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IDENTIFICATION OF CRITICAL CURVES PART II: DISCRETIZATION AND NUMERICAL REALIZATION

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Summary. We consider the finite element approximation of the identification problem, where one wishes to identify a curve along which a given solution of the boundary value problem possesses some specific property. We prove the convergence of FE-approximation and give some results of numerical tests.

Keywords: Identification of a curve, approximation by FEM, convergence.

AMS classification: 49E30, 65N30

INTRODUCTION

In practice we often meet problems when we wish to identify a curve along which a given solution of a boundary value problem possesses some specific property. In [1] the problem of identification of a curve along which the "flux" functional $\int_\varphi (\partial u/\partial n)\,ds$ attains its maximum, is analysed. The present paper deals with the approximation of this problem. Some numerical results are presented.

1. SETTING OF THE PROBLEM

This paper deals with the finite element approximation of an identification problem, the continuous version of which has been already introduced in [1]. Let us mention its definition. We shall assume the following mixed boundary value problem:

$$(\mathscr{P}') \qquad \begin{cases} -\Delta u = f & \text{in} \quad \Omega\,, \\ u = 0 & \text{on} \quad \Gamma_1\,, \\ \dfrac{\partial u}{\partial n} = g & \text{on} \quad \Gamma_2\,, \end{cases}$$

where

$$\Omega = \{(x_1, x_2) \in \mathbf{R}^2 \mid 0 < x_2 < p(x_1),\, x_1 \in (0, 1)\}$$

is a bounded domain, the Lipschitz boundary $\partial\Omega$ of which is decomposed as follows

$$\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 ,$$

$$\Gamma_2 = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_2 = p(x_1), x_1 \in (0, 1)\} ,$$

$$\Gamma_1 = \partial\Omega \setminus \bar{\Gamma}_2 .$$

Here p is a Lipschitz continuous function on $[0, 1]$. Moreover, $f \in L^2(\Omega)$, $g \in L^2(\Gamma_2)$. In order to give the variational form of $(\mathscr{P}')$, we introduce the space

$$V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_1\} .$$

The variational formulation of $(\mathscr{P}')$ reads as follows:

$$(\mathscr{P}) \qquad \begin{cases} \text{Find} \quad u \in V \quad \text{such that} \\ (\nabla u, \nabla v)_{0,\Omega} = (f, v)_{0,\Omega} + \int_{\Gamma_2} gv \, ds \quad \forall v \in V . \end{cases}$$

The symbol $(\ , \ )_{0,\Omega}$ denotes the usual scalar product in $L^2(\Omega)$.

Let $0 < \bar{\alpha} < \bar{\beta} < 1$, $\delta > 0$ be given. By U_{ad} we denote a subset of Lipschitz continuous functions, defined by

$$U_{ad} = \{\varphi \mid \exists\alpha \in [0, \bar{\alpha}], \beta \in [\bar{\beta}, 1]: \varphi \in C^{0,1}([\alpha, \beta])] ,$$

$$\varphi(\alpha) = p(\alpha), \varphi(\beta) = p(\beta), \delta \leqq \varphi \leqq p \text{ on } [\alpha, \beta],$$

$$|\varphi(x_1) - \varphi(\bar{x}_1)| \leqq C_1|x_1 - \bar{x}_1| \quad \forall x_1, \bar{x}_1 \in [\alpha, \beta], \quad \text{meas} \quad \Omega(\varphi) = C_2\}$$

where

$$\bar{\Omega}(\varphi) = \{(x_1, x_2) \in \mathbf{R}^2 \mid 0 \leqq x_2 \leqq p(x_1) \quad x_1 \in [0, \alpha] \cup [\beta, 1]$$

$$0 \leqq x_2 \leqq \varphi(x_1) \quad x_1 \in [\alpha, \beta]\}$$

and C_1, C_2 are positive constants such that $U_{ad} \neq \emptyset$.

Finally, set

$$J(\varphi) = \left\langle \frac{\partial u}{\partial n}, 1 \right\rangle_{\partial\Omega(\varphi)} - \int_{\Gamma_2^1(\varphi)} g \, ds - \int_{\Gamma_2^2(\varphi)} g \, ds ,$$

where

$$\Gamma_2^1(\varphi) = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_2 = p(x_1), x_1 \in (0, \alpha)\}$$

$$\Gamma_2^2(\varphi) = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_2 = p(x_1), x_1 \in (\beta, 1)\}$$

and $\langle \ , \ \rangle_{\partial\Omega(\varphi)}$ denotes the duality pairing between $H^{-1/2}(\partial\Omega(\varphi))$ and $H^{1/2}(\partial\Omega(\varphi))$.

In [1] the following problem has been introduced:

$$(\mathbf{P})' \qquad \begin{cases} \text{Find} \quad \varphi^* \in U_{ad} \quad \text{such that} \\ J(\varphi^*) = \max\limits_{\varphi \in U_{ad}} J(\varphi) . \end{cases}$$

This problem can be equivalently formulated (see [1]) as follows:

$$(\mathbf{P}) \qquad \begin{cases} \text{Find} \quad \varphi^* \in U_{\mathrm{ad}} \quad \text{such that} \\ \mathscr{I}(\varphi^*) = \min\limits_{\varphi \in U_{\mathrm{ad}}} \mathscr{I}(\varphi)\,, \end{cases}$$

where

$$\mathscr{I}(\varphi) = \int_{\Omega(\varphi)} f \,\mathrm{d}x + \int_{\Gamma_2{}^1(\varphi)} g \,\mathrm{d}s + \int_{\Gamma_2{}^2(\varphi)} g \,\mathrm{d}s\,.$$

The existence of at least one solution φ^* of $(\mathbf{P})$ has been established in [1].

2. APPROXIMATION OF (P)

In what follows we shall assume $\Omega \subset \mathbf{R}^2$ to be of a special type, namely such that the function p describing Γ_2 is piecewise linear in $[0, 1]$ with nodes included in $[\bar{\alpha}, \bar{\beta}]$ only (see Fig. 1)*)

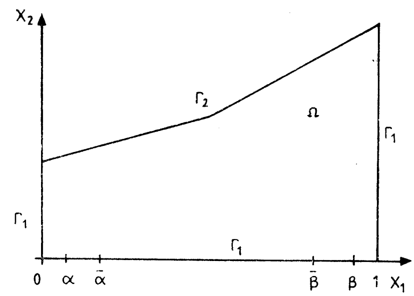

Figure 1.

Let $\mathscr{D}_n(\alpha, \beta)$: $\alpha = a_1 < a_2 < \ldots < a_n = \beta$ be a partition of $[\alpha, \beta]$, where the number of nodal points n doesn't depend on $\alpha \in [0, \bar{\alpha}]$ and $\beta \in [\bar{\beta}, 1]$. The x_1-coordinates of the vertices of p are included in $\mathscr{D}_n(\alpha, \beta)$. Any $\mathscr{D}_n(\alpha, \beta)$ will be characterized by two numbers $h_{\max}(\alpha, \beta) = \max_i |\, a_{i+1} - a_i|$, $h_{\min}(\alpha, \beta) = \min_i |a_{i+1} - a_i|$. We shall assume that the position of a_i, $i = 1, \ldots, n$ depends continuously on α, β.

The approximation of U_{ad} will be constructed by means of piecewise linear functions:

$$\begin{aligned} U_{\mathrm{ad}}^n = \{\varphi \mid \exists \alpha \in [0, \bar{\alpha}], \beta \in [\bar{\beta}, 1]: \varphi \in C([\alpha, \beta]), \\ \varphi|_{\overline{a_{i-1}a_i}} \in P_1(\overline{a_{i-1}a_i}), \varphi(\alpha) = p(\alpha), \varphi(\beta) = p(\beta)\,, \\ \delta \leqq \varphi \leqq p \text{ on } [\alpha, \beta], |\varphi(\bar{x}_1) - \varphi(x_1)| \leqq C_1 |x_1 - \bar{x}_1| \\ \forall x_1, \bar{x}_1 \in [\alpha, \beta], \tilde{C}_2 \leqq \operatorname{meas} \Omega(\varphi) \leqq \tilde{C}_3\}\,. \end{aligned}$$

$C_1, \tilde{C}_2, \tilde{C}_3$ are positive constants chosen in such a way that $U_{\mathrm{ad}}^n \neq \emptyset$.

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*) This assumption is only for technical reasons.

Remark 2.1. The equality constraint meas $\Omega(\varphi) = C_2$ in the definition of U_{ad} is now replaced by two sided inequality constraints $\tilde{C}_2 \leqq$ meas $\Omega(\varphi) \leqq \tilde{C}_3$ (for the choice of $\tilde{C}_2$, $\tilde{C}_3$ see below). This means that $U^n_{ad} \not\subset U_{ad}$.

The approximation of $(\mathbf{P})$ is now defined as follows:

$$(\mathbf{P})_n \qquad \begin{cases} \text{Find } \varphi_n^* \in U^n_{ad} \text{ such that} \\ \mathscr{I}(\varphi_n^*) = \min_{\varphi \in U^n_{ad}} \mathscr{I}(\varphi). \end{cases}$$

Next, we shall analyse

(i) the existence of at least one solution φ_n^* of $(\mathbf{P})_n$;

(ii) the relation between $(\mathbf{P})_n$ and $(\mathbf{P})$ when $n \to \infty^+$.

3. EXISTENCE OF A SOLUTION FOR $(\mathbf{P})_n$

In order to prove the existence of a solution for $(\mathbf{P})_n$, we formulate this problem in the language of discrete design variables. The vector of discrete design variables $\omega = (\omega_1, \ldots, \omega_{2n})$ contains

(j) the nodes of $\mathscr{D}_n(\alpha, \beta)$;

(jj) the x_2-coordinates of vertices of $\varphi \in U^n_{ad}$ at a_i, $i = 1, \ldots, n$.

For the sake of simplicity of notation we shall suppose that the first n components of ω are the nodes of $\mathscr{D}_n(\alpha, \beta)$, while the elements of (jj) are listed last, i.e.

$$\omega_i = a_i \qquad i = 1, \ldots, n;$$
$$\omega_i = \varphi(a_{i-n}) \quad i = n+1, \ldots, 2n.$$

Let the parameter n be fixed. Then U^n_{ad} can be identified with a compact subset U of $\mathbf{R}^{2n}$ as follows:

$$U = \Big\{\omega \in \mathbf{R}^{2n} \mid h_{\max}(\omega_1, \omega_n) \geqq \omega_{i+1} - \omega_i \geqq h_{\min}(\omega_1, \omega_n),\ i = 1, \ldots, n-1;$$
$$\omega_1 \in [0, \bar{\alpha}],\ \omega_n \in [\bar{\beta}, 1],\ \omega_{n+1} = p(\omega_1),\ \omega_{2n} = p(\omega_n);$$
$$\delta \leqq \omega_i \leqq p(\omega_{i-n}),\ i = n+2, \ldots, 2n-1;$$
$$|\omega_{i+1} - \omega_i| \leqq C_1 |\omega_{i+1-n} - \omega_{i-n}|,\ i = n+1, \ldots, 2n-1;$$
$$\tilde{C}_2 \leqq \omega_1 \frac{(p(0) + p(\omega_1))}{2} + \sum_{i=n+1}^{2n-1} \frac{(\omega_{i+1} + \omega_i)}{2} (\omega_{i+1-n} - \omega_{i-n}) +$$
$$+ (1 - \omega_n) \frac{(p(\omega_n) + p(1))}{2} \leqq \tilde{C}_3 \Big\}.$$

Let $\mathscr{T}_n \colon U^n_{ad} \to \mathbf{R}^{2n}$ be a mapping defined through the relation

$$\mathscr{T}_n(\varphi) = (a_1, \ldots, a_n, \varphi(a_1), \ldots, \varphi(a_n)), \quad \varphi \in U^n_{ad}.$$

It is easy to see that $\mathscr{T}_n(U^n_{\mathrm{ad}}) = U$ and the inverse mapping $\mathscr{T}_n^{-1}: U \to U^n_{\mathrm{ad}}$ is given by

$$\mathscr{T}_n^{-1}(\omega) = \sum_{j=1}^{n} \omega_{j+n}\ell_j\,, \quad \omega = (\omega_1, \ldots, \omega_{2n}) \in U\,,$$

where ℓ_j, $j = 1, \ldots, n$ are piecewise linear functions satisfying $\ell_j(a_i) = \delta_{ij}$. Finally, set

$$\mathscr{L}(\omega) = \mathscr{I}(\mathscr{T}_n^{-1}\omega) = \int_{\Omega(\varphi)} f\,\mathrm{d}x + \int_{\Gamma_2{}^1(\varphi)} g\,\mathrm{d}s + \int_{\Gamma_2{}^2(\varphi)} g\,\mathrm{d}s\,.$$

The equivalent formulation of $(\mathbf{P})_n$ is given by

$$(\tilde{\mathbf{P}})_n \qquad \begin{cases} \text{Find} \quad \omega^* \in U \quad \text{such that} \\ \mathscr{L}(\omega^*) = \min\limits_{\omega \in U} \mathscr{L}(\omega)\,. \end{cases}$$

The main result of this section is

Theorem 3.1. *For any n there exists at least one solution φ_n^** of $(\mathbf{P})_n$.

Proof. U is a compact subset of $\mathbf{R}^{2n}$, $\omega \to \mathscr{L}(\omega)$ is a continuous function. Using the classical compactness argument, we obtain the existence of at least one $\omega^* = (\omega_1^*, \omega_2^*, \ldots, \omega_{2n}^*) \in U$ solving $(\tilde{\mathbf{P}})_n$. Setting $\varphi_n^* = \sum_{j=1}^{n} \omega_{j+n}^*\ell_j$ we arrive at the assertion of Theorem 3.1.

4. RELATION BETWEEN (P) AND $(\mathbf{P})_n$, $n \to \infty$

Let $\{\mathscr{D}_n(\alpha, \beta)\}$, $n \to \infty$ be a regular family of partitions of $[\alpha, \beta]$ in the following sense:

$$\begin{cases} \text{there exists a positive number } q \text{ independent of} \\ n,\ \alpha,\ \beta \text{ and such that} \\ \dfrac{h_{\max}(\alpha, \beta)}{h_{\min}(\alpha, \beta)} \leqq q\,. \end{cases}$$

Let $\varphi_n^* \in U^n_{\mathrm{ad}}$ be a solution of $(\mathbf{P})_n$. Now we shall analyse what happens when $n \to \infty$.

First of all we shall specify the choice of constants $\tilde{C}_2$, $\tilde{C}_3$ appearing in the definition of U^n_{ad}. We set

$$\tilde{C}_2 = C_2\left(1 - \left(\frac{1}{n}\right)^{\gamma}\right),$$

$$\tilde{C}_3 = C_2\left(1 + \left(\frac{1}{n}\right)^{\gamma}\right),$$

where $\gamma \in (0, 1)$ and the meaning of C_2 is given by the definition of U_{ad}. First we prove

Lemma 4.1. *For any $\varphi \in U_{\text{ad}}$, $\varphi: [\alpha, \beta] \to \mathbf{R}^1$ there exists a sequence $\varphi_n \in U^n_{\text{ad}}$ defined on $[\alpha, \beta]$ and such that*

$$\varphi_n \rightrightarrows \varphi \quad \text{(uniformly) on } [\alpha, \beta]\,. \tag{4.1}$$

Proof. Let $\varphi \in U_{\text{ad}}$ be defined on $[\alpha, \beta]$, $\alpha \in [0, \bar{\alpha}]$, $\beta \in [\bar{\beta}, 1]$. Set $\varphi_n = r_n\varphi$, where $r_n\varphi$ denotes the piecewise linear Lagrange interpolation of φ. Using the classical approximation properties of φ_n we have

$$\|\varphi_n - \varphi\|_{L^\infty((\alpha,\beta))} \leqq c\frac{1}{n}\|\varphi\|_{W^{1,\infty}((\alpha,\beta))} \leqq c\frac{1}{n}, \tag{4.2}$$

which yields (4.1). Let us prove that $\varphi_n \in U^n_{\text{ad}}$. Clearly, it is necessary only to verify that

$$C_2\left(1 - \left(\frac{1}{n}\right)^\gamma\right) \leqq \text{meas } \Omega(\varphi_n) \leqq C_2\left(1 + \left(\frac{1}{n}\right)^\gamma\right)$$

(the other properties of φ_n appearing in the definition of U^n_{ad} are satisfied because of the definition of φ_n). We have

$$\begin{aligned}
\text{meas } \Omega(\varphi_n) &= \int_{\Omega(\varphi_n)} \mathrm{d}x = \text{meas } \Omega - \int_\alpha^\beta \int_{\varphi_n}^p \mathrm{d}x = \\
&= \text{meas } \Omega - \int_\alpha^\beta \int_{\varphi_n}^p \mathrm{d}x \pm \int_\alpha^\beta \int_\varphi^p \mathrm{d}x = \\
&= \text{meas } \Omega(\varphi) + \int_\alpha^\beta \left(\int_\varphi^p \mathrm{d}x_1 - \int_{\varphi_n}^p \mathrm{d}x_1\right)\mathrm{d}x_2 \leqq C_2 + \int_\alpha^\beta \left|\int_{\varphi_n}^\varphi \mathrm{d}x_1\right| \mathrm{d}x_2 \\
&\leqq C_2 + c\frac{1}{n} \leqq C_2\left(1 + \left(\frac{1}{n}\right)^\gamma\right)
\end{aligned}$$

because of (4.2), provided n is sufficiently large.

Similarly

$$\text{meas } \Omega(\varphi_n) \geqq C_2\left(1 - \left(\frac{1}{n}\right)^\gamma\right).$$

The main of this section is

Theorem 4.1. *Let $\varphi_n^* \in U^n_{\text{ad}}$, $\varphi_n^*: [\alpha_n^*, \beta_n^*] \to \mathbf{R}^1$ be a solution of $(\mathbf{P})_n$. Then there exist subsequences of $\{\alpha_n^*\}$, $\{\beta_n^*\}$, $\{\varphi_n^*\}$ (denoted by the same symbol) and $\alpha^* \in [0, \bar{\alpha}]$, $\beta^* \in [\bar{\beta}, 1]$, $\varphi^* \in U_{\text{ad}}$, $\varphi^*: [\alpha^*, \beta^*] \to \mathbf{R}^1$ such that*

$$\begin{cases} \alpha_n^* \to \alpha^*\,, \quad \beta_n^* \to \beta^*\,, \quad n \to \infty \\ \varphi_n^* \rightrightarrows \varphi^* \quad \textit{on} \quad I_m = \left[\alpha^* + \dfrac{1}{m}, \ \beta^* - \dfrac{1}{m}\right] \end{cases} \tag{4.3}$$

for any integer m and

$$\varphi^* \textit{ is a solution of } (\mathbf{P})\,. \tag{4.4}$$

Proof. Let $\varphi_n^* \in U^n_{\text{ad}}$, $\varphi_n^*: [\alpha_n^*, \beta_n^*] \to \mathbf{R}^1$ be a solution of $(\mathbf{P})_n$:

$$\mathscr{J}(\varphi_n^*) \leqq \mathscr{J}(\varphi) \quad \forall \varphi \in U^n_{\text{ad}}\,. \tag{4.5}$$

As $\alpha_n^* \in [0, \bar{\alpha}]$, $\beta_n^* \in [\bar{\beta}, 1]$ there exist subsequences of $\{\alpha_n^*\}$, $\{\beta_n^*\}$ (still denoted by the same symbol) and numbers $\alpha^* \in [0, \bar{\alpha}]$, $\beta^* \in [\bar{\beta}, 1]$ such that

$$\alpha_n^* \to \alpha^* , \quad \beta_n^* \to \beta^* , \quad n \to \infty . \tag{4.6}$$

Let m be an integer and I_m as above. Then φ_n^* are defined on I_m (m being fixed) for n sufficiently large. As the sequence $\{\varphi_n^* |_{I_m}\}$ satisfies on I_m all assumptions of the Ascoli-Arzela theorem, there exist a subsequence $\{\varphi_{n^1}^*\} \subset \{\varphi_n^*\}$ and a function $\varphi^{*(m)} \in C(I_m)$ such that

$$\varphi_{n^1}^* \rightrightarrows \varphi^{*(m)} \quad \text{on} \quad I_m .$$

Replacing m by $(m + 1)$, one can find a subsequence $\{\varphi_{n^2}^*\} \subset \{\varphi_{n^1}^*\}$ defined on I_{m+1} and a function $\varphi^{*(m+1)} \in C(I_{m+1})$ such that

$$\varphi_{n^2}^* \rightrightarrows \varphi^{*(m+1)} \quad \text{on } I_{m+1} .$$

Clearly $\varphi^{*(m+1)} = \varphi^{*(m)}$ on I_m. Repeating the same procedure for any integer m and passing to the diagonal subsequence defined by means of $\{\varphi_{n^1}^*\}$, $\{\varphi_{n^2}^*\}, \ldots$ one construct a sequence, denoted by $\{\varphi_n^*\}$, such that

$$\varphi_n^* \rightrightarrows \varphi^* \quad \text{on} \quad I_m \quad \text{where} \quad \varphi^* = \varphi^{*(m)} \quad \text{on} \quad I_m , \quad m \text{ integer.}$$

It is easy to see that $\varphi^* \in U_{\text{ad}}$. Indeed, as

$$C_2\left(1 - \left(\frac{1}{n}\right)^q\right) \leqq \text{meas } \Omega(\varphi_n^*) \leqq C_2\left(1 + \left(\frac{1}{n}\right)^q\right)$$

then

$$\lim_{n \to \infty} \text{meas } \Omega(\varphi_n^*) = C_2 . \tag{4.7}$$

But

$$\text{meas } \Omega(\varphi_n^*) = \text{meas } \Omega - \int_{\alpha_n^*}^{\beta_n^*} \int_{\varphi_n^*}^{p} \mathrm{d}x =$$
$$= \text{meas } \Omega - \int_{I_m} \int_{\varphi_n^*}^{p} \mathrm{d}x - \int_{G_m} \int_{\varphi_n^*}^{p} \mathrm{d}x ,$$

where meas $G_m \to 0$ as $m \to \infty$. Keeping m fixed and $n \to \infty$ we have

$$\lim_{n \to \infty} \text{meas } \Omega(\varphi_n^*) = \text{meas } \Omega - \int_{I_m} \int_{\varphi^*}^{p} \mathrm{d}x - \int_{G_m} \int_{\varphi^*}^{p} \mathrm{d}x .$$

Letting $m \to \infty$ we finally obtain

$$\lim_{n \to \infty} \text{meas } \Omega(\varphi_n^*) = \text{meas } \Omega - \int_{\alpha^*}^{\beta^*} \int_{\varphi^*}^{p} \mathrm{d}x = \text{meas } \Omega(\varphi^*) .$$

Comparing this with (4.7) we see that meas $\Omega(\varphi^*) = C_2$. Further

$$\varphi^*\left(\alpha^* + \frac{1}{m}\right) = \lim_{n \to \infty}\left(\varphi_n^*\left(\alpha^* + \frac{1}{m}\right) - \varphi_n^*(\alpha_n^*)\right) + \lim_{n \to \infty} \varphi_n^*(\alpha_n^*) =$$

$$= \lim_{n \to \infty}\left(\varphi_n^*\left(\alpha^* + \frac{1}{m}\right) - \varphi_n^*(\alpha_n^*)\right) + \lim_{n \to \infty} p(\alpha_n^*) = p(\alpha^*) + c(m) ,$$

where $c(m) \to 0$ if $m \to \infty$. Thus $\varphi^*(\alpha^*) = p(\alpha^*)$ and similarly $\varphi^*(\beta^*) = p(\beta^*)$. The other conditions appearing in the definition of U_{ad} are easily satisfied. Let us prove that

$$(4.8) \qquad \lim_{n\to\infty} \mathscr{I}(\varphi_n^*) = \mathscr{I}(\varphi^*) .$$

Indeed,

$$\int_{\Omega(\varphi_n^*)} f \, dx = \int_\Omega f \, dx - \int_{\alpha_n^*}^{\beta_n^*} \int_{\varphi_n^*}^p f \, dx =$$
$$= \int_\Omega f \, dx - \int_{I_m} \int_{\varphi_n^*}^p f \, dx - \int_{G_m} \int_{\varphi_n^*}^p f \, dx ,$$

where meas $G_m \to 0$ as $m \to \infty$. For m fixed and $n \to \infty$ we have

$$\lim_{n\to\infty} \int_{\Omega(\varphi_n^*)} f \, dx = \int_\Omega f \, dx - \int_{I_m} \int_{\varphi^*}^p f \, dx - \int_{G_m} \int_{\varphi^*}^p f \, dx .$$

Letting $m \to \infty$ we finally obtain

$$(4.9) \qquad \lim_{n\to\infty} \int_{\Omega(\varphi_n^*)} f \, dx = \int_\Omega f \, dx - \int_{\alpha^*}^{\beta^*} \int_{\varphi^*}^p f \, dx = \int_{\Omega(\varphi^*)} f \, dx .$$

Similarly

$$\int_{\Gamma_2^1(\varphi_n^*)} g \, ds = \int_0^{\alpha_n^*} g \sqrt{(1 + (p')^2)} \, dx_1 \overset{n\to\infty}{\to} \int_0^{\alpha^*} g \sqrt{(1 + (p')^2)} \, dx_1 =$$
$$= \int_{\Gamma_2^1(\varphi^*)} g \, ds$$

and

$$\int_{\Gamma_2^2(\varphi_n^*)} g \, ds \overset{n\to\infty}{\to} \int_{\Gamma_2^2(\varphi^*)} g \, ds .$$

Taking into account this, (4.9) and the definition of $\mathscr{I}$, we arrive at (4.8).

Let $\varphi \in U_{ad}$, φ: $[\alpha, \beta] \mapsto \mathbf{R}^1$ be fixed. According to Lemma 4.1 there exists a sequence $\varphi_n \in U_{ad}^n$, φ_n: $[\alpha, \beta] \mapsto \mathbf{R}^1$ and such that (4.1) holds. In the same way as before one can prove that

$$\lim_{n\to\infty} \mathscr{I}(\varphi_n) = \mathscr{I}(\varphi) .$$

From this, (4.8) and (4.5) we get

$$\mathscr{I}(\varphi^*) \leqq \mathscr{I}(\varphi) .$$

As $\varphi \in U_{ad}$ is an arbitrary element this means that φ^* solves (**P**).

5. NUMERICAL EXAMPLES

Let us suppose for simplicity that p is a linear function defined on $[0, 1]$, and let $0 < \bar{\alpha} < \bar{\beta} < 1$. Let n be fixed. The partition $\mathscr{D}_n(\alpha, \beta)$ will contain moving nodes, forming a partition of $[\alpha, \beta]$ (see Fig. 2).

The vector of discrete design variables $\omega = (\omega_1, \ldots, \omega_{2n})$ now contains

(j) the nodes of $\mathscr{D}_n(\alpha, \beta)$;

(jj) the x_2-coordinates of vertices of $\varphi \in U^n_{\mathrm{ad}}$ at a_i, $i = 1, \ldots, n$.

The set U, introduced in Section 3, is a compact subset of $\mathbf{R}^{2n}$.

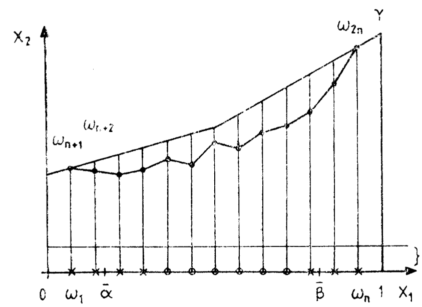

Figure 2.

From the definition of U we see that all constraints with the exception of the last one are linear. In U we take the constants to be

$$\bar{\alpha} = 0{\cdot}15\,, \quad \bar{\beta} = 0{\cdot}85\,, \quad C_1 = 1{\cdot}5\,, \quad C_2 = \tilde{C}_2 = \tilde{C}_3 = 0{\cdot}5894\,,$$

$$\delta = 0{\cdot}1\,, \quad h_{\max}(\omega_1, \omega_n) = 1\,, \quad h_{\min}(\omega_1, \omega_n) = 0{\cdot}02\,, \quad n = 13\,.$$

In order to solve the problem $(P)_n$ numerically one uses some iterative method; typically a gradient type method.

In optimization we apply the Sequential Quadratic Programming method (subroutine E04VDE of NAG-library). Domain integrals are computed using Gaussian quadrature and line integrals with the trapezoidal formula. A sufficient subdivision is performed dynamically in the domains of integration to get accurate results. In sensitivity analysis the method of part I was compared with the finite difference and the algebraic method and all three methods gave the same gradient. In optimization the algebraic gradient, obtained through analytical differentiation of the cost, was used.

In the examples we take Ω to be given by

$$\Omega = \{(x_1, x_2) \in \mathbf{R}^2 \mid 0 < x_2 < p(x_1) = \tfrac{1}{2}x_1 + \tfrac{1}{2}\,, \quad x_1 \in (0, 1)\}\,.$$

We consider two cases:

Example 5.1. Let $x_3 = f(x_1, x_2) = -2\pi^2 \sin(\pi x_1) \sin(\pi x_2)$, $g(x_1, x_2) = -2{\cdot}5\pi\,.$ $.\, \sin(\pi x_1) \cos(\pi x_2)$ and let Ω be given as above. As an initial guess we choose

$$\omega_i = 0{\cdot}125 + (i - 1)\, 0{\cdot}0625\,, \quad i = 1, \ldots, 13$$

and

$$\omega_{14} = 0{\cdot}5625\,, \quad \omega_{15} = 0{\cdot}5\,, \quad \omega_{16} = 0{\cdot}4375\,, \quad \omega_{17} = 0{\cdot}375,$$
$$\omega_{18} = 0{\cdot}3125\,, \quad \omega_{19} = 0{\cdot}3125\,, \quad \omega_{20} = 0{\cdot}40178\,, \quad \omega_{21} = 0{\cdot}49107\,,$$
$$\omega_{22} = 0{\cdot}58035\,, \quad \omega_{23} = 0{\cdot}66964\,, \quad \omega_{24} = 0{\cdot}75892\,, \quad \omega_{25} = 0{\cdot}84821\,,$$
$$\omega_{26} = 0{\cdot}9375\,.$$

In Figure 3 below we have the initial and optimal curves φ fot this example.

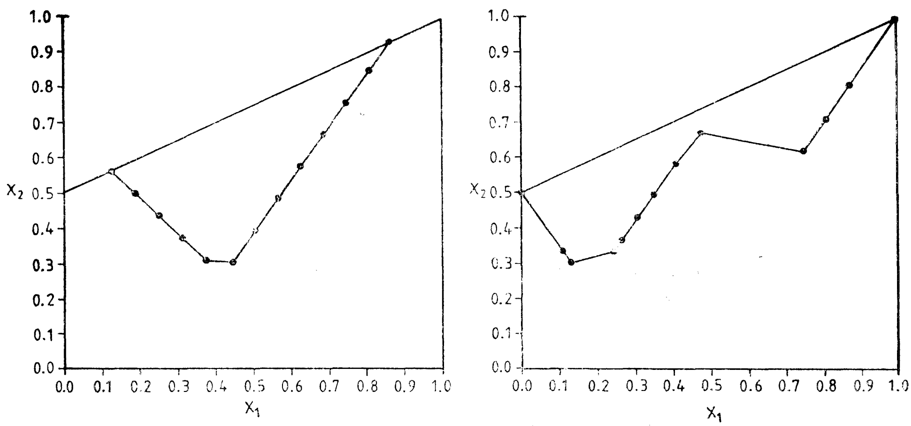

Figure 3.

In Table 1 below we have the values of f along the optimal curve φ_n^*.

Table 1.

x_1	x_2	x_3
0·0000000	0·5000000	0·0000000
0·1092079	0·3361881	− 5·780113
0·1313176	0·3030236	− 6·446706
0·2441563	0·3359311	−11·91932
0·2645934	0·3665868	−13·32052
0·3071481	0·4304188	−15·83973
0·3508292	0·4959405	−17·60962
0·4093626	0·5837405	−18·29257
0·4750012	0·6689263	−16·97169
0·7462183	0·6193624	−13·14121
0·8078977	0·7118464	− 8·811865
0·8717889	0·8076833	− 4·395551
1·000000	1·000000	0·000000

The cost in optimization was reduced from $J_0 = -3{\cdot}93614$ to $J_{opt} = -4{\cdot}88278$.

Example 5.2. Let $x_3 = f(x_1, x_2) = x_1^2 + x_2^2$, $g(x_1, x_2) = 1$ and let Ω be given as above. Moreover, let the initial guess be the same as in Example 5.1. In Figure 4 below we have the initial and optimal curves φ for this example.

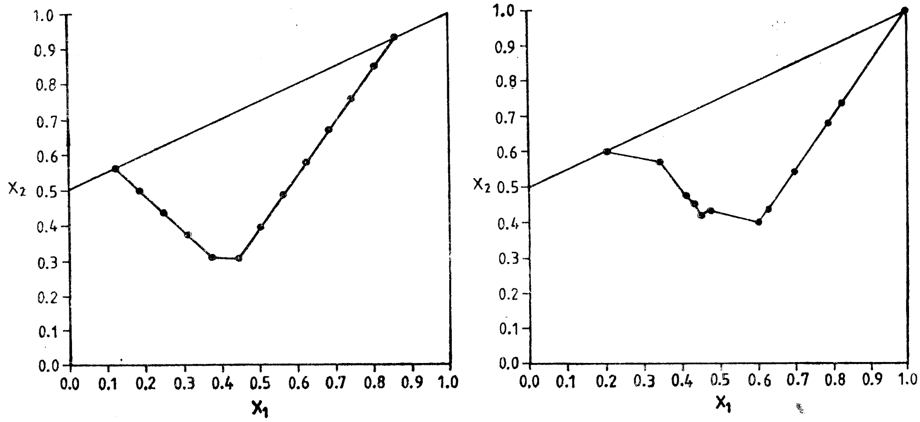

Figure 4

In Table 2 below we have the values of f along the optimal curve φ_n^*.

Table 2.

x_1	x_2	x_3
0·0000000	0·5000000	0·2500000
0·2072488	0·5999357	0·4028749
0·3436032	0·5744058	0·4480052
0·4082979	0·4785570	0·3957239
0·4289429	0·4560512	0·3919747
0·4492524	0·4260519	0·3833479
0·4692871	0·4344718	0·4089961
0·5995702	0·4016983	0·5208459
0·6263743	0·4395616	0·5855591
0·6976938	0·5465407	0·7854833
0·7892034	0·6838051	1·090431
0·8258656	0·7387984	1·227877
1·000000	1·000000	2·000000

From the table we see that f is constant along most of the optimal curve φ, which corresponds to the theory from [1]. The cost in optimization was reduced from $J_0 = 0·6616$ to $J_{opt} = 0·3124$.

References

[1] *J. Haslinger, V. Horák:* Identification of critical curves. Part I: Continuous case, Apl. Mat. 35 (1990), 169—177.

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