpha)} = \langle \sigma^0, \mathbf{e}(\boldsymbol{v}^\sim) \rangle_{\hat{D}} = \mathscr{F}_{\hat{D}}(\boldsymbol{v}^\sim) = \mathscr{F}_{D(\alpha)}(\boldsymbol{v})$$

follows from $(A1)$.

The sets $E(D(\alpha))$ and $P(D(\alpha))$ are convex and closed in $S(D(\alpha))$, the functional of complementary energy is quadratic, strictly convex. Hence the existence of a unique minimizer $\sigma(\alpha)$ follows.

2. Approximations by piecewise linear stress fields

Let N be a positive integer and $h = 1/N$. We define $\Delta_j = [(j-1)h, jh]$, $j = 1, 2, \ldots, N$,

$$U_{\mathrm{ad}}^h = \{\alpha_h \in C^{(0),1}([0, 1]) | \alpha_{\min} \leqq \alpha_h(z) \leqq \alpha_{\max},$$
$$\alpha_h|_{\Delta_j} \in P_1(\Delta_j) \quad \forall j, |\mathrm{d}\alpha_h/\mathrm{d}z| \leqq C_1,$$
$$|\delta_h^2 \alpha_h(jh)| \leqq C_2, \quad j = 1, \ldots, N-1, \quad \textstyle\int_0^1 \alpha_h^2 \,\mathrm{d}z = C_3\},$$

where

$$\delta_h^2 \alpha_h(jh) = h^{-2}[\alpha_h((j+1)h) - 2\alpha_h(jh) + \alpha((j-1)h)]$$

and $P_1(\Delta_j)$ denotes the set of linear polynomials on Δ_j. (Note that $U_{\mathrm{ad}}^h \not\subset U_{\mathrm{ad}}$).

Let $D_h \equiv D(\alpha_h)$ for $\alpha_h \in U_{\mathrm{ad}}^h$. The polygonal domain D_h will be carved into triangles by the following way. We chose $\alpha_0 \in (0, \alpha_{\min})$ and introduce a uniform triangulation of the rectangle $\mathscr{R} = [0, \alpha_0] \times [0, 1]$, which is independent of α_h, if h is fixed.

In the remaining part $D_h \dot{-} \mathscr{R}$ let the nodal points divide the segments $[\alpha_0, \alpha_h(jh)]$, $(j = 0, 1, \ldots, N)$ into M equal segments, where

$$M = 1 + [(\alpha_{\max} - \alpha_0) N]$$

and the brackets denote the integer part. Consequently, one obtains a strongly regular family $\{\mathscr{T}_h(\alpha_h)\}$, $h \to 0$, $\alpha_h \in U_{\mathrm{ad}}^h$, of triangulations.

Let us consider the space of linear finite elements

$$V_h(D_h) = \{\mathbf{v}_h = (u_h, w_h) \in [C(\mathscr{C}\ell D_h)]^2 | (u_h/r)|_T \in P_1(T),$$
$$w_h|_T \in P_1(T) \quad \forall T \in \mathscr{T}_h(\alpha_h), \quad \mathbf{v}_h = 0 \quad \text{on} \quad \Gamma_u\}.$$

(Here T denotes any triangle.)

Note that $V_h(D_h) \subset V(D_h)$ and $u_h = 0$ on Γ_0 follows from $\mathbf{v}_h \in V(D_h)$.

The stress field will be approximated by means of the following space

$$H_h(D_h) = \{\tau^h \in S(D_h) | \tau_{ij}^h|_T \in P_1(T), \quad i, j = 1, 2, 3, \quad \forall T \in \mathscr{T}_h(\alpha_h)\}.$$

We introduce an external approximation of the set $E(D_h)$ as follows

$$E_h(D_h) = \{\tau^h \in H_h(D_h) | \langle \tau^h, \mathbf{e}(\mathbf{v}_h) \rangle_{D_h} = \mathscr{F}_{D_h}(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h(D_h)\}.$$

Instead of the problem (3) we shall solve the following *Approximate State Problem*: find $\sigma^{h\eta} \in E_h(D_h) \cap P_{h\eta}(D_h)$ such that

$$(\sigma^{h\eta}, \tau^h - \sigma^{h\eta})_{D_h} \geqq 0 \quad \forall \tau^h \in E_h(D_h) \cap P_{h\eta}(D_h), \tag{6}$$

where η is a positive parameter and

$$P_{h\eta}(D_h) = \{\tau^h \in H_h(D_h) | \textstyle\int_{D_h} (f(\tau^h) - 1)^+ r \,\mathrm{d}r \,\mathrm{d}z \leqq \eta\}.$$

Lemma 2.1. *Let r_h: $S(D_h) \to H_h(D_h)$ be the projection mapping defined by means of the equation*

(7) $$\langle r_h \tau - \tau, \sigma^h \rangle_{D_h} = 0 \quad \forall \sigma^h \in H_h(D_h) .$$

Then for any $\tau^0 \in P(\hat{D})$ a function $h_0(\eta, \tau^0)$ exists such that

$$h < h_0(\eta, \tau^0) \Rightarrow r_h(\tau^0|_{D_h}) \in P_{h\eta}(D_h) .$$

Proof. The triangulations $\mathscr{T}_h(\alpha_h)$ of D_h can be extended to $\hat{D} \dot{-} D_h$ in such a way that the extended triangulations $\{\mathscr{T}_h(\alpha_h, \hat{D})\}$, $h \to 0$, $\alpha_h \in U^h_{ad}$, generate a regular family. Let the projections r_h be extended to map $S(\hat{D})$ into $H_h(\hat{D})$ in the corresponding way.

Then

(8) $$\|r_h \tau^0 - \tau^0\|_{0,r,\hat{D}} \to 0 , \quad h \to 0 .$$

For any $g \in L^2_r(\hat{D})$ and any $\varepsilon > 0$ there exists $g_\varepsilon \in C^\infty(\mathscr{Cl}\ \hat{D})$ such that

$$\|g - g_\varepsilon\|_{0,r,\hat{D}} < \sqrt{(\varepsilon)}/2 .$$

It is readily seen that $r_h g|_T$, where T is any triangle of $\mathscr{T}_h(\alpha_h, \hat{D})$, coincides with the $L^2_r(T)$-projection of $g|_T$ into $P_1(T)$. Therefore

(9) $$\|g - r_h g\|_{0,r,T} \leqq \|g - r_h g_\varepsilon\|_{0,r,T} \leqq \|g - g_\varepsilon\|_{0,r,T} + $$
$$+ \|g_\varepsilon - r_h g_\varepsilon\|_{0,r,T}$$

follows from the optimality of the $L^2_r(T)$-projection.

We may compare $r_h g_\varepsilon$ with the linear Lagrange interpolate $\pi_h g_\varepsilon$ on any triangle $T \in \mathscr{T}_h(\alpha_h, \hat{D})$. Thus we obtain

(10) $$\sum_T \|g_\varepsilon - r_h g_\varepsilon\|^2_{0,r,T} \leqq \sum_T \|g_\varepsilon - \pi_h g_\varepsilon\|^2_{0,r,T} =$$
$$= \|g_\varepsilon - \pi_h g_\varepsilon\|^2_{0,r,\hat{D}} \leqq C h^4 |g_\varepsilon|^2_{2,r,\hat{D}}$$

where the latter estimate can be derived via the argument of Lemma 6.1 and 6.2 of [8]. Combining (9), (10), we arrive at

(11) $$\|g - r_h g\|_{0,r,\hat{D}} \to 0 , \quad h \to 0 .$$

Applying (11) to every components of the tensor τ^0, we obtain (8).

Since

$$(f(r_h \tau^0) - 1)^+ \leqq (f(r_h \tau^0) - f(\tau^0))^+ + (f(\tau^0) - 1)^+ \leqq$$
$$\leqq |f(r_h \tau^0) - f(\tau^0)| ,$$

where

$$(f(\tau^0) - 1)^+ = 0 \quad \text{a.e.} \quad \text{in } D_h$$

by assumption, we obtain

$$\int_{D_h} (f(r_h \tau^0) - 1)^+ \, r \, dr \, dz \leqq \int_{D_h} |f(r_h \tau^0) - f(\tau^0)| \, r \, dr \, dz \leqq$$
$$\leqq C_f C \|r_h \tau^0 - \tau^0\|_{0,r,\hat{D}} \leqq \eta$$

for $h < h_0(\eta, \tau^0)$, using (8).

Lemma 2.2. *The set $P_{h\eta}(D_h)$ is closed and convex in $S(D_h)$.*

Proof. 1° Let $\tau^n \in P_{h\eta}(D_h)$, $\tau^n \to \tau$ in $S(D_h)$ for $n \to \infty$. Since

$$(f(\tau) - 1)^+ \leqq |f(\tau) - f(\tau^n)| + (f(\tau^n) - 1)^+ ,$$

we may write

$$\int_{D_h} (f(\tau) - 1)^+ \, r \, dr \, dz \leqq \int_{D_h} |f(\tau) - f(\tau^n)| \, r \, dr \, dz + \eta \quad \forall n .$$

Passing with $n \to \infty$, the integral on the right-hand side tends to zero, since

$$\int_{D_h} |f(\tau) - f(\tau^n)| \, r \, dr \, dz \leqq C \|\tau - \tau^n\|_{0,r,D_h} \to 0 .$$

Consequently, $\tau \in P_{h\eta}(D_h)$ follows, as $H_h(D_h)$ is a closed subspace of $S(D_h)$.

2° Let us consider $\sigma \in P_{h\eta}(D_h)$ and $\tau \in P_{h\eta}(D_h)$, $t \in [0, 1]$.
Then

$$f(t\sigma + (1 - t)\tau) - 1 \leqq t[f(\sigma) - 1] + (1 - t)[f(\tau) - 1]$$

and therefore

$$[f(t\sigma + (1 - t)\tau) - 1]^+ \leqq t[f(\sigma) - 1]^+ + (1 - t)[f(\tau) - 1]^+$$

holds a.e. in D_h. Integrating, we obtain

$$t\sigma + (1 - t)\tau \in P_{h\eta}(D_h) .$$

Lemma 2.3. *There exists a function $h_1(\eta)$ such that the problem (6) has a unique solution for any $\eta > 0$, $h < h_1(\eta)$, $\alpha_h \in U^h_{\text{ad}}$.*

Proof. By assumptions $(A1)$, $(A4)$, $(A3)$, there exists

$$\sigma^0(\alpha_h) \in E(D_h) \cap P(D_h)$$

for any $h \to 0$, $\alpha_h \in U^h_{\text{ad}}$. (We define

(12) $$\sigma^0(\alpha_h) = \sigma^0|_{D_h} .)$$

Using Lemma 2.1, we obtain

$$r_h\sigma^0(\alpha_h) \in P_{h\eta}(D_h) \quad \text{if} \quad h < h_0(\eta, \sigma_0) \equiv h_1(\eta) .$$

Moreover, since $\mathbf{e}(\mathbf{v}_h) \in H_h(D_h)$ for $\mathbf{v}_h \in V_h(D_h) \subset V(D_h)$,

$$\langle r_h\sigma^0(\alpha_h), \mathbf{e}(\mathbf{v}_h)\rangle_{D_h} = \langle \sigma^0(\alpha_h), \mathbf{e}(\mathbf{v}_h)\rangle_{D_h} = \mathscr{F}_{D_h}(\mathbf{v}_h) .$$

Consequently, $r_h\sigma^0(\alpha_h) \in E_h(D_h)$ and the set $E_h(D_h) \cap P_{h\eta}(D_h)$ is non-empty. Since it is also closed and convex in $S(D_h)$ by virtue of Lemma 2.2, the unique solvability of the problem (6) follows.

Proposition 2.1. *Let $\{\alpha_h\}$, $h \to 0$, be a sequence of $\alpha_h \in U^h_{\text{ad}}$ such that $\alpha_h \to \alpha$ in $C([0, 1])$.*

Then a function $h_1(\eta)$ exists such that if $\eta \to 0$, $h \to 0$, $h < h_1(\eta)$, then

$$\tilde{\sigma}^{h\eta} \to \sigma(\alpha) \quad in \quad S(\hat{D}),$$

where $\tilde{\sigma}^{h\eta}$ is the solution of the approximate problem (6), extended by zero to the domain $\hat{D} \dot{-} D(\alpha_h)$ and $\sigma(\alpha)$ is the solution of the problem (3) on $D(\alpha)$, extended by zero to $\hat{D} \dot{-} D(\alpha)$.

Proof. In what follows we assume that $\eta \to 0$, $h \to 0$, $h < h_1(\eta)$.

1° We may insert

$$\tau^h = r_h \sigma^0(\alpha_h)$$

into (6) (cf. the proof of Lemma 2.3) to obtain

$$\|\sigma^{h\eta}\|^2_{D_h} \leqq (\sigma^{h\eta}, r_h\sigma^0(\alpha_h))_{D_h} \leqq \|\sigma^{h\eta}\|_{D_h} \|r_h\sigma^0(\alpha_h)\|_{D_h} .$$

Therefore, cancelling and using the inequalities (2), (7) and ($A1$), ($A4$), we have (using (12) again)

$$b_0^{1/2}\|\sigma^{h\eta}\|_{0,r,D_h} \leqq \|\sigma^{h\eta}\|_{D_h} \leqq \|r_h\sigma^0(\alpha_h)\|_{D_h} \leqq$$
$$\leqq b_1^{1/2}\|\sigma^0(\alpha_h)\|_{0,r,D_h} \leqq C b_1^{1/2} .$$

Consequently, the sequence $\{\tilde{\sigma}^{h\eta}\}$ is bounded, as

(14) $$\|\tilde{\sigma}^{h\eta}\|_{0,r,\hat{D}} \leqq C(b_1/b_0)^{1/2} \quad \forall h, \eta .$$

There exists a subsequence (and we shall denote it by the same symbol) such that

(15) $$\tilde{\sigma}^{h\eta} \rightharpoonup \sigma \text{ (weakly) in } S(\hat{D}), \quad \sigma \in S(\hat{D}) .$$

2° We can show that

(16) $$\sigma = 0 \quad \text{a.e. in} \quad \hat{D} \dot{-} D(\alpha) .$$

Assume that

$$\|\sigma\|_{0,r,M} > 0$$

on some set $M \subset \hat{D} \dot{-} D(\alpha)$, meas $M > 0$.

Introducing the characteristic function χ_M of M, we may write

$$|\langle\tilde{\sigma}^{h\eta}, \chi_M\sigma\rangle_{\hat{D}}| = |\langle\tilde{\sigma}^{h\eta}, \sigma\rangle_{D_h \cap M}| \leqq \|\tilde{\sigma}^{h\eta}\|_{0,r,\hat{D}} \|\sigma\|_{0,r,D_h\cap M} \to 0$$

by virtue of (14) and because meas $(D_h \cap M) \to 0$. On the other hand,

$$\langle\tilde{\sigma}^{h\eta}, \chi_M\sigma\rangle_{\hat{D}} \to \langle\sigma, \chi_M\sigma\rangle_{\hat{D}} = \|\sigma\|^2_{0,r,M} > 0$$

and we arrive at a contradiction.

3° Let us show that

$$\sigma|_{D(\alpha)} \in E(D(\alpha)) .$$

Let a $\mathbf{v} \in V(D(\alpha))$ be given. We construct an extension $\mathbf{v}^\sim \in \mathscr{H}(\hat{D})$ by zero in Case I and symmetric with respect to $\Gamma(\alpha)$ in the r-direction in Case II, respectively.

There exists a sequence $\{\mathbf{v}_\varkappa\}$, $\mathbf{v}_\varkappa \equiv (u_\varkappa, w_\varkappa)$, $\varkappa \to 0$, such that $\mathbf{v}_\varkappa \in [C^\infty(\mathscr{C}\ell\ D)]^2$, $\mathbf{v}_\varkappa \to \mathbf{v}^\sim$ in $\mathscr{H}(\hat{D})$ and

Case I: $\quad \operatorname{supp} \mathbf{v}_\varkappa \cap (\hat{D} \dot{-} D(\alpha)) = \emptyset\,,$

Case II: $\quad \operatorname{supp} u_\varkappa \cap (\partial\hat{D} \dot{-} \Gamma_\delta) = \emptyset$

$\quad \operatorname{supp} w_\varkappa \cap (\partial\hat{D} \dot{-} \Gamma_\delta \dot{-} \Gamma_0) = \emptyset\,.$

(For the proof – see [5] – Lemma 1.3, p. 227).

Then $\mathbf{v}_\varkappa|_{D_h} \in V(D_h)\ \forall h, \varkappa, (h < h_2(\varkappa)$ in Case I). Let us introduce the interpolation

$$\pi_h\colon [C(\mathscr{C}\ell\ \hat{D})]^2 \cap \mathscr{H}(\hat{D})) \to V_h(\hat{D})\,,$$

using the triangulation $\mathscr{T}_h(\alpha_h, \hat{D})$, introduced in the proof of Lemma 2.1. Let $\varkappa$ be fixed. Then $u_\varkappa/r \equiv \omega_\varkappa \in C^\infty(\mathscr{C}\ell\ \hat{D})$ and we define

$$\pi_h(\mathbf{v}_\varkappa) \equiv (\pi_h^r u_\varkappa, \pi_h^z w_\varkappa)\,,$$

$$\pi_h^r u_\varkappa = r(I_h \omega_\varkappa)\,,$$

$$\pi_h^z w_\varkappa = I_h w_\varkappa$$

and I_h is the standard piecewise linear interpolation.

We have

$$\pi_h(\mathbf{v}_\varkappa)|_{D_h} \in V_h(D_h)\,, \quad \mathbf{e}(\pi_h(\mathbf{v}_\varkappa))|_{D_h} \in H_h(D_h)$$

and

$$\langle \tilde{\sigma}^{h\eta}, \mathbf{e}(\pi_h(\mathbf{v}_\varkappa))\rangle_{\hat{D}} = \mathscr{F}_{D_h}(\pi_h(\mathbf{v}_\varkappa))\,. \tag{17}$$

We can prove that

$$\mathbf{e}(\pi_h(\mathbf{v}_\varkappa)) \to \mathbf{e}(\mathbf{v}_\varkappa) \quad \text{in} \quad S(\hat{D}) \quad \text{for} \quad h \to 0\,. \tag{18}$$

In fact, denoting

$$\left(\int_{\hat{D}} (u^2/r^2 + (\partial u/\partial r)^2 + (\partial u/\partial z)^2)\, r\, \mathrm{d}r\, \mathrm{d}z\right)^{1/2} = \|u\|_{X_1(\hat{D})}\,,$$

we have (cf. [6] – Lemmas 5.1 and 5.9)

$$\|\pi_h^r u_\varkappa - u_\varkappa\|_{X_1(\hat{D})} = \|r(I_h\omega_\varkappa - \omega_\varkappa)\|_{X_1(\hat{D})} \leqq Ch|\omega_\varkappa|_{2,\hat{D}} \tag{19}$$

and (cf. [8] – Lemma 5.9)

$$\|\pi_h^z w - w_\varkappa\|_{1,r,\hat{D}} \leqq Ch\|w_\varkappa\|_{2,r,\hat{D}}\,. \tag{20}$$

Combining (18) and (19), (20), the convergence (18) can be derived. Moreover, we may write e.g. in Case II – Example 2

$$\begin{aligned} &|\mathscr{F}_{D_h}(\pi_h(\mathbf{v}_\varkappa)) - \mathscr{F}_{D(\alpha)}(\mathbf{v}_\varkappa)| \leqq \\ &\leqq \left|\int_{D_h} F_r(\pi_h^r u_\varkappa - u_\varkappa)\, r\, \mathrm{d}r\, \mathrm{d}z\right| + \\ &+ \left|\int_{D_h} F_r u_\varkappa r\, \mathrm{d}r\, \mathrm{d}z - \int_{D(\alpha)} F_r u_\varkappa r\, \mathrm{d}r\, \mathrm{d}z\right| \leqq \\ &\leqq \|F_r\|_{0,r,\hat{D}}\, \|\pi_h^r u_\varkappa - u_\varkappa\|_{0,r,\hat{D}} + \\ &\int_{\Delta(D_h, D(\alpha))} |F_r u_\varkappa|\, r\, \mathrm{d}r\, \mathrm{d}z \to 0\,, \quad h \to 0\,, \end{aligned} \tag{21}$$

where $\Delta(D_h, D(\alpha)) = (D_h \dot{-} D(\alpha)) \cup (D(\alpha) \dot{-} D_h)$ and

$$\|\pi_h^r u_\varkappa - u_\varkappa\|_{0,r,\hat{D}} \leqq Ch|\omega_\varkappa|_{2,\hat{D}}$$

holds due to the fact, that

$$\operatorname{supp} \omega_\varkappa \cap \Gamma_0 = \emptyset .$$

The other integrals occurring in $\mathscr{F}_{D_h}(\pi_h(\mathbf{v}_\varkappa))$ (in Case I) can be treated in a parallel way, using also the Trace theorem (cf. e.g. [4] – Lemma 1) and (18).

Passing to the limit with $\eta \to 0$, $h \to 0$ in (17) and using the weak convergence (15) together with (18), (21), we obtain

$$\langle \sigma, \mathbf{e}(\mathbf{v}_\varkappa) \rangle_{D(\alpha)} = \mathscr{F}_{D(\alpha)}(\mathbf{v}_\varkappa) .$$

Passing to the limit with $\varkappa \to 0$, we arrive at

$$\langle \sigma, \mathbf{e}(\mathbf{v}) \rangle_{D(\alpha)} = \mathscr{F}_{D(\alpha)}(\mathbf{v}) ,$$

since

$$\mathbf{v}_\varkappa \to \mathbf{v}^\sim \quad \text{in} \quad \mathscr{H}(\hat{D}) \Rightarrow \mathbf{e}(\mathbf{v}_\varkappa) \to \mathbf{e}(\mathbf{v}^\sim) \quad \text{in} \quad S(\hat{D}) ,$$

$$\|u_\varkappa - u\|_{0,r,D(\alpha)} \leqq \|u_\varkappa - \tilde{u}\|_{0,r,\hat{D}} \leqq \delta \|u_\varkappa - \tilde{u}\|_{0,1/r,\hat{D}}$$

and

$$\mathscr{F}_{D(\alpha)}(\mathbf{v}_\varkappa) \to \mathscr{F}_{D(\alpha)}(\mathbf{v}) .$$

Consequently,

$$\sigma|_{D(\alpha)} \in E(D(\alpha)) .$$

4° We can show that

$$\sigma|_{D(\alpha)} \in P(D(\alpha)) .$$

In fact, the functional

$$j: \tau \to \int_D (f(\tau) - 1)^+ \, r \, \mathrm{d}r \, \mathrm{d}z$$

is convex and continuous on $S(\hat{D})$, since for $\tau^n \to \tau$ we have

$$|j(\tau) - j(\tau^n)| \leqq \int_{\hat{D}} |(f(\tau) - 1)^+ - (f(\tau^n) - 1)^+| r \, \mathrm{d}r \, \mathrm{d}z \leqq$$

$$\leqq \int_{\hat{D}} |f(\tau) - f(\tau^n)| \, r \, \mathrm{d}r \, \mathrm{d}z \leqq C_f C \|\tau - \tau^n\|_{0,r,\hat{D}} \to 0 .$$

Consequently, j is weakly lower semicontinuous. Using also (16), we may write

$$\int_{D(\alpha)} (f(\sigma) - 1)^+ \, r \, \mathrm{d}r \, \mathrm{d}z = j(\sigma) \leqq \liminf_{\eta \to 0, h \to 0} j(\tilde{\sigma}^{h\eta}) =$$

$$= \liminf \int_{D_h} (f(\sigma^{h\eta}) - 1)^+ \, r \, \mathrm{d}r \, \mathrm{d}z \leqq \lim \eta = 0 .$$

Therefore

$$f(\sigma) \leqq 1 \quad \text{a.e. in} \quad D(\alpha) .$$

5° Let us verify that $\sigma|_{D(\alpha)}$ is a solution of the problem (3).

Let a $\tau \in P(D(\alpha) \cap E(D(\alpha))$ be given. We define $\sigma^0(\alpha) = \sigma^0|_{D(\alpha)}$,

$$\omega = \tau - \sigma^0(\alpha) \quad \text{in} \quad D(\alpha)\,, \qquad \omega = 0 \quad \text{in} \quad \hat{D} \dot{-} D(\alpha)\,,$$

and distinguish the two cases of boundary conditions, in what follows.

Case I. Let us introduce a positive parameter λ and introduce $k = 1 + \lambda$,

$$\omega^\lambda(r, z) = \mathscr{A}\omega(r/k, z)$$

by means of the formulas

$$\omega^\lambda_{ij}(r, z) = a_{ij}(\lambda)\,\omega_{ij}(r/k, z) \text{ (no sum)}\,, \quad i, j = 1, 2, 3\,,$$

where

$$a_{11}(\lambda) = a_{33}(\lambda) = 1\,, \quad a_{22}(\lambda) = k^{-2}\,, \quad a_{12}(\lambda) = k^{-1}\,.$$

Then we deduce

$$\int_{D_h} \omega^\lambda_{ij} e_{ij}(\mathbf{v})\, r\, \mathrm{d}r\, \mathrm{d}z = 0 \quad \forall \mathbf{v} \in V(D_h) \tag{22}$$

for all $h < h_0(\lambda)$.

In fact, denoting

$$D_\lambda \equiv k\, D(\alpha) = \{(r, z)|\, 0 < r < k\,\alpha(z),\ 0 < z < 1\}\,,$$

and using new variables

$$\varrho = r/k\,, \quad \zeta = z\,,$$

we obtain

$$\begin{aligned}
\mathbf{v} \in V(D_\lambda) \Rightarrow & \int_{D_\lambda} \omega^\lambda_{ij} e_{ij}(\mathbf{v})\, r\, \mathrm{d}r\, \mathrm{d}z = \\
&= \int_{D_\lambda} (\mathscr{A}\omega)_{ij}\,(r/k, z)\; e_{ij}(\mathbf{v}(r, z))\, r\, \mathrm{d}r\, \mathrm{d}z = \\
&= \int_{D(\alpha)} a_{ij}\omega_{ij}(\varrho, \zeta)\; e_{ij}(\mathbf{v}^*(\varrho, \zeta))\, k^2 \varrho\, \mathrm{d}\varrho\, \mathrm{d}\zeta = \\
&= \int_{D(\alpha)} k^{-2}\omega_{ij} e_{ij}(\mathbf{v}^*)\, k^2 \varrho\, \mathrm{d}\varrho\, \mathrm{d}\zeta = 0\,,
\end{aligned} \tag{23}$$

where

$$\mathbf{v}^* = (u^*, w^*)\,, \quad u^*(\varrho, \zeta) = k u(k\varrho, \zeta)\,, \quad w^*(\varrho, \zeta) = w(k\varrho, \zeta)\,.$$

Here we have used the fact, that

$$D_h \subset D_\lambda \quad \forall h < h_1(\lambda) \quad \text{and} \quad \mathbf{v}^* \in V(D(\alpha))\,.$$

Extending $\mathbf{v} \in V(D_h)$ by zero to $\mathbf{v}^\sim \in V(D_\lambda)$, (22) follows from (23).

Case II. Let us define

$$\omega^\lambda(r, z) = \mathscr{A}\omega(kr, z)$$

by means of the formulas

$$\omega^\lambda_{ij}(r, z) = a_{ij}(\lambda)\,\omega_{ij}\,(kr, z) \text{ (no sum)}\,, \quad i,j = 1, 2, 3,$$

where

$$a_{11}(\lambda) = a_{33}(\lambda) = 1\,, \quad a_{22}(\lambda) = k^2\,, \quad a_{12}(\lambda) = k\,.$$

We shall again verify (22) for $h < h_0(\lambda)$. Denoting

$$k^{-1} D(\alpha) = \{(r, z) \mid 0 < r < k^{-1} \alpha(z),\ 0 < z < 1\},$$

and using new variables

$$\varrho = kr, \quad \zeta = z,$$

we obtain for $\mathbf{v} \in V(D_h)$

$$\int_{D_h} \omega_{ij}^{\lambda} e_{ij}(\mathbf{v})\, r \,\mathrm{d}r\, \mathrm{d}z =$$
$$\int_{k^{-1}D(\alpha)} (\mathscr{A}\omega)_{ij}\,(kr, z)\, e_{ij}(\mathbf{v}(r, z))\, r\,\mathrm{d}r\,\mathrm{d}z =$$
$$= \int_{D(\alpha)} k^2 \omega_{ij}(\varrho, \zeta)\, e_{ij}(\mathbf{v}^*(\varrho, \zeta))\, k^{-2} \varrho\, \mathrm{d}\varrho\, \mathrm{d}\zeta = 0,$$

since $k^{-1} D(\alpha) \subset D_h\ \forall h < h_0(\lambda)$ and $\mathbf{v}^* \in V(D(\alpha))$. Here we defined $\mathbf{v}^* = (u^*, w^*)$,

$$u^*(\varrho, \zeta) = k^{-1} u(\varrho/k, \zeta), \quad w^*(\varrho, \zeta) = w(\varrho/k, \zeta). \qquad \square$$

Let us introduce

$$\gamma(\lambda) = (1 - 2\lambda^{1/2}/\varepsilon)/(1 + \lambda^{1/2})$$

and

$$\tau^\lambda = \sigma^0 + \gamma(\lambda)\, \omega^\lambda \quad (\text{in } \hat{D}).$$

Then

$$\tau^\lambda|_{D_h} \in E(D_h) \quad \forall h < h_0(\lambda). \tag{24}$$

In fact, for $\mathbf{v} \in V(D_h)$ we use (22) and ($A1$), ($A4$) to obtain

$$\langle \tau^\lambda, \mathbf{e}(\mathbf{v}) \rangle_{D_h} = \langle \sigma^0(\alpha_h), \mathbf{e}(\mathbf{v}) \rangle_{D_h} + \gamma(\lambda) \langle \omega^\lambda, \mathbf{e}(\mathbf{v}) \rangle_{D_h} = \mathscr{F}_{D_h}(\mathbf{v}).$$

Next we can show that

$$\tau^\lambda|_{D_h} \in P(D_h) \quad \forall \lambda < \lambda_1(\omega) \quad \forall h < h_3(\lambda). \tag{25}$$

Let us denote $x = (r, z)$ and

$$y \equiv \begin{cases} (r/k, z) & \text{in Case I}, \\ (kr, z) & \text{in Case II}, \end{cases}$$

$\sigma^\lambda(y) = \sigma^0(y) + \gamma(\lambda)\, \omega^\lambda(x)$.

On the basis of ($A2$), we may write for a.a. $z \in (0, 1)$

$$\|\tau^\lambda(x) - \sigma^\lambda(y)\| = \|\sigma^0(x) - \sigma^0(y)\| \leqq C_L |t - r| \leqq C_L \delta \lambda \tag{26}$$

where $t = r/k$ or $t = rk$, respectively.

Next we shall prove that

$$f((1 + \lambda^{1/2})\, \sigma^\lambda(y)) \leqq 1 \quad \forall \lambda < \lambda_0(\varepsilon, \omega), \quad \forall h < h_2(\lambda) \tag{27}$$

for $y \in D(\alpha)$.

In fact, since for $y \in D(\alpha)$

$$\omega^{\lambda}(x) = \mathscr{A}\,\omega(y) = \omega(y) + [\mathscr{A}\,\omega(y) - \omega(y)] =$$
$$= \tau(y) - \sigma^0(y) + \mathscr{A}\,\omega(y) - \omega(y)\,,$$

we may write

$$(28) \qquad (1 + \lambda^{1/2})\,\sigma^{\lambda} = (1 + \lambda^{1/2})\,[\sigma^0 + \gamma(\lambda)\,(\tau - \sigma^0)] +$$
$$+ (1 + \lambda^{1/2})\,\gamma(\lambda)\,[\mathscr{A}\omega - \omega] =$$
$$= (1 + \lambda^{1/2})\,[\sigma^0(1 - \gamma) + \tau\gamma] + (1 - 2\lambda^{1/2}/\varepsilon)\,[\mathscr{A}\omega - \omega] \equiv$$
$$\equiv \mathscr{B}_1 + \mathscr{B}_2\,.$$

By assumption $(A\ 3)$ and the convexity of the yield function,

$$(29) \qquad f(\mathscr{B}_1) \leqq (1 + \lambda^{1/2})\,[(1 - \gamma)\,f(\sigma^0) + \gamma\,f(\tau)] \leqq$$
$$\leqq (1 + \lambda^{1/2})\,[(1 - \gamma(\lambda))\,(1 + \varepsilon)^{-1} + \gamma(\lambda)] =$$
$$= 1 - \lambda^{1/2}(1 + \varepsilon)^{-1}\,.$$

Since the function f is Lipschitz, we have

$$(30) \qquad f(\mathscr{B}_1 + \mathscr{B}_2) \leqq f(\mathscr{B}_1) + C_f\|\mathscr{B}_2\|\,.$$

For an estimation of the term $\mathscr{B}_2$ we have

$$(31) \qquad \|\mathscr{A}\sigma - \sigma\| = (2[k^{-1}\sigma_{12} - \sigma_{12}]^2 + [k^{-2}\sigma_{22} - \sigma_{22}]^2)^{1/2} \leqq 3\lambda\|\sigma\|$$
(in Case I)

and the same upper bound in Case II.

Altogether, from (20), (29) and (31), we obtain

$$(32) \qquad f((1 + \lambda^{1/2})\,\sigma^{\lambda}) \leqq 1 - \lambda^{1/2}(1 + \varepsilon)^{-1} + 3C_f\lambda\|\omega\| \leqq 1$$

if $\lambda < \lambda_0(\varepsilon, \omega)$ and $h < h_2(\lambda)$, for a.a. $y \in D(\alpha)$.

Using (32) and (26), we derive that

$$(33) \qquad f(\tau^{\lambda}(x)) \leqq f(\sigma^{\lambda}(y)) + C_f C_L \delta\lambda = (1 + \lambda^{1/2})^{-1} + \tilde{C}\lambda \leqq 1$$

if $\lambda < \lambda_1(\omega)$, $h < h_2(\lambda)$ and for a.a. $y \in D(\alpha)$, i.e., a.a. $x \in k\,D(\alpha)$ in Case I and a.a. $x \in k^{-1}\,D(\alpha)$ in Case II, respectively.

Having Case II in mind, we realize that

$$x \in D_h \dot{-} k^{-1}\,D(\alpha) \Rightarrow y \notin D(\alpha)\,, \quad \omega(y) = 0 \Rightarrow \omega^{\lambda}(x) = 0\,,$$
$$\tau^{\lambda}(\alpha_h)\,(x) = \sigma^0(\alpha_h)\,, \quad f(\tau^{\lambda}(\alpha_h)\,(x)) \leqq (1 + \varepsilon)^{-1}$$

almost everywhere by assumption $(A\ 3)$. Combining this result with (33), we are led to the conclusion (25).

We may apply Lemma 2.1 to the function τ^λ. Consequently (denoting $\tau^\lambda|_{D_h} = \tau^\lambda(\alpha_h)$, we obtain

$$r_h \tau^\lambda(\alpha_h) \in P_{h\eta}(D_h) \quad \forall h < h_4(\eta, \tau^\lambda, \lambda) .$$

We have

$$r_h \tau^\lambda(\alpha_h) \in E_h(D_h) \quad \forall h < h_1(\lambda) .$$

In fact, since

$$V_h(D_h) \subset V(D_h) ,$$

$$\mathbf{v}_h \in V_h(D_h) \Rightarrow \langle r_h \tau^\lambda(\alpha_h), \mathbf{e}(\mathbf{v}_h)\rangle_{D_h} = \langle \tau^\lambda(\alpha_h), \mathbf{e}(\mathbf{v}_h)\rangle_{D_h} = \mathscr{F}_{D_h(\mathbf{v}_h)}$$

follows from (24).

Therefore $r_h \tau^\lambda(\alpha_h)$ can be substituted into (6) to obtain

$$(\sigma^{h\eta}, r_h \tau^\lambda(\alpha_h))_{D_h} = (\tilde{\sigma}^{h\eta}, r_h \tau^\lambda)_{\hat{D}} \geqq \|\tilde{\sigma}^{h\eta}\|^2_{\hat{D}} \quad \forall h < h_4(\eta, \tau^\lambda(\alpha), \lambda) .$$

Let $h \to 0$, $\eta \to 0$, $h < h_1(\eta)$. Then

$$\|r_h \tau^\lambda - \tau^\lambda\|_{0,r,\hat{D}} \to 0$$

by virtue of (8).

Moreover, from (15), (16),

$$\text{(34)} \qquad \liminf \|\tilde{\sigma}^{h\eta}\|^2_{\hat{D}} \geqq \|\tilde{\sigma}\|^2_{\hat{D}} = \|\sigma\|^2_{D(\alpha)} .$$

Consequently,

$$\text{(35)} \qquad \|\sigma\|^2_{D(\alpha)} \leqq (\sigma, \tau^\lambda)_{D(\alpha)} .$$

Then we may write

$$\text{(36)} \qquad \|\tau^\lambda - \tau\|_{0,r,D(\alpha)} \leqq \|\gamma(\omega^\lambda - \omega)\|_{0,r,D(\alpha)} + |\gamma - 1| \, \|\omega\|_{0,r,D(\alpha)} \to 0$$

if $\lambda \to 0$, since $\gamma(\lambda) \to 1$ and, by virtue of (31),

$$\|\omega^\lambda(x) - \omega(x)\| = \|(\mathscr{A}\omega)(y) - \omega(x)\| \leqq \|\mathscr{A}\,\omega(y) - \omega(y)\| + $$
$$+ \|\omega(y) - \omega(x)\| \leqq 3\lambda\|\omega(y)\| + \|\omega(y) - \omega(x)\| ,$$

$$\text{(37)} \qquad \int_{D(\alpha)} \|\omega(y) - \omega(x)\|^2 \, r \, \mathrm{d}r \, \mathrm{d}z \to 0 \quad \text{for} \quad \lambda \to 0 .$$

In order to verify (37), we apply the following argument. There exists a sequence $\{\omega^n\}$, $n = 1, 2, \ldots$, such that

$$\omega^n \in [C^\infty_{0,r}(D(\alpha))]^4 \cap S(D(\alpha)) , \quad \omega^n \to \omega \text{ in } S(D(\alpha)) .$$

Then

$$\text{(38)} \qquad \int_{D(\alpha)} \|\omega^n(y) - \omega^n(x)\|^2 \, r \, \mathrm{d}r \, \mathrm{d}z \leqq C\lambda^2 \|\omega^n\|_{C^1(D(\alpha))}$$

can be deduced on the basis of the mean value theorem.

Moreover, we easily derive that

$$(39)\qquad \left(\int_{D(\alpha)} \|\omega(y) - \omega^n(y)\|^2 \, r \, dr \, dz\right)^{1/2} \leqq$$
$$\leqq C\left(\|\omega - \omega^n\|_{0,r,D(\alpha)} + \|\omega\|_{0,r,kD(\alpha) \dot{-} D(\alpha)}\right).$$

Finally, we may write

$$\|\omega(y) - \omega\|_{0,r,D(\alpha)} \leqq \|\omega(y) - \omega^n(y)\|_{0,r,D(\alpha)} +$$
$$+ \|\omega^n(y) - \omega^n\|_{0,r,D(\alpha)} + \|\omega^n - \omega\|_{0,r,D(\alpha)}$$

and using (39), (38), we arrive at (37).

Passing with $\lambda \to 0$ in (35), we obtain from (36)

$$\|\sigma\|^2_{D(\alpha)} \leqq (\sigma, \tau)_{D(\alpha)} .$$

Thus $\sigma|_{D(\alpha)}$ is a solution of the variational inequality (3). Since the solution is unique, $\sigma|_{D(\alpha)} = \sigma(\alpha)$ and the whole sequence $\tilde{\sigma}^{h\eta}$ tends to σ weakly in $S(\hat{D})$.

6° To prove the strong convergence, we insert $\tau \equiv \sigma(\alpha)$ into the argument of 5°. Thus we obtain

$$(\tilde{\sigma}^{h\eta}, r_h\sigma^\lambda(\alpha_h))_{\hat{D}} \geqq \|\tilde{\sigma}^{h\eta}\|^2_{\hat{D}} ,$$
$$\limsup_{h\to 0,\eta\to 0} \|\tilde{\sigma}^{h\eta}\|^2_{\hat{D}} \leqq (\sigma, \sigma(\alpha))_{D(\alpha)} = \|\sigma(\alpha)\|^2_{D(\alpha)} .$$

Combining this result with (34), we may write

$$(40)\qquad \lim \|\tilde{\sigma}^h\|^2_{\hat{D}} = \|\sigma(\alpha)\|^2_{D(\alpha)} .$$

The weak convergence (15), convergence of norms (40) and the equivalence of norms (2) imply the strong convergence

$$\|\tilde{\sigma}^{h\eta} - \sigma(\alpha)\|_{0,r,\hat{D}} \to 0 .$$

Proposition 2.2. *Let $\{\alpha_h\}$, $h \to 0$, be a sequence of $\alpha_h \in U^h_{\mathrm{ad}}$ such that $\alpha_h \to \alpha$ in $C([0, 1])$.*

Then a function $h_1(\eta)$ exists such that if $\eta \to 0$, $h \to 0$, $h < h_1(\eta)$, then

$$\mathscr{J}(\alpha_h, \sigma^{h\eta}(\alpha_h)) \to \mathscr{J}(\alpha, \sigma(\alpha)) ,$$

where $\sigma^{h\eta}(\alpha_h)$ and $\sigma(\alpha)$ is the solution of the problem (6) and (3), respectively.

Proof. Since $f(0) = 0$, we may write (cf. Proposition 2.1)

$$\mathscr{J}(\alpha_h, \sigma^{h\eta}) = \int_{\hat{D}} f^2(\tilde{\sigma}^{h\eta}) \, r \, dr \, dz ,$$
$$\mathscr{J}(\alpha, \sigma(\alpha)) = \int_{\hat{D}} f^2(\sigma) \, r \, dr \, dz .$$

By assumption, we have

$$|f^2(\tilde{\sigma}^{h\eta}) - f^2(\sigma)| \leqq |f(\tilde{\sigma}^{h\eta}) - f(\sigma)| \, |f(\tilde{\sigma}^{h\eta}) + f(\sigma)| \leqq$$
$$\leqq C_f\|\tilde{\sigma}^{h\eta} - \sigma\| \, (2f(\sigma) + C_f\|\tilde{\sigma}^{h\eta} - \sigma\|).$$

Therefore, we may write

$$\left|\mathscr{J}(\alpha_h, \tilde{\sigma}^{h\eta}) - \mathscr{J}(\alpha, \sigma(\alpha))\right| \leqq \int_{\hat{D}} \left|f^2(\tilde{\sigma}^{h\eta}) - f^2(\sigma)\right| r \,\mathrm{d}r \,\mathrm{d}z \leqq$$
$$\leqq C \int_{\hat{D}} \left\|\tilde{\sigma}^{h\eta} - \sigma\right\| f(\sigma)\, r \,\mathrm{d}r \,\mathrm{d}z + C \int_{\hat{D}} \left\|\tilde{\sigma}^{h\eta} - \sigma\right\|^2 r \mathrm{d}r \,\mathrm{d}z \leqq$$
$$\leqq C\left(\left\|\tilde{\sigma}^{h\eta} - \sigma\right\|_{0,r,\hat{D}} + \left\|\tilde{\sigma}^{h\eta} - \sigma\right\|^2_{0,r,\hat{D}}\right) \to 0$$

using Proposition 2.1. Q.E.D.

We define the *Approximate Optimal Design Problem*: given h, η, find $\alpha_{h\eta} \in U^h_{\mathrm{ad}}$, such that

(41) $$\alpha_{h\eta} = \underset{\beta_h \in U_{\mathrm{ad}}{}^h}{\operatorname{argmin}} \mathscr{J}(\beta_h, \sigma^{h\eta}(\beta_h)) .$$

Theorem 1. *There exists a function $h_2(\eta)$ such that the problem* (41) *has a solution for any $\eta > 0$, $h < h_2(\eta)$.*

Proof. First we establish two auxiliary lemmas.

Lemma 2.4. *Let h and η be fixed, $\beta_n \in U^h_{\mathrm{ad}}$ and $\lim_{n \to \infty} \beta_n = \alpha$ in $C([0, 1])$.*

Denote by $\tilde{\sigma}(\beta_n)$ the solution $\tilde{\sigma}^{h\eta}(\beta_n)$ of the variational inequality (6) *on $D_h \equiv D(\beta_n)$, extended by zero to $\hat{D} \dot{-} D(\beta_n)$.*

Then a (positive) function $h_2(\eta)$ exists such that

$$\tilde{\sigma}(\beta_n) \to \tilde{\sigma}(\alpha) \quad in \quad S(\hat{D}) \quad for \quad n \to \infty$$

if $h < h_2(\eta)$.

Proof. 1° Following the argument of Proposition 2.1 and using the mapping r^n_h defined on $D(\beta_n)$, we can show that

$$\left\|\sigma(\beta_n)\right\|_{0,r,D(\beta_n)} \leqq C \quad \forall n .$$

Let us define $\hat{\sigma}(\beta_n)$ as the vector of coefficients in the formula

$$\sigma(\beta_n) = \sum_{i=1}^{M} \hat{\sigma}_i(\beta_n)\, \vartheta_i(\beta_n) ,$$

where $\vartheta_i(\beta_n)$ are the basis functions of the space $H_h(D(\beta_n))$.

One can show that positive n_0 and $C_0 = \mathrm{const}$ exist such that

(42) $$\left\|\sigma\right\|_{0,r,D(\beta_n)} \geqq C_0 \left\|\hat{\sigma}\right\|_{\mathbf{R}^M}$$

holds for all $n > n_0$ and all $\sigma \in H_h(D(\beta_n))$.

Consequently, a subsequence of $\{\sigma(\beta_n)\}$ (which will be denoted by the same symbol) and $\sigma \in S(\hat{D})$ exist such that

(43) $$\tilde{\sigma}(\beta_n) \to \text{(weakly) in } S(\hat{D}) .$$

Making use of (42), we prove that

$$\hat{\sigma}(\beta_n) \to \hat{\sigma} \quad \text{in} \quad \mathbb{R}^M ,$$

where $\hat{\sigma}$ is the vector of coefficients of σ on $D(\alpha)$, $\sigma = 0$ in $D \dot{-} D(\alpha)$ and the convergence (43) is even strong.

2° Next we realize that

$$\sigma \in E_h(D(\alpha)) \Leftrightarrow A(\alpha)\, \hat{\sigma} = \mathscr{F}(\alpha) ,$$

where the matrix $A(\alpha)$ has the entries

$$\langle \vartheta_k^{lm}(T_j), \mathbf{e}(\mathbf{v}_i) \rangle_{T_j(\alpha)} .$$

Here ϑ_k^{lm} are the basis functions (barycentrix coordinates) of $H_h(D(\alpha))$, $T_j(\alpha) \in \in \mathscr{T}_h(D(\alpha))$ the triangles and $\mathbf{v}_i$ the basis functions of the space $V_h(D(\alpha))$.

Since

$$\sigma(\beta_n) \in E_h(D(\beta_n)) \Rightarrow A(\beta_n)\, \hat{\sigma}(\beta_n) = \mathscr{F}(\beta_n) ,$$

passing to the limit with $n \to \infty$, we obtain that

$$A(\alpha)\, \hat{\sigma} = \mathscr{F}(\alpha) , \quad \text{i.e.} \quad \sigma \in E_h(D(\alpha)) , \tag{44}$$

since the matrices $A(\beta_n)$ and $\mathscr{F}(\beta_n)$ depend continuously on β_n.

3° Next we show that

$$\sigma \in P_{\eta h}(D(\alpha)) . \tag{45}$$

In fact, the functional

$$j\colon \tau \to \int_{\hat{D}} [f(\tau) - 1]^+ \, r \, \mathrm{d}r \, \mathrm{d}z$$

in continuous on $S(\hat{D})$ (cf. the proof of Propos. 2.1 — 4°). Consequently, we have

$$j(\sigma) = \lim_{n \to \infty} j(\tilde{\sigma}(\beta_n)) \leqq \eta .$$

4° Let us show that $\sigma|_{D(\alpha)}$ is a solution of the variational inequality

$$(\sigma, \tau - \sigma)_{D(\alpha)} \geqq 0 \quad \forall \tau \in E_h(D(\alpha) \cap P_{\eta h}(D(\alpha)) .$$

Let a test function τ be given. We can write

$$\tau = \sum_{i=1}^{M} \hat{\tau}_{\alpha i}\, \vartheta_i(\alpha) ,$$

$$A(\alpha)\, \hat{\tau}_\alpha = \mathscr{F}(\alpha) .$$

One can prove that the matrix $A(\alpha)$ has a full rang

$$n_A = \dim V_h(D(\alpha)) .$$

(It is easy to realize that $n_A < \dim H_h/3$). Thus a suitable renumbering leads to the equation

$$A_1(\alpha)\, \hat{\tau}_\alpha^1 + A_2(\alpha)\, \hat{\tau}_\alpha^2 = \mathscr{F}(\alpha) ,$$

where $A_1(\alpha)$ is regular, so that

$$\hat{\tau}_\alpha^1 = A_1^{-1}(\alpha)\,\mathscr{F}(\alpha) - A_1^{-1}(\alpha)\,A_2(\alpha)\,\hat{\tau}_\alpha^2\,.$$

Let us define

$$\hat{\tau}_n^1(\hat{\tau}^2) = A_1^{-1}(\beta_n)\,(\mathscr{F}(\beta_n) - A_2(\beta_n)\,\hat{\tau}^2)\,,$$

$$\hat{f}(\beta_n, \hat{\tau}^2) \equiv f\Big(\sum_{i=1}^{n_A} \hat{\tau}_{ni}^1(\hat{\tau}^2)\vartheta_i(\beta_n) + \sum_{i>n_A}^{M} \hat{\tau}_i^2\vartheta_i(\beta_n)\Big)\,,$$

$$j(\beta_n, \hat{\tau}^2) = \int_{D(\beta_n)} [\hat{f}(\beta_n, \hat{\tau}^2) - 1]^+\, r\,\mathrm{d}r\,\mathrm{d}z\,.$$

Obviously, we have

$$\tau \equiv \sum_{i \leqq n_A} \hat{\tau}_{ni}^1(\hat{\tau}^2)\,\vartheta_i(\beta_n) + \sum_{i>n_A}^{M} \hat{\tau}_i^2\,\vartheta_i(\beta_n) \in E_h(D(\beta_n))$$

and

$$j(\beta_n, \hat{\tau}^2) = \int_{D(\beta_n)} [f(\tau) - 1]^+\, r\,\mathrm{d}r\,\mathrm{d}z\,.$$

Let us define $m = M - n_A$ and the set

$$P_n^2 = \{\hat{\tau}^2 \in \mathbb{R}^m \mid j(\beta_n, \hat{\tau}^2) \leqq \eta\}\,.$$

The latter set is convex and closed in $\mathbb{R}^m$ for any β_n. Indeed, the convexity follows from the convexity of the functions $f(\beta_n, \cdot)$ and $j(\beta_n, \cdot)$, the closedness from their continuity.

Consequently, we may define the projection

$$\pi_n\colon \mathbb{R}^m \to P_n^2 \quad \text{on the set} \quad P_n^2 \quad \text{in} \quad \mathbb{R}^m\,.$$

Let us consider the vectors

$$\hat{\tau}_n = [\hat{\tau}_\alpha^1(\pi_n\hat{\tau}_n^2), \pi_n\hat{\tau}_\alpha^2]\,, \quad n = 1, 2, \ldots$$

and the functions

$$\tau(\beta_n) = \sum_{i=1}^{M} \hat{\tau}_{ni}\,\vartheta_i(\beta_n)\,.$$

It is readily seen that

$$\tau(\beta_n) \in E_h(D(\beta_n)) \cap P_{\eta h}(D(\beta_n))$$

so that we may insert it into the inequality (6).

Defining also the extensions of $\tau(\beta_n)$ and τ as follows

$$\hat{\tau}(\beta_n) = 0 \quad \text{in} \quad \hat{D} \dot{-} D(\beta_n)\,, \qquad \tilde{\tau} = 0 \quad \text{in} \quad \hat{D} \dot{-} D(\alpha)\,,$$

we obtain

(46) $$(\tilde{\sigma}(\beta_n), \tilde{\tau}(\beta_n))_{\hat{D}} \geqq \|\tilde{\sigma}(\beta_n)\|_{\hat{D}}^2\,.$$

Next we can show that

(47) $$\tilde{\tau}(\beta_n) \to \tilde{\tau} \quad \text{in} \quad S(\hat{D})\,.$$

In fact, we may write

$$(48) \qquad \|\tilde{\tau}(\beta_n) - \tilde{\tau}\|^2_{0,r,\hat{D}} \leqq 2 \int_{\hat{D}} \left[\sum_{i=1}^{M} (\hat{\tau}_{ni} - \hat{\tau}_{\alpha i})\, \vartheta_i(\beta_n)\right]^2 r \, dr \, dz \; +$$

$$+ \; 2 \int_{\hat{D}} \left[\sum_{i=1}^{M} \hat{\tau}_{\alpha i}(\vartheta_i(\beta_n) - \vartheta_i(\alpha))\right]^2 r \, dr \, dz = J_{1n} + J_{2n} \, .$$

Using some results of Pironneau [9], we obtain

$$(49) \qquad J_{2n} \leqq 2\|\hat{\tau}_\alpha\|^2_{\mathbf{R}^M} \int_{\hat{D}} \sum_{i=1}^{M} (\vartheta_i(\beta_n) - \vartheta_i(\alpha))^2 \, r \, dr \, dz \to 0 \, .$$

Moreover,

$$(50) \qquad J_{1n} \leqq 2\|\hat{\tau}_n - \hat{\tau}_\alpha\|^2_{\mathbf{R}^M} \int_{\hat{D}} \sum_{i=1}^{M} \vartheta_i^2(\beta_n) \, r \, dr \, dz \leqq C\|\hat{\tau}_n - \hat{\tau}_\alpha\|^2_{\mathbf{R}^M} =$$

$$= C\left(\|\hat{\tau}^1_n(\pi_n \hat{\tau}^2_\alpha) - \hat{\tau}^1_\alpha\|^2_{\mathbf{R}^{n_A}} + \|\pi_n \hat{\tau}^2_\alpha - \hat{\tau}^2_\alpha\|^2_{\mathbf{R}^m} = C(K^2_{1n} + K^2_{2n}\right) .$$

By definition of π_n we have

$$(51) \qquad K_{2n} \leqq \|y_n - \hat{\tau}^2_\alpha\|_{\mathbf{R}^m} \quad \forall y_n \in P^2_n \, .$$

Let us construct a suitable sequence $\{y_n\}$. From the proof of Lemma 2.1 we conclude that

$$(52) \qquad h < h_0(\eta\Theta, \sigma^0) \Rightarrow \int_{D_h(\beta_n)} [f(r^n_h \sigma^0) - 1]^+ \, r \, dr \, dz \leqq \eta\Theta \, ,$$

where $\Theta \in (0, 1)$ is an arbitrary parameter and

$$r^n_h \, \sigma^0(\beta_n) = \sum_{i=1}^{M} \varrho_{ni} \, \vartheta_i(\beta_n)$$

is the projection corresponding with the triangulation $\mathscr{T}_h(\beta_n)$.

On the basis of (42) and the assumptions (A 1), (A 3), (A 4), we obtain for $\varrho_n = [\varrho^1_n, \varrho^2_n]$

$$(53) \qquad C_0\|\varrho^2_n\|_{\mathbf{R}^m} \leqq C_0\|\varrho_n\|_{\mathbf{R}^M} \leqq \|r^n_h \, \sigma^0(\beta_n)\|_{0,r,D(\beta_n)} \leqq \tilde{C} \, .$$

Moreover,

$$r^n_h \, \sigma^0(\beta_n) \in E_h(D(\beta_n)) \cap P_{\eta h}(D(\beta_n))$$

(cf. the proof of Lemma 2.3), so that (52) implies

$$j(\beta_n, \varrho^2_n) \leqq \eta\Theta \, .$$

Since the function $\beta \mapsto j(\beta, \hat{\tau}^2_\alpha)$ is continuous, there exists a sequence $\{\delta_n\}$, $\delta_n > 0$, $\delta_n \to 0$, such that

$$j(\beta_n, \hat{\tau}^2_\alpha) \leqq j(\alpha, \hat{\tau}^2_\alpha) + \delta_n \leqq \eta + \delta_n \, .$$

Let us define

$$(54) \qquad t_n = \delta_n(\delta_n + \eta - \eta\Theta)^{-1} \, ,$$

$$y_n = \hat{\tau}^2_\alpha + t_n(\varrho^2_n - \hat{\tau}^2_\alpha) \, .$$

From the convexity of the function $y \mapsto j(\beta_n, y)$, we deduce that

$$j(\beta_n, y_n) \leqq t_n j(\beta_n, \varrho_n^2) + (1 - t_n) j(\beta_n, \hat{\tau}_\alpha^2) \leqq$$
$$\leqq t_n \eta \Theta + (1 - t_n)(\eta + \delta_n) = \eta$$

and consequently, $y_n \in P_n^2$.

We may write, using (51) and (53), (54),

$$K_{2n} \leqq \|t_n(\varrho_n^2 - \hat{\tau}_\alpha^2)\|_{\mathbf{R}^m} \to 0\,. \tag{55}$$

Furthermore, we have

$$\begin{aligned} K_{1n} = \; & \|A_1^{-1}(\beta_n)\,\mathscr{F}(\beta_n) - A_1^{-1}(\alpha)\,\mathscr{F}(\alpha) - \\ & - A_1^{-1}(\beta_n)\,A_2(\beta_n)\,(\pi_n\hat{\tau}_\alpha^2) + A_1^{-1}(\alpha)\,A_2(\alpha)\,\hat{\tau}_\alpha^2\|_{\mathbf{R}^{n_A}} \leqq \\ & \leqq \|A_1^{-1}(\beta_n)\,\mathscr{F}(\beta_n) - A_1^{-1}(\alpha)\,\mathscr{F}(\alpha)\|_{\mathbf{R}^{n_A}} + \\ & + \|A_1^{-1}(\beta_n)\,A_2(\beta_n)\|_* \,\|\pi_n\hat{\tau}_\alpha^2 - \hat{\tau}_\alpha^2\|_{\mathbf{R}^m} + \\ & + \|A_1^{-1}(\beta_n)\,A_2(\beta_n) - A_1^{-1}(\alpha)\,A_2(\alpha)\|_* \,\|\hat{\tau}_\alpha^2\|_{\mathbf{R}^m} \to 0\,, \end{aligned} \tag{56}$$

using (55) and the continuous dependence of the matrices A_1^{-1}, A_2 and $\mathscr{F}$ on the variable β.

Combining (50), (55) and (56), we arrive at

$$\lim_{n\to\infty} J_{1n} = 0\,.$$

Inserting this result and (49) into (48), we obtain (47). Using the strong convergence (43) and (47) in (46) leads to

$$(\sigma, \tilde{\tau})_{\hat{D}} \geqq \|\sigma\|_{\hat{D}}^2\,,$$

which can be rewritten as follows

$$(\sigma, \tau - \sigma)_{D(\alpha)} \geqq 0\,.$$

Since (44) and (45) hold for σ and the solution of the variational inequality (6) is unique (cf. Lemma 2.3 and its proof), if $h < h_2(\eta) = h_0(\eta\Theta, \sigma^0) \leqq h_1(\eta) = h_0(\eta, \sigma^0)$, we conclude that

$$\sigma = \sigma(\alpha)$$

and the whole sequence $\{\tilde{\sigma}(\beta_n)\}$ tends to $\tilde{\sigma}(\alpha)$ in $S(\hat{D})$.

Lemma 2.5. *Let the assumptions of Lemma* 2.4 *be fulfilled and* $h < h_2(\eta)$. *Then*

$$\lim_{n\to\infty} \mathscr{J}(\beta_n, \sigma(\beta_n)) = \mathscr{J}(\alpha, \sigma(\alpha))\,.$$

Proof is analogous to that of Proposition 2.2. Since $f(0) = 0$, we have

$$\mathscr{I}(\beta_n, \sigma(\beta_n)) = \int_{\hat{D}} f^2(\tilde{\sigma}(\beta_n))\, r\, \mathrm{d}r\, \mathrm{d}z\,,$$

$$|\mathscr{I}(\beta_n, \sigma(\beta_n)) - \mathscr{I}(\alpha, \sigma(\alpha))| \leqq$$

$$\leqq C[\|\tilde{\sigma}(\beta_n) - \tilde{\sigma}(\alpha)\|_{0,r,\hat{D}} + \|\tilde{\sigma}(\beta_n) - \tilde{\sigma}(\alpha)\|_{0,r,\hat{D}}^2] \to 0$$

by virtue of Lemma 2.4.

Proof of Theorem 1. Let us denote by

$$\boldsymbol{a} = \{\alpha(0), \alpha(h), \ldots, \alpha(1)\}$$

the vector of nodal values of the function $\alpha \in U_{\mathrm{ad}}^h$. Then

$$\alpha \in U_{\mathrm{ad}}^h \Leftrightarrow \boldsymbol{a} \in \mathscr{A}\,,$$

where $\mathscr{A}$ is a compact subset of $\mathbb{R}^{N+1}$.
By Lemma 2.5, the function

$$j_0(\boldsymbol{a}) \equiv \mathscr{I}(\alpha, \sigma(\alpha))$$

is continuous in $\mathscr{A}$.
Consequently, j_0 attains its minimum in the set $\mathscr{A}$.

Theorem 2. *Let $\{\alpha_{h\eta}\}$, $h \to 0$, $\eta \to 0$, $h < h_2(\eta)$, be a sequence of solutions of the Approximate Optimal Design Problems* (41).
Then a subsequence $\{\alpha_{\hat{h}\hat{\eta}}\}$ exists such that

$$\alpha_{\hat{h}\hat{\eta}} \to \alpha \quad in \quad C([0, 1])\,, \tag{57}$$

$$\tilde{\sigma}^{\hat{h}\hat{\eta}}(\alpha_{\hat{h}\hat{\eta}}) \to \sigma(\alpha) \quad in \quad [L_r^2(\hat{D})]^4\,, \tag{58}$$

where α is a solution of the Optimal Design Problem (4), *$\tilde{\sigma}^{\hat{h}\hat{\eta}}(\alpha_{\hat{h}\hat{\eta}})$ is the solution of the approximate problem* (6), *extended by zero to $\hat{D} \dot{-} D(\alpha_{\hat{h}\hat{\eta}})$ and $\sigma(\alpha)$ is the solution of the problem* (3), *extended by zero to $\hat{D} \dot{-} D(\alpha)$.*
Any uniformly convergent subsequence of $\{\alpha_{h\eta}\}$ tends to a solution of the problem (4) *and an analogue of* (58) *holds.*

Proof. Let us consider a $\beta \in U_{\mathrm{ad}}$. There exists a sequence $\{\beta_h\}$, $h \to 0$, $\beta_h \in U_{\mathrm{ad}}^h$, such that $\beta_h \to \beta$ in $C([0, 1])$ (for the proof – see [7] – Lemma 3.1).
A subsequence $\{\alpha_{\hat{h}\hat{\eta}}\} \subset \{\alpha_{h\eta}\}$ exists such that (57) holds and $\alpha \in U_{\mathrm{ad}}$ (see [7] – Lemma 3.2). We have

$$\mathscr{I}(\alpha_{\hat{h}\hat{\eta}}, \sigma^{\hat{h}\hat{\eta}}(\alpha_{\hat{h}\hat{\eta}})) \leqq \mathscr{I}(\beta_{\hat{h}}, \sigma^{\hat{h}\hat{\eta}}(\beta_{\hat{h}}))$$

by virtue of (41).

Applying Proposition 2.2 on both sides, we are led to the inequality

$$\mathscr{J}(\alpha, \sigma(\alpha)) \leqq \mathscr{J}(\beta, \sigma(\beta)) .$$

Consequently, α is a solution of the problem (4). The convergence (58) follows from Proposition 2.1. The rest of the assertion is obvious.

Corollary. *There exists at least one solution of the Optimal Design Problem* (4).

Proof is an immediate consequence of Theorems 1 and 2.

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Souhrn

OPTIMALIZACE TVARU OSOVĚ SYMETRICKÝCH PRUŽNĚ PLASTICKÝCH TĚLES

IVAN HLAVÁČEK

Uvažuje se pružně plastické těleso, jehož stav napjatosti se řídí Henckyovým zákonem. K řešení stavové úlohy se používá princip Haara-Kármána a po částech lineární aproximace napětí. Tvar meridiánového řezu je optimalizován na základě integrálního kritéria. Dokazuje se konvergence přibližných řešení a existence optimálního meridiánového řezu.

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