Applications of Mathematics

Ivan Hlaváček

Dual finite element analysis of axisymmetric elliptic problems with an absolute term

Applications of Mathematics, Vol. 36 (1991), No. 5, 392–406

Persistent URL: http://dml.cz/dmlcz/104475

DUAL FINITE ELEMENT ANALYSIS OF AXISYMMETRIC ELLIPTIC PROBLEMS WITH AN ABSOLUTE TERM

IVAN HLAVÁČEK

(Received April 23, 1990)

Summary: A model second order elliptic equation in cylindrical coordinates with mixed boundary conditions is considered. A dual variational formulation is employed to calculate the cogradient of the solution directly. Approximations are defined on the basis of standard finite elements spaces. Convergence analysis and some a posteriori error estimates are presented.

Keywords: Finite elements, elliptic problems, dual analysis, axisymmetric problem.

AMS Subject classification: 65N30, 65N15.

INTRODUCTION

We consider a second order elliptic problem in an axisymmetric bounded domain, supposing all data are axisymmetric, the boundary conditions are of a mixed type and the equation contains an absolute term. Extending the ideas of the paper [1] (cf. also [2], [3]), we introduce a dual variational formulation of the problem, which enables us to calculate the cogradient of the solution directly. Finite element approximations to the solution of the dual problem are defined and an error analysis presented, within the framework of the theory of weighted Sobolev spaces (cf. [4] for the primal variational solution).

The equation under consideration occurs e.g. in the theory of a vector potential distribution for a transformer with magnetic shielding (see [6]).

We shall consider a domain $\Omega \subset \mathbb{R}^3$, which is generated by the rotation of a bounded domain

$$D \subset \{(r, z) \in \mathbb{R}^2 \mid r \geqq 0\}$$

about the axis $x_3 \equiv z$. Assume that Ω has a Lipschitz boundary. Hence, the set $\partial D \cap \mathcal{O}$, where $\mathcal{O}$ is the z-axis, cannot contain isolated points. Let Γ_0 denote the interior of the set $\partial D \cap \mathcal{O}$. We assume

$$\partial D = \Gamma_0 \cup \bar{\Gamma}_u \cup \bar{\Gamma}_g ,$$

where Γ_0, Γ_u, Γ_g have at most a finite number of components, being mutually disjoint and open in the boundary ∂D and $\Gamma_u \neq \emptyset$.

Let k be a non-negative integer. By $W_r^{k,2}(D)$ we shall denote the weighted Sobolev space with the weight r, the norm

$$\|u\|_{k,r,D} = \Big(\sum_{|\alpha| \leq k} \int_D |D^\alpha u|^2 \, r \, dr \, dz\Big)^{1/2}$$

and the seminorm

$$|u|_{k,r,D} = \Big(\sum_{|\alpha| = k} \int_D |D^\alpha u|^2 \, r \, dr \, dz\Big)^{1/2} .$$

If $k = 0$, we denote $W_r^{0,2}(D) \equiv L_r^2(D)$.

By $L^2_{1/r}(D)$ we shall denote the weighted space of functions with the norm

$$\|u\|_{0,1/r,D} = \Big(\int_D u^2 r^{-1} \, dr \, dz\Big)^{1/2} .$$

Let $\Gamma \subset \partial D \dot{-} \Gamma_0$ be a measurable part of the boundary. By $L_r^2(\Gamma)$ we shall denote the space of functions with the norm

$$\|u\|_{0,r,\Gamma} = \Big(\int_\Gamma u^2 r \, ds\Big)^{1/2} .$$

There exists a linear continuous mapping

$$\gamma \colon W_r^{1,2}(D) \to L_r^2(\Gamma)$$

such that $\gamma u = u|_\Gamma$ for any $u \in C^1(\bar{D})$. (For the proof see e.g. [5] – Section 1. It follows easily from the Trace Theorem in the standard Sobolev space $H^1(\Omega)$.)

Let us introduce the set

$$C^\infty_{0,r}(D) = \{v \in C^\infty(\bar{D}) \mid \operatorname{supp} v \cap (\partial D - \Gamma_0) = \emptyset\} .$$

We say that the divergence of a vector-function $(q_r, q_z) \equiv \boldsymbol{q} \in [L_r^2(D)]^2$ exists and belongs to $L_r^2(D)$, if there exists $f \in L_r^2(D)$ such that

$$\int_D \left(q_r \frac{\partial v}{\partial r} + q_z \frac{\partial v}{\partial z}\right) r \, dr \, dz = - \int_D f v r \, dr \, dz \quad \forall v \in C^\infty_{0,r}(D) .$$

Then we set $\operatorname{div} \boldsymbol{q} = f$; for $\boldsymbol{q}$ smooth, we have

$$\operatorname{div} \boldsymbol{q} = \frac{1}{r} q_r + \frac{\partial}{\partial r} q_r + \frac{\partial}{\partial z} q_z .$$

We introduce the subspace

$$H_r(\operatorname{div}, D) = \{\boldsymbol{q} \in [L_r^2(D)]^2 \mid \operatorname{div} \boldsymbol{q} \in L_r^2(D)\}$$

with the following norm

$$\|\boldsymbol{p}\|_{\operatorname{div},D} = \big(\|p_r\|^2_{0,r,D} + \|p_z\|^2_{0,r,D} + \|\operatorname{div} \boldsymbol{p}\|^2_{0,r,D}\big)^{1/2} .$$

Next let us denote $\Gamma = \partial D \dot{-} \Gamma_0$ and define the following subspace of $L^2_r(\Gamma)$:

$$H^{1/2}(\Gamma) = \gamma(W^{1,2}_r(D)) .$$

In $H^{1/2}(\Gamma)$ we shall use the norm

$$\|w\|_{1/2,r,\Gamma} = \inf_{v \in W_r^{1,2}(D), \gamma v = w} \|v\|_{1,r,D} .$$

The intersection

$$W^{1,2}_r(D) \cap L^2_{1/r}(D)$$

will be denoted by $X_1(D)$ and equipped with the following norm

$$\|u\|_{X_1(D)} = (|u|^2_{1,r,D} + \|u\|^2_{0,1/r,D})^{1/2} .$$

1. MODEL PROBLEM

As a model problem, we shall consider the following boundary value problem

$$\text{(1.1)} \qquad -\left[\frac{1}{r} a_r \frac{\partial u}{\partial r} + \frac{\partial}{\partial r}\left(a_r \frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(a_z \frac{\partial u}{\partial z}\right)\right] + a_0 u = f \quad \text{in} \quad D ,$$

$$\text{(1.2)} \qquad u = u_0 \quad \text{on} \quad \Gamma_u ,$$

$$\text{(1.3)} \qquad \nu \,.\, \text{cograd}\, u = g \quad \text{on} \quad \Gamma_g ,$$

where $\nu = (\nu_r, \nu_z)$ denotes the unit outward normal with respect to Γ_g,

$$\text{cograd}\, u = \left(a_r \frac{\partial u}{\partial r}, a_z \frac{\partial u}{\partial z}\right),$$

$a_0, a_r, a_z \in L^\infty(D)$ are coefficients such that there exists a positive constant c such that

$$\text{(1.4)} \qquad a_0 \geqq c , \quad a_r \geqq c , \quad a_z \geqq c \quad \text{a.e. in } D ,$$

$$f \in L^2_r(D) , \quad u_0 \in W^{1,2}_r(D) , \quad g \in L^2_r(\Gamma_g)$$

are given functions.

Let us introduce the following forms

$$a(u, v) = \int_D \left(a_r \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + a_z \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + a_0 uv\right) r \,\mathrm{d}r \,\mathrm{d}z ,$$

$$L(v) = \int_D fvr \,\mathrm{d}r \,\mathrm{d}z + \int_{\Gamma_g} gvr \,\mathrm{d}\Gamma .$$

We say that $u \in W^{1,2}_r(D)$ is a *weak solution* of the primal problem if $u - u_0 \in V$ and

$$\text{(1.5)} \qquad a(u, v) = L(v) \quad \forall v \in V ,$$

where

$$V = \{u \in W^{1,2}_r(D) \mid \gamma v = 0 \text{ on } \Gamma_u\} .$$

There exists a unique weak solution. This follows from (1.4) and the Riesz representation theorem, since L is a continuous linear functional on V.

In order to derive a dual variational formulation of the problem (1.1)–(1.3), we employ the method of orthogonal projections in the space

$$\mathscr{H} = [L_r^2(D)]^3 .$$

Let $\boldsymbol{q} = (q_r, q_z, q_0)$ denote elements of the space $\mathscr{H}$. We introduce the following bilinear form on $\mathscr{H} \times \mathscr{H}$:

$$(\boldsymbol{q}, \boldsymbol{p})_{\mathscr{H}} = \int_D (a_r^{-1} p_r q_r + a_z^{-1} p_z q_z + a_0^{-1} p_0 q_0)\, r \,\mathrm{d}r \,\mathrm{d}z . \tag{2.1}$$

It is easy to see that the associated norm

$$\|\boldsymbol{q}\|_{\mathscr{H}} = (\boldsymbol{q}, \boldsymbol{q})_{\mathscr{H}}^{1/2}$$

is equivalent with the standard norm

$$\|\boldsymbol{q}\|_{0,r,D} = \left(\int_D (q_r^2 + q_z^2 + q_0^2)\, r \,\mathrm{d}r \,\mathrm{d}z\right)^{1/2} .$$

Thus $\mathscr{H}$ equipped with the scalar product (2.1) is a Hilbert space.

Let us define the following subsets of $\mathscr{H}$:

$$\mathscr{H}_1 = \{\boldsymbol{q} \in \mathscr{H} \mid \exists v \in V: \boldsymbol{q} = (\operatorname{cograd} v, a_0 v)\}$$

(and we shall write $\boldsymbol{q} = \boldsymbol{q}(v)$ whenever $\boldsymbol{q}$ is constructed according to the definition of $\mathscr{H}_1$),

$$\mathscr{H}_2 = \{\boldsymbol{q} \in \mathscr{H} \mid B(\boldsymbol{q}, v) = 0 \quad \forall v \in V\} ,$$

where

$$B(\boldsymbol{q}, v) = \int_D \left(q_r \frac{\partial v}{\partial r} + q_z \frac{\partial v}{\partial z} + q_0 v\right) r \,\mathrm{d}r \,\mathrm{d}z . \tag{2.2}$$

Lemma 2.1.

$$\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2 .$$

Proof. Let us consider a sequence

$$\{\boldsymbol{q}^m\}_{m=1}^{\infty} \subset \mathscr{H}_1 , \quad \boldsymbol{q}^m \to \boldsymbol{q} \quad \text{in } \mathscr{H} \quad \text{as} \quad m \to \infty .$$

By definition, there exist $v_m \in V$ such that $\boldsymbol{q}^m = \boldsymbol{q}(v_m)$ and

$$\|\boldsymbol{q}^m - \boldsymbol{q}^n\|_{\mathscr{H}}^2 = \int_D [a_r^{-1}(q_r^m - q_r^n)^2 + \ldots \quad]\, r \,\mathrm{d}r \,\mathrm{d}z =$$

$$= \int_D \left[a_r \left(\frac{\partial}{\partial r}(v_m - v_n)\right)^2 + a_z \left(\frac{\partial}{\partial z}(v_m - v_n)\right)^2 + \right.$$

$$\left. + a_0 (v_m - v_n)^2\right] r \,\mathrm{d}r \,\mathrm{d}z \geqq c \|v_m - v_n\|_{1,r,D}^2 .$$

Since V is a Hilbert space, there exists a limit $v \in V$ such that $v_m \to v$ in V. We easily find that

$$\|\boldsymbol{q}^m - \boldsymbol{q}(v)\|_{\mathscr{H}}^2 \leqq C\|v_m - v\|_{1,r,D}^2 \to 0\,,$$

which yields $\boldsymbol{q} = \boldsymbol{q}(v)$.

Hence $\mathscr{H}_1$ is a closed subspace. Since the form $\boldsymbol{q} \to B(\boldsymbol{q}, v)$ is continuous in $\mathscr{H}$, $\mathscr{H}_2$ is a closed subspace, as well.

Let $\boldsymbol{q} \in \mathscr{H}_1$ and $\boldsymbol{p} \in \mathscr{H}_2$. Then $\boldsymbol{q} = \boldsymbol{q}(v)$ for some $v \in V$ and we have

$$(2.3) \qquad (\boldsymbol{q}, \boldsymbol{p})_{\mathscr{H}} = \int_D \left(\frac{\partial v}{\partial r} p_r + \frac{\partial v}{\partial z} p_z + v p_0\right) r \,\mathrm{d}r \,\mathrm{d}z = B(p, v) = 0\,,$$

i.e., $\mathscr{H}_1$ is orthogonal to $\mathscr{H}_2$.

Denoting by $\mathscr{H}_1^\perp$ the orthocomplement to $\mathscr{H}_1$ in the space $\mathscr{H}$, we see by (2.3) that $\mathscr{H}_2 \subset \mathscr{H}_1^\perp$. Let $\mathbf{t} \in \mathscr{H}_1^\perp$ and $v \in V$. Then we have

$$0 = \int_D \left(a_r^{-1} t_r a_r \frac{\partial v}{\partial r} + a_z^{-1} t_z a_z \frac{\partial v}{\partial z} + a_0^{-1} t_0 a_0 v\right) r \,\mathrm{d}r \,\mathrm{d}z = B(\mathbf{t}, v)\,,$$

which implies $\mathbf{t} \in \mathscr{H}_2$ and $\mathscr{H}_2 = \mathscr{H}_1^\perp$. Q.E.D.

Let us define the following set

$$Q_{fg} = \{\boldsymbol{q} \in \mathscr{H} \mid B(\boldsymbol{q}, v) = L(v) \quad \forall v \in V\}\,.$$

Theorem 2.1. *Principle of minimium complementary energy.*

Let u be the weak solution of the primal problem. Then

$$(2.4) \qquad \boldsymbol{q}^0 = \operatorname*{argmin}_{\boldsymbol{q} \in Q_{fg}} \mathscr{S}(\boldsymbol{q})\,,$$

where

$$\mathscr{S}(\boldsymbol{q}) = \tfrac{1}{2}\|\boldsymbol{q} - \boldsymbol{q}(u_0)\|_{\mathscr{H}}^2$$

if and only if $\boldsymbol{q}^0 = \boldsymbol{q}(u)$.

Proof. Let us write $u = u_0 + w$, $w \in V$, For $\boldsymbol{q} \in Q_{fg}$ let us define the functional

$$I(\boldsymbol{q}) = \|\boldsymbol{q} - \boldsymbol{q}(u_0)\|_{\mathscr{H}}^2 = \|\boldsymbol{q} - \boldsymbol{q}(u) + \boldsymbol{q}(w)\|_{\mathscr{H}}^2 =$$
$$= \|\boldsymbol{q} - \boldsymbol{q}(u)\|_{\mathscr{H}}^2 + \|\boldsymbol{q}(w)\|_{\mathscr{H}}^2\,.$$

Here the orthogonality of $\mathscr{H}_1$ and $\mathscr{H}_2$ has been employed. Indeed,

$$B(\boldsymbol{q} - \boldsymbol{q}(u), v) = 0 \quad \forall v \in V$$

holds due to the definition of Q_{fg} and

$$B(\boldsymbol{q}(u), v) = a(u, v) = L(v) \quad \forall v \in V\,.$$

Consequently, $\boldsymbol{q} - \boldsymbol{q}(u) \in \mathscr{H}_2$ and $\boldsymbol{q}(u) \in Q_{fg}$ by definition. Obviously, the functional I attains its minimum over Q_{fg} at the point $\boldsymbol{q}^0$ iff $\boldsymbol{q}^0 = \boldsymbol{q}(u)$.

Remark 2.1. If the boundary condition on Γ_u is homogeneous, we set $u_0 \equiv 0$ and then $\mathscr{S}(\boldsymbol{q}) = \frac{1}{2}\|\boldsymbol{q}\|^2_{\mathscr{H}}$.

Since the primal problem has a unique solution, the dual problem (2.4) has a unique solution, as well.

Simplification of the set Q_{fg}

Let us denote $\boldsymbol{q}^* = (q_r, q_z)$ the "reduced" vector associated with $\boldsymbol{q} = (q_r, q_z, q_0)$. From the definition we deduce

$$(2.5) \qquad \boldsymbol{q} \in Q_{fg} \Rightarrow \boldsymbol{q}^* \in H_r(\mathrm{div}, D),\ \mathrm{div}\, \boldsymbol{q}^* = -f + q_0 .$$

Let us denote $H^{-1/2}(\Gamma) \equiv [\gamma(W_r^{1,2}(D))]'$ (i.e., the dual space).

For $\boldsymbol{q}^* \in H_r(\mathrm{div}, D)$, we define a functional $\boldsymbol{q}^* . \nu \in H^{-1/2}(\Gamma)$ by means of the following formula

$$(2.6) \qquad \langle \boldsymbol{q}^* . \nu, w \rangle = \int_D \left(q_r \frac{\partial v}{\partial r} + q_z \frac{\partial v}{\partial z} + v\, \mathrm{div}\, \boldsymbol{q}^* \right) r\, \mathrm{d}r\, \mathrm{d}z\ ,\ w \in \gamma(W_r^{1,2}(D))\, ,$$

where $v \in W_r^{1,2}(D)$ is any extension of the function w such that $\gamma v = w$ on Γ.

Note that the integral does not depend on the kind of extension. In fact, for the difference $\omega \equiv v' - v''$ of any two extensions $\gamma\omega = 0$ holds and since

$$\int_D \left(q_r \frac{\partial \varphi}{\partial r} + q_z \frac{\partial \varphi}{\partial z} + \varphi\, \mathrm{div}\, \boldsymbol{q}^* \right) r\, \mathrm{d}r\, \mathrm{d}z = 0 \quad \forall \varphi \in C^\infty_{0,r}(D)$$

(cf. the definition of the divergence),

$$\langle \boldsymbol{q}^* . \nu, \gamma\omega \rangle = 0$$

follows, using the density of the set $C^\infty_{0,r}(D)$ in the subspace

$$W^{1,2}_{0,r}(D) = \{ u \in W_r^{1,2}(D) \mid \gamma u = 0 \} .$$

Moreover, there exists an extension $\mathscr{E} w \in W_r^{1,2}(D)$ such that

$$\|\mathscr{E} w\|_{1,r,D} = \|w\|_{1/2,r,\Gamma} ;$$

consequently, the continuity of the functional $\boldsymbol{q}^* . \nu$ follows. We easily find that

$$(2.7) \qquad \boldsymbol{q} \in \mathscr{H}_2 \Rightarrow \boldsymbol{q}^* \in H_r(\mathrm{div}, D)\, , \quad \langle \boldsymbol{q}^* . \nu, \gamma v \rangle = 0 \quad \forall v \in V .$$

In fact

$$(2.8) \qquad \boldsymbol{q} \in \mathscr{H}_2 \Rightarrow \mathrm{div}\, \boldsymbol{q}^* = \boldsymbol{q}_0 \in L^2_r(D)$$

so that $\boldsymbol{q}^* \in H_r(\mathrm{div}, D)$ and

$$0 = B(\boldsymbol{q}, v) = \int_D \left(q_r \frac{\partial v}{\partial r} + q_z \frac{\partial v}{\partial z} + v\, \mathrm{div}\, \boldsymbol{q}^* \right) r\, \mathrm{d}r\, \mathrm{d}z = \langle \boldsymbol{q}^* . \nu, \gamma v \rangle$$

holds for any $v \in V$.

Let us introduce the set

$$Q_0 = \{q^* \in H_r(\text{div}, D) \mid \langle q^* . \nu, \gamma v \rangle = 0 \quad \forall v \in V\} .$$

Then Q_0 is a (closed) subspace of $H_r(\text{div}, D)$ and we easily deduce

$$q^* \in Q_0 \Rightarrow q = (q_r^*, q_z^*, \text{div } q^*) \in \mathcal{H}_2 , \tag{2.9}$$

$$q \in \mathcal{H}_2 \Rightarrow q = (q_r, q_z, \text{div } q^*) , \quad q^* \in Q_0 . \tag{2.10}$$

Next we introduce the following bilinear form on $[H_r(\text{div}, D)]^2$:

$$(q, p)_C = \int_D (a_r^{-1} q_r p_r + a_z^{-1} q_z p_z + a_0^{-1} \text{div } q \text{ div } p) \, r \, dr \, dz .$$

The latter form is a scalar product in the space $H_r(\text{div}, D)$ and the associated norm $\|\cdot\|_C$ is equivalent to the norm $\|\cdot\|_{\text{div},D}$.

Having numerical methods in mind, we replace the affine hyperplane Q_{fg} by the sum of a particular element $\lambda^0 \in Q_{fg}$ and the subspace $\mathcal{H}_2$, i.e., we set

$$Q_{fg} = \lambda^0 + \mathcal{H}_2 .$$

Remark 2.2. *Construction of λ^0.*

1° Consider first the case $\Gamma = \Gamma_u$. Then we may set

$$\lambda_r^0 = \lambda_z^0 = 0 , \quad \lambda_0^0 = f .$$

2° Let $\Gamma_g \neq \emptyset$. If we find $p \in H_r(\text{div}, D)$ such that $p . \nu = g$ on Γ_g, then we may set

$$\lambda_r^0 = p_r , \quad \lambda_z^0 = p_z , \quad \lambda_0^0 = f + \text{div } p .$$

In fact, for any $v \in V$ we obtain

$$B(\lambda^0, v) = \int_D \left(p_r \frac{\partial v}{\partial r} + p_z \frac{\partial v}{\partial z} + (f + \text{div } p) \, v \right) r \, dr \, dz =$$

$$= \langle p . \nu, \gamma v \rangle + \int_D f v r \, dr \, dz = L(v) .$$

3° In case that g is piecewise polynomial on Γ_g, we can look for a p from a finite-element space, satisfying $p . \nu = g$ on Γ_g.

(Otherwise, we can first find a piecewise polynomial approximation g_h and then apply the previous technique.) ■

Theorem 2.2. (*Equivalent version of the principle.*)
Let us define the element $q^{*0} = (\lambda_r^0, \lambda_z^0)$ and the following functional

$$\psi(p) = \tfrac{1}{2}\|p\|_C^2 + (q^{*0}, p)_C + \int_D a_0^{-1} f \text{ div } p \, r \, dr \, dz - \langle p . \nu, \gamma u_0 \rangle .$$

Then

$$q^* = \arg\min_{p \in Q_0} \psi(p) \tag{2.11}$$

if and only if

$$(q_r^*, q_z^*, \text{div } q^*) = q(u) - \lambda^0 ,$$

where u is the weak solution of the primal problem (1.5).

Proof. Let us substitute

$$q \in Q_{fg} \Rightarrow q = \lambda^0 + \chi\,, \quad \chi \in \mathscr{H}_2$$

so that

$$\mathscr{S}(q) = \mathscr{S}(\lambda^0 + \chi) = \tfrac{1}{2}\|\lambda^0 + \chi - q(u_0)\|^2_{\mathscr{H}} =$$
$$= \tfrac{1}{2}\|\chi\|^2_{\mathscr{H}} + (\chi, \lambda^1)_{\mathscr{H}} + \tfrac{1}{2}\|\lambda^1\|^2_{\mathscr{H}},\ \lambda^1 = \lambda^0 - q(u_0)\,.$$

The last term can be omitted and

$$\chi = (p_r, p_z, \operatorname{div} p)\,, \quad p \in Q_0$$

inserted by virtue of (2.10). We obtain

$$\mathscr{S}(q) - \tfrac{1}{2}\|\lambda^1\|^2_{\mathscr{H}} = \tfrac{1}{2}\|p\|^2_C + \int_D \left[a_r^{-1}\, p_r \left(\lambda_r^0 - a_r \frac{\partial u_0}{\partial r}\right) + \right.$$
$$\left. + a_z^{-1}\, p_z \left(\lambda_z^0 - a_z \frac{\partial u_0}{\partial z}\right) + a_0^{-1}(\lambda_0^0 - a_0 u_0) \operatorname{div} p \right] r\, \mathrm{d}r\, \mathrm{d}z =$$
$$= \tfrac{1}{2}\|p\|^2_C + (p, q^{*0})_C + \textstyle\int_D a_0^{-1} f \operatorname{div} p\, r\, \mathrm{d}r\, \mathrm{d}z - \langle p \cdot \nu, \gamma u_0 \rangle\,,$$

using also that

$$\lambda_0^0 = f + \operatorname{div} q^{*0}$$

follows from the definition of Q_{fg}.

Furthermore, $q(u) - \lambda^0 \in \mathscr{H}_2$ and by (2.10) we may write

$$q(u) - \lambda^0 = [q_r^*, q_z^*, \operatorname{div} q^*]\,, \quad q^* \in Q_0\,.$$

Consequently, we are led to the assertion of the theorem.

3. APPROXIMATIONS TO THE DUAL PROBLEM

We discuss here only applications of *internal* finite-element approximations of the set Q_0, i.e., the construction and approximation properties of subspaces $Q_{0h} \subset Q_0$.

Assume that two families $\{V_h^r\}$, $\{V_h^z\}$ of finite-dimensional subspaces are given, such that for any parameter h, $0 < h \leqq 1$, the following conditions are satisfied:

(A1) $\quad V_h^r \subset W_r^{1,2}(D) \cap L^2_{1/r}(D)\,, \quad V_h^z \subset W_r^{1,2}(D)\,;$

(A2) $\quad$ an integer $k \geqq 1$ and a positive constant C exist, independent of h and u, v such that

$\quad \forall v \in W_r^{k+1,2}(D) \cap X_1(D) \quad \exists v_h \in V_h^r$:

(3.1) $\quad \|v - v_h\|_{X_1(D)} \leqq C h^k \|v\|_{k+1,r,D}\,;$

$\quad \forall u \in W_r^{k+1,2}(D) \quad \exists u_h \in V_h^z$:

(3.2) $\quad \|u - u_h\|_{1,r,D} \leqq C h^k \|u\|_{k+1,r,D}\,;$

(3.3) $\quad$ if $\; v\nu_r + u\nu_z = 0$ on Γ_g, then $\; v_h\nu_r + u_h\nu_z = 0$ on Γ_g.

Let us define

$$V(h) = \{(p_r, p_z) \in V_h^r \times V_h^z\}\ ; \quad Q_{0h} = V(h) \cap Q_0 \,.$$

It is readily seen that for any $\boldsymbol{p} \in V(h)$ we obtain div $\boldsymbol{p} \in L_r^2(D)$, so that $\boldsymbol{p} \in H_r(\text{div}, D)$. Moreover,

$$Q_{0h} = \{\boldsymbol{p} \in V(h) \mid \boldsymbol{p} \,.\, \nu = 0 \text{ on } \Gamma_g\} \,.$$

We call $\boldsymbol{q}^h \in Q_{0h}$ a finite-element approximation to the dual problem (2.11), if

$$\boldsymbol{q}^h = \underset{\boldsymbol{p} \in Q_{0h}}{\operatorname{argmin}}\, \psi(\boldsymbol{p}) \,. \tag{3.4}$$

In order to prove the convergence of approximations, we shall need the following

Proposition 3.1. *Let D be a bounded domain with a Lipschitz boundary ∂D, which consists of a finite number of infinitely smooth parts. Then the set*

$$\mathcal{M} = \{\boldsymbol{q} \in [C^\infty(\bar{D})]^2 \mid \operatorname{supp} \boldsymbol{q} \cap \Gamma_g = \emptyset\}$$

is dense in the subspace Q_0.

Proof is based on the following property of any Banach space (cf. [7] – Thm. 2.6, p. 29): A subspace $\mathcal{M}$ of the space B is dense in B if and only if every element of the dual space B' that vanishes on $\mathcal{M}$ also vanishes on B.

Let $f \in Q_0'$. There exist $\boldsymbol{F} \in H_r(\text{div}, D)$ and $F_0 \in L_r^2(D)$ such that

$$f(\boldsymbol{q}) = (F_r, q_r)_{0,r,D} + (F_z, q_z)_{0,r,D} + (F_0, \operatorname{div} \boldsymbol{q})_{0,r,D} \,. \tag{3.5}$$

Assume that f vanishes on the set $\mathcal{M}$. Since

$$[C_{0,r}^\infty(D)]^2 \subset \mathcal{M} \,,$$

we obtain

$$\boldsymbol{F} = \operatorname{grad} F_0$$

in the sense of distributions, so that $F_0 \in W_r^{1,2}(D)$. By definition and (3.5), we have for any $\boldsymbol{q} \in Q_0$

$$f(\boldsymbol{q}) = (\operatorname{grad} F_0, \boldsymbol{q})_{0,r,D} + (F_0, \operatorname{div} \boldsymbol{q})_{0,r,D} = \langle \boldsymbol{q} \,.\, \nu, \gamma F_0 \rangle \,. \tag{3.6}$$

Moreover, we can verify that

$$\gamma F_0 = 0 \quad \text{on } \Gamma_u \,, \quad \text{so that} \quad F_0 \in V \,. \tag{3.7}$$

In fact, assume that

$$\|\gamma F_0\|_{0,r,\Gamma_u \cap S} > 0$$

on some infinitely smooth part S of ∂D.

Let us define

$$C_{0,r}^\infty(\Gamma) = \{\varphi \in C^\infty(\bar{\Gamma}) \mid \operatorname{supp} \varphi \cap (\partial\Gamma \dot{-} \bar{\Gamma}_0) = \emptyset\} \,.$$

It is easy to verify that: (i) $C^{\infty}_{0,r}(\Gamma_u \cap S)$ is dense in $L^2_r(\Gamma_u \cap S)$ and (ii) for any $\varphi \in C^{\infty}_{0,r}(\Gamma_u \cap S)$ there exists $\boldsymbol{p} \in \mathcal{M}$ such that $\boldsymbol{p} \, . \, \nu = \varphi$ on $\Gamma_u \cap S$, $\boldsymbol{p} \, . \, \nu = 0$ outside $\Gamma_u \cap S$.

Then using (3.6), we arrive at a contradiction with the assumption that

$$f(\boldsymbol{p}) = \langle \boldsymbol{p} \, . \, \nu, \gamma F_0 \rangle = \int_{\Gamma_u \cap S} \boldsymbol{p} \, . \, \nu \gamma F_0 r \, \mathrm{d}s = 0 \quad \forall \boldsymbol{p} \in \mathcal{M} \, .$$

For $\boldsymbol{q} \in Q_0$, we have

$$f(\boldsymbol{q}) = \langle \boldsymbol{q} \, . \, \nu, \gamma F_0 \rangle = 0$$

by virtue of (3.6), (3.7) and the definition of Q_0. Hence f vanishes on the whole space Q_0 and therefore $\mathcal{M}$ is dense in Q_0.

Theorem 3.1. *Let D be a bounded domain with (Lipschitz) polygonal boundary. Then*

$$\lim_{h \to 0} \|\boldsymbol{q}^h - \boldsymbol{q}^*\|_{\mathrm{div},D} = 0$$

where $\boldsymbol{q}^h$ and $\boldsymbol{q}^*$ is the finite-element approximation (3.4) and the solution of the dual problem (2.11), respectively.

Proof. The definition (2.11) of $\boldsymbol{q}^*$ results in

$$(\boldsymbol{q}^*, \boldsymbol{p})_C = l(\boldsymbol{p}) \quad \forall \boldsymbol{p} \in Q_0 \, ,$$

whereas the definition (3.4) of $\boldsymbol{q}^h$ implies

$$(\boldsymbol{q}^h, \boldsymbol{p}^h)_C = l(\boldsymbol{p}^h) \, , \quad \forall \boldsymbol{p}^h \in Q_{0h} \, ,$$

where

$$l(\boldsymbol{p}) = -(\boldsymbol{p}, \boldsymbol{q}^{*0})_C - \int_D a_0^{-1} f \operatorname{div} \boldsymbol{p} \, r \, \mathrm{d}r \, \mathrm{d}z + \langle \boldsymbol{p} \, . \, \nu, \gamma u_0 \rangle \, .$$

By subtraction we obtain

$$(\boldsymbol{q}^* - \boldsymbol{q}^h, \boldsymbol{p}^h)_C = 0 \quad \forall \boldsymbol{p}^h \in Q_{0h} \, ,$$

i.e., $\boldsymbol{q}^h$ is the orthogonal projection of $\boldsymbol{q}^*$ onto Q_{0h} in $H_r(\mathrm{div}, D)$. Therefore

$$\|\boldsymbol{q}^* - \boldsymbol{q}^h\|_C \leqq \|\boldsymbol{q}^* - \boldsymbol{p}^h\|_C \quad \forall \boldsymbol{p}^h \in Q_{0h} \, . \tag{3.8}$$

Using Proposition 3.1, we obtain

$$\forall \eta > 0 \quad \exists \mathbf{t} \equiv (t_r, t_z) \in [C^{\infty}(\bar{D})]^2 \cap Q_0$$

such that

$$\|\boldsymbol{q}^* - \mathbf{t}\|_{\mathrm{div},D} < \eta/2 \, . \tag{3.9}$$

By virtue of the assumption ($A2$), one can find an element

$$\mathbf{t}^h \equiv (t_r^h, t_z^h) \in Q_{0h}$$

such that

$$\|\mathbf{t}^h - \mathbf{t}\|_{\mathrm{div},D} \leqq Ch^k(\|t_r\|_{k+1,r,D} + \|t_z\|_{k+1,r,D}) \, . \tag{3.10}$$

In fact, we have

$$\|\operatorname{div} \boldsymbol{p}\|^2_{0,r,D} = \int_D \left(\frac{1}{r} p_r + \frac{\partial p_r}{\partial r} + \frac{\partial p_z}{\partial z}\right)^2 r\,\mathrm{d}r\,\mathrm{d}z \leqq$$
$$\leqq 3(\|p_r\|^2_{0,1/r,D} + |p_r|^2_{1,r,D} + |p_z|^2_{1,r,D})$$

and

$$\|p_r\|_{0,r,D} \leqq C\|p_r\|_{0,1/r,D}\,,$$

so that we may write

$$\|\boldsymbol{p}\|^2_{\mathrm{div},D} \leqq (3 + C)\,\|p_r\|^2_{0,1/r,D} + 3|p_r|^2_{1,r,D} + 3\|p_z\|^2_{1,r,D} \leqq$$
$$\leqq C(\|p_r\|^2_{X_1(D)} + \|p_z\|^2_{1,r,D})\,.$$

Therefore, using $(A2)$, we obtain (3.10) as follows

$$\|t^h - \boldsymbol{t}\|_{\mathrm{div},D} \leqq C(\|t_r^h - t_r\|_{X_1(D)} + \|t_z^h - t_z\|_{1,r,D}) \leqq$$
$$\leqq Ch^k(\|t_r\|_{k+1,r,D} + \|t_z\|_{k+1,r,D})\,.$$

Combining (3.8), the equivalence of norms and (3.10), we may write

$$\|\boldsymbol{q}^h - \boldsymbol{q}^*\|_{\mathrm{div},D} \leqq C_1\|\boldsymbol{q}^h - \boldsymbol{q}^*\|_C \leqq C_1\|\boldsymbol{q}^* - \boldsymbol{t}^h\|_C \leqq$$
$$\leqq C_2\|\boldsymbol{q}^* - \boldsymbol{t}^h\|_{\mathrm{div},D} \leqq C_2(\|\boldsymbol{q}^* - \boldsymbol{t}\|_{\mathrm{div},D} + \|\boldsymbol{t} - \boldsymbol{t}^h\|_{\mathrm{div},D}) \leqq$$
$$\leqq C_2\eta/2 + C_3h^k(\|t_r\|_{k+1,r,D} + \|t_z\|_{k+1,r,D}) < \varepsilon$$

for η and h sufficiently small. Q.E.D.

Corollary 3.1. *Let us assume that the solution of the dual problem* (2.11) $\boldsymbol{q}^* \in [W_r^{k+1,2}(D)]^2$, $k \geqq 1$. *Then*

$$\|\boldsymbol{q}^* - \boldsymbol{q}^h\|_{\mathrm{div},D} \leqq Ch^k(\|q_r^*\|_{k+1,r,D} + [q_z^*\|_{k+1,r,D})\,.$$

Proof follows immediately from the argument of Theorem 3.1, setting $\boldsymbol{t} \equiv \boldsymbol{q}^*$.

Remark 3.1. From Theorem 3.1 (or Corollary 3.1) we obtain approximation to the cogradient and to the solution, as well. In fact, Theorem 2.2 implies that

$$\operatorname{cograd} u = \boldsymbol{q}^* + \boldsymbol{q}^{*0}\,,$$
$$u = a_0^{-1}(\operatorname{div} \boldsymbol{q}^* + \lambda_0^0)\,.$$

Inserting $\boldsymbol{q}^h$ instead of $\boldsymbol{q}^*$, we arrive at the following approximations

$$(\operatorname{cograd} u)_h = \boldsymbol{q}^h + \boldsymbol{q}^{*0}\,, \tag{3.11}$$
$$u_h = a_0^{-1}(\operatorname{div} \boldsymbol{q}^h + \lambda_0^0)\,.$$

Using Theorem 3.1, we obtain

$$\|\operatorname{cograd} u - (\operatorname{cograd} u)_h\|_{0,r,D} + \|u - u_h\|_{0,r,D} = \tag{3.12}$$
$$= \|\boldsymbol{q}^* - \boldsymbol{q}^h\|_{0,r,D} + \|a_0^{-1} \operatorname{div}(\boldsymbol{q}^* - \boldsymbol{q}^h)\|_{0,r,D} \leqq C\|\boldsymbol{q}^* - \boldsymbol{q}^h\|_{\mathrm{div},D} \to 0\,.$$

In the present Section we show examples of subspaces V_h^r and V_h^z, satisfying assumptions (A1), (A2). For details we refer to the papers [4] and [8].

Let us consider a bounded domain D with polygonal (Lipschitz) boundary ∂D and triangulations $\mathscr{T}_h$, which are consistent with the decomposition $\partial D = \Gamma_0 \cup \bar{\Gamma}_u \cup \cup \bar{\Gamma}_g$. For any triangle $K \subset \mathscr{T}_h$ we introduce a local interpolation mapping

$$\Pi_K^k \colon C(K) \to P_k(K),$$

where $k = 1$ or $k = 2$ and $P_k(K)$ denotes the space of polynomial functions of the degree at most k, and such that $\Pi_K^k\, u(Q) = u(Q)$ at the nodal points $Q \in \partial K$. If $k = 1$, the nodal points are the vertices, if $k = 2$, vertices and the midpoints of sides.

We define the spaces

$$\Sigma_h^k = \{v \in C(\bar{D}) \,|v|_K \in P_k(K) \quad \forall K \in \mathscr{T}_h\}, \quad k = 1, 2$$

and global interpolation mappings

$$\Pi_h^k \colon C(\bar{D}) \to \Sigma_h^k,$$

$$\Pi_h^k u|_K = \Pi_K^k u \quad \forall K \in \mathscr{T}_h.$$

Let us define

$$h = \max_{K \in \mathscr{T}_h} (\operatorname{diam} K)$$

and assume that the family $\{\mathscr{T}_h\}$, $h \to 0$ of triangulations is **regular**, i.e., a positive ϑ exists such that all internal angles in $\mathscr{T}_h$ are not less than ϑ.

Then we have the following estimate

$$|u - \Pi_h^k u|_{1,r,D} + \|u - \Pi_h^k u\|_{0,1/r,D} \leqq Ch^k |u|_{k+1,r,D} \tag{4.1}$$

$$\forall u \in W_r^{k+1,2}(D) \cap X_1(D).$$

For the proof we refer to [4]-Lemma 6.1, 6.2, 6.3. Recall that $W_r^{2,2}(D) \hookrightarrow C(\bar{D})$ holds and if $u \in X_1(D) \cap C(\bar{D})$, then $u = 0$ on Γ_0. Therefore, $\Pi_h^k u = 0$ on Γ_0, as well.

Thus setting

$$V_h^r = \{w \in \Sigma_h^k \,|w = 0 \text{ on } \Gamma_0\},$$

$v_h = \Pi_h^k v$, the assumptions $(A1)$ and $(A2) - (3.1)$ will be satisfied, as far as V_h^r is concerned.

If we define

$$V_h^z = \Sigma_h^k, \quad u_h = \Pi_h^k u,$$

then $(A2) - (3.2)$ is fulfilled, since

$$|u - \Pi_h^k u|_{1,r,D} \leqq Ch^k |u|_{k+1,r,D} \quad \forall u \in W_r^{k+1,2}(D)$$

holds (cf. [4]-Lemma 6.1, 6.2) and

$$\|u - \Pi_h^k u\|_{0,r,D} \leqq C h^{k+1} |u|_{k+1,r,D} \quad \forall u \in W_r^{k+1,2}(D)$$

can be derived following the same way as in the proof of Lemma 6.1, 6.2 of [4].

Let us verify (A2) – (3.3). Let

$$v v_r + u v_z = 0 \quad \text{on} \quad \Gamma_g \cap K = S$$

where S is any side on the polygonal boundary Γ_g. By definition, we have

$$\varphi_h \equiv (\Pi_h^k v)\, v_r + (\Pi_h^k u)\, v_z = v v_r + u v_z = 0$$

at the nodes $A_j \in S$. Since the restriction $\varphi_h|_S \in P_k(S)$, φ_h vanishes on the side S, i.e., (3.3) is true.

5. A POSTERIORI ERROR ESTIMATES AND TWO-SIDED BOUNDS OF ENERGY

Suppose that we have solved the problem from two sides:

(i) by the primal method, using a standard finite element model [4], which yields the approximation

$$u_{h_1} = u_0 + w_{h_1}\,, \quad w_{h_1} \in \Sigma_{h_1}^k \cap V\,, \quad (k = 1 \text{ or } k = 2)\,,$$

(u_{h_1} is defined by the condition

$$a(u_{h_1}, v_{h_1}) = L(v_{h_1}) \quad \forall v_{h_1} \in \Sigma_{h_1}^k \cap V)\,;$$

(ii) by the dual method of Section 3, which yields the approximations

$$\lambda^h = \lambda^0 + (q_r^h, q_z^h, \operatorname{div} \boldsymbol{q}^h)\,,$$

where $\lambda^0 \in Q_{fg}$ and $\boldsymbol{q}^h$ is the solution of (3.4).

Lemma 5.1. *Let* $C = \max \{\|a_r\|_{\infty,D};\ \|a_z\|_{\infty,D};\ \|a_0\|_{\infty,D}\}$. Then

$$(5.1) \qquad c^{1/2} \|u_{h_1} - u\|_{1,r,D} \leqq \|\boldsymbol{q}(u_{h_1}) - \boldsymbol{q}(u)\|_{\mathscr{H}} \leqq \|\boldsymbol{q}(u_{h_1}) - \lambda^h\|_{\mathscr{H}}\,,$$

$$(5.2) \qquad C^{-1/2} \|\lambda^h - \boldsymbol{q}(u)\|_{0,r,D} \leqq \|\lambda^h - \boldsymbol{q}(u)\|_{\mathscr{H}} \leqq \|\boldsymbol{q}(u_{h_1}) - \lambda^h\|_{\mathscr{H}}\,.$$

Proof. It is readily seen that (cf. Section 2)

$$\boldsymbol{q}(u_{h_1}) - \boldsymbol{q}(u) \in \mathscr{H}_1\,, \quad \boldsymbol{q}(u) \in Q_{fg} \quad \text{and} \quad \lambda^h \in Q_{fg}$$

by virtue of (2.9).

Using Lemma 2.1, we may write

$$(5.3) \qquad \|\boldsymbol{q}(u_{h_1}) - \lambda^h\|_{\mathscr{H}}^2 = \|\boldsymbol{q}(u_{h_1}) - \boldsymbol{q}(u)\|_{\mathscr{H}}^2 + \|\boldsymbol{q}(u) - \lambda^h\|_{\mathscr{H}}^2\,.$$

Moreover, from (1.4) and the definition of $\mathbf{q}(v)$ we obtain

$$c\|v\|^2_{1,r,D} \leqq \|\mathbf{q}(v)\|^2_{\mathscr{H}} \quad \forall v \in W^{1,2}_r(D)\,. \tag{5.4}$$

$$\|\mathbf{q}\|^2_{\mathscr{H}} \geqq C^{-1}\|\mathbf{q}\|^2_{0,r,D} \quad \forall \mathbf{q} \in \mathscr{H}\,. \tag{5.5}$$

The estimates (5.1), (5.2) follow from (5.3)–(5.5) immediately.

Lemma 5.2. *Let us denote*

$$a(v, v) \equiv \|v\|^2_A\,.$$

Then we have

$$\|u_{h_1} - u_0\|_A \leqq \|u - u_0\|_A \leqq \|\lambda^h - \mathbf{q}(u_0)\|_{\mathscr{H}}\,. \tag{5.6}$$

Proof. Recall that we may write

$$u = u_0 + w\,, \quad w \in V \quad \text{and} \quad u_{h_1} = u_0 + w_{h_1}\,, \quad w_{h_1} \in V \cap \Sigma^k_{h_1}\,,$$

$$a(u, w) = L(w) \quad \text{and} \quad a(u_{h_1}, w_{h_1}) = L(w_{h_1})\,.$$

Denoting by

$$\mathscr{L}(v) = \tfrac{1}{2}\|v\|^2_A - L(v)$$

the potential energy, we obtain that

$$\begin{aligned} 2\,\mathscr{L}(u) + 2\,L(u_0) &= \|u\|^2_A - 2\,L(w) = \|u\|^2_A = 2\,a(u, w) = \\ &= \|u - w\|^2_A - \|w\|^2_A = \|u_0\|^2_A - \|w\|^2_A\,. \end{aligned} \tag{5.7}$$

In the same way, we deduce that

$$2\,\mathscr{L}(u_{h_1}) + 2\,L(u_0) = \|u_0\|^2_A - \|w_{h_1}\|^2_A\,.$$

Then

$$0 \leqq 2[\mathscr{L}(u_{h_1}) - \mathscr{L}(u)] = -\|w_{h_1}\|^2_A + \|w\|^2_A\,,$$

since $u_{h_1} \in u_0 + V$ and u is the minimizer of $\mathscr{L}$ over $u_0 + V$. Consequently, the left-hand inequality in (5.6) follows.

By a direct calculation we can derive that

$$\mathscr{L}(u) + \mathscr{S}(\mathbf{q}(u)) + L(u_0) - \tfrac{1}{2}\|u_0\|^2_A = 0\,.$$

Using also Theorem 2.1 and recalling that $\lambda^h \in Q_{fg}$, we obtain

$$\begin{aligned} -\,2[\mathscr{L}(u) + L(u_0)] &= 2\,\mathscr{S}(\mathbf{q}(u)) - \|u_0\|^2_A \leqq 2\,\mathscr{S}(\lambda^h) - \|u_0\|^2_A = \\ &= \|\lambda^h - \mathbf{q}(u_0)\|^2_{\mathscr{H}} - \|u_0\|^2_A\,. \end{aligned}$$

The left-hand side, however, is equal to

$$\|w\|^2_A - \|u_0\|^2_A\,,$$

by virtue of (5.7). Consequently, the right-hand inequality of (5.6) follows.

References

[1] *I. Hlaváček:* Dual finite element analysis for some elliptic variational equations and inequalities. Acta Appl. Math. 1 (1983), 121—150.

[2] *I. Hlaváček:* Some equilibrium and mixed models in the finite element method. Banach Center Publications, 3 (1977), 147—165.

[3] *J.-P. Aubin, H. Burchard:* Some aspects of the method of hypercircle applied to elliptic variational problems. Proceed. of SYNSPADE, 1971, Academic Press, 1—67.

[4] *B. Mercier, G. Raugel:* Resolution d'un problème aux limites dans un ouvert axisymétrique par éléments finis. R.A.I.R.O. Anal. numér. 16, (1982), 405—461.

[5] *I. Hlaváček:* Domain optimization in axisymmetric elliptic boundary value problems by finite elements. Apl. Mat. 33 (1988), 213—244.

[6] *I. Doležel:* Numerical calculation of the leakage field in the window of a transformer with magnetic shielding. Acta Technica ČSAV, (1981), 563—588.

[7] *V. Girault, P. A. Raviart:* Finite element methods for Navier-Stokes equations. Springer-Verlag, Berlin 1986.

[8] *I. Hlaváček, M. Křížek:* Dual finite element analysis of 3D-axisymmetric elliptic problems. Numer. Meth. in Part. Dif. Eqs. (To appear.)

Souhrn

DUÁLNÍ ANALÝZA OSOVĚ SYMETRICKÝCH ELIPTICKÝCH PROBLÉMŮ S ABSOLUTNÍM ČLENEM METODOU KONEČNÝCH PRVKŮ

IVAN HLAVÁČEK

Uvažuje se osově symetrická eliptická úloha se smíšenými okrajovými podmínkami v cylindrických souřadnicích. K přímému výpočtu kogradientu řešení je aplikována duální variační formulace. Aproximace se definují na základě standardních prostorů konečných prvků. Dokazuje se konvergence přibližných řešení a některé aposteriorní odhady.

Author's address: Ing. *Ivan Hlaváček*, DrSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.