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HYSTERESIS MEMORY PRESERVING OPERATORS

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Summary. The recent development of mathematical methods of investigation of problems with hysteresis has shown that the structure of the hysteresis memory plays a substantial role. In this paper we characterize the hysteresis operators which exhibit a memory effect of the Preisach type (memory preserving operators). We investigate their properties (continuity, invertibility) and we establish some relations between special classes of such operators (Preisach, Ishlinskii and Nemytskii operators). For a general memory preserving operator we derive an energy inequality.

Keywords: Hysteresis memory, Preisach operators, memory preserving operators, energy inequality.

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The latest development of the mathematical theory of hysteresis has shown that the structure of the memory plays an important role in solving partial differential equations arising in various problems of mathematical physics. Hysteresis memory appears naturally in evolution differential inequalities. The construction of the solution ([6], [7], [8]) is based on estimated of the dissipation of energy which is closely related to the memory effect.

One of the classical models of hysteresis is the Preisach operator. It turns out that the inversion and superposition of Preisach operators are in general not Preisach, but they exhibit the same memory effects. This leads us to consider a wider class of operators (memory preserving operators) characterized by the "Preisach" structure of memory. We specify the relationship between various models of hysteresis (Prandtl, Ishlinskii, Preisach, mh-hysterons, moving model etc.) and we find sufficient conditions for the validity of an energy inequality.

The concept of memory preserving operator is developed here only for odd operators, but in special cases (for instance in the Preisach case) the oddness is not substantial. Let us note that the preservation of memory implies the wiping-out property (cf. [1], [9]).

(1.1) **Definition** ([5], [6]). *For a piecewise monotone function* $u \in C([0, T])$ *and for* $h \geqq 0$ *we put*

$$l_h(u)(0) = \operatorname{sign}(u(0)) \max\{0, |u(0)| - h\},$$

$$l_h(u)(t) = \begin{cases} \max\{l_h(u)(t_0), u(t) - h\}, & t \in (t_0, t_1], \\ \quad \text{if } u \text{ is nondecreasing in } [t_0, t_1], \\ \min\{l_h(u)(t_0), u(t) + h\}, & t \in (t_0, t_1], \\ \quad \text{if } u \text{ is nonincreasing in } [t_0, t_1]. \end{cases}$$

(1.2) **Remarks**

(i) The operator l_h is sometimes called "play". It represents the plastic deformation in Prandtl's model of elastoplasticity.

(ii) We have $l_0(u)(t) = u(t)$.

(iii) The function $\xi(t) = l_h(u)(t)$ for $u \in W^{1,1}(0, T)$ is the unique solution of the inequality (see [8])

$$\xi'(u - \xi - \phi) \geqq 0 \quad \forall \phi \in [-h, h],$$
$$|u - \xi| \leqq h,$$
$$\xi(0) = \operatorname{sign}(u(0)) \max\{0, |u(0)| - h\}.$$

(iv) The operator l_h can be extended to a Lipschitz continuous operator in $C([0, T])$. More precisely, for every $u, v \in C([0, T])$ and $t \in [0, T]$ we have

$$|l_h(u)(t) - l_h(v)(t)| \leqq \|u - v\|_{[0,t]} \equiv$$
$$\equiv \max\{|u(s) - v(s)|, s \in [0, t]\} \quad (\text{see } [5], [8]).$$

In [6] we can find another formula for the operator l_h which reflects the structure of the memory: it shows which values from the "past" of the input function u are necessary for determining the value of $l_h(u)(t)$.

(1.3) **Definition.** Let $u \in C([0, T])$, $t \in [0, T]$ be given. The *memory sequence of* u *at the point* t (denoted by $MS(u)(t)$) is the sequence (finite or infinite) $\{(t_j, h_j)\}$, $0 \leqq t_j < t_{j+1} \leqq t$, $0 < h_{j+1} < h_j \leqq \|u\|_{[0,t]}$ given by the formulas:

$$\boldsymbol{t} = \max\{\tau \in [0, t];\ |u(\tau)| = \|u\|_{[0,t]}\},$$
$$\boldsymbol{t} = t_0 \quad \text{for} \quad u(\boldsymbol{t}) < 0, \quad \boldsymbol{t} = t_1 \quad \text{for} \quad u(\boldsymbol{t}) \geqq 0,$$

and by the induction

$$t_{2k} = \max\{\tau \in [t_{2k-1}, t];\ u(\tau) = \min\{u(\sigma);\ \sigma \in [t_{2k-1}, t]\}\},$$
$$t_{2k+1} = \max\{\tau \in [t_{2k}, t];\ u(\tau) = \max\{u(\sigma);\ \sigma \in [t_{2k}, t]\}\}$$

until $t_n = t$, and

$$h_i = \boldsymbol{h} = |u(\boldsymbol{t})| \quad \text{for} \quad \boldsymbol{t} = t_i, \quad i = 0 \quad \text{or} \quad 1;$$

for higher indices we put

$$h_j = \tfrac{1}{2}(-1)^j (u(t_{j-1}) - u(t_j)) .$$

(1.4) **Lemma** (see [6]). *For $u \in C([0, T])$, $t \in [0, T]$, $h \geqq 0$ we have*

$$l_h(u)(t) = \operatorname{sign}(u(t)) \max\{0, |u(t)| - h\} ,$$
$$l_h(u)(t_{2n}) = l_h(u)(t_{2k-1}) - 2\max\{0, h_{2k} - h\} ,$$
$$l_h(u)(t_{2k+1}) = l_h(u)(t_{2k}) + 2\max\{0. h_{2k+1} - h\} .$$

If the sequence $MS(u)(t)$ is infinite, then $u(t_j) \to u(t)$, hence $h_j \searrow 0$ as $j \to \infty$ If $MS(u)(t)$ is finite (for instance $t_n = t$), we define $h_{n+1} = h_{n+2} = \ldots = 0$. This enables us to rewrite the formulas in Lemma (1.4):

$$(1.5) \qquad \begin{cases} l_h(u)(t) = (-1)^{k+1}(h_k - h) + \sum_{j=0/1}^{k-1} (-1)^{j+1}(h_j - h_{j+1}) \\ \qquad \text{for} \quad h \in [h_{k+1}, h_k] , \\ l_h(u)(t) = 0 \quad \text{for} \quad h \geqq \boldsymbol{h} , \end{cases}$$

$$(1.6) \qquad u(t) = \sum_{j=0/1}^{\infty} (-1)^{j+1}(h_j - h_{j+1}) .$$

2. MEMORY PRESERVING OPERATORS

(2.1) **Definition.** An operator P: $C([0, T]) \to C([0, T])$ is called:

rate independent, if for every $u \in C([0, T])$ and every increasing bijection s: $[0, T] \to [0, T]$ we have $P(u \circ s) = P(u) \circ s$;

memory preserving, if there exist functions Φ_k: $D_k \to (0, \infty)$, $D_k = \{(p_k, \ldots, p_1) \in \in R^k, 0 < p_k < \ldots < p_1\}$, $k = 1, 2, \ldots$, such that the following implication holds:

If $MS(u)(t) = \{(t_j, h_j)\}$, then $MS(P(u))(t) = \{(t_j, r_j)\}$, where

$$(2.2) \qquad \begin{cases} r_1 = \Phi_1(h_1) \\ r_2 = \Phi_2(h_2, h_1) \\ \quad \vdots \\ r_k = \Phi_k(h_k, \ldots, h_1) \end{cases} \quad \text{or} \quad \begin{cases} r_0 = \Phi_1(h_0) \\ r_1 = \Phi_2(h_1, h_0) \\ \\ r_{k-1} = \Phi_k(h_{k-1}, \ldots, h_0) \end{cases}$$

for $\boldsymbol{t} = t_1$ or $\boldsymbol{t} = t_0$, respectively.

(2.3) **Lemma.** *Every memory preserving operator is odd and rate independent.*

Proof. Applying formula (1.6) to $P(u)$ we obtain

$$(2.4) \qquad P(u)(t) = \sum_{j=0/1}^{\infty} (-1)^{j+1}(r_j - r_{j+1})$$

and Lemma (2.3) follows easily. ∎

Motivation. The functions Φ_k are those which we obtain from experimental data by making measurements of the output. The function Φ_1 (in ferromagnetism) describes the "primary magnetization curve", Φ_2 characterizes the "first order loop" corresponding to simple harmonic inputs, and the functions Φ_k for $k \geqq 3$ determine minor loops of k-th memory level. We use the term *identification functions*.

(2.5) **Lemma.** *The function $h \mapsto \Phi_k(h, h_{k-1}, \ldots, h_1)$ is nondecreasing in $(0, h_{k-1})$ for every $k \geqq 1$ and every $(h_{k-1}, \ldots, h_1) \in D_{k-1}$. Moreover, if the operator P is injective, then this function is increasing.*

Proof. Let $0 = t_1 < t_2 < \ldots < t_k = T$ be a partition of $[0, T]$. We choose the function $u \in C([0, T])$ to be linear in $[t_{j-1}, t_j]$, $j = 2, \ldots, k$, $u(0) = h_1$, $u(t_j) = (-1)^{j+1} h_j + \sum_{i=1}^{j-1} (-1)^{i+1} (h_i - h_{i+1})$ for $j = 2, \ldots, k-1$, $u(t_k) = u(t_{k-2})$. For $t \in (t_{k-1}, t_k)$ we have

$$MS(u)(t) = \{(t_1, h_1), \ldots, (t_{k-1}, h_{k-1}), (t, \hat{h}_k(t))\},$$

where $\hat{h}_k(t) = \frac{1}{2}(-1)^{k+1}(u(t) - u(t_{k-1}))$, $\hat{h}_k(t_k) = h_{k-1}$.

Put $w = P(u)$. For $t \in (t_{k-1}, t_k)$ we have

$$MS(w)(t) = \{(t_1, r_1), \ldots, (t_{k-1}, r_{k-1}), (t, \hat{r}_k(t))\},$$

where $r_j = \Phi_j(h_j, \ldots, h_1)$ for $j = 1, \ldots, k_1$,

$$\hat{r}_k(t) = \Phi_k(\hat{h}_k(t), h_{k-1}, \ldots, h_1).$$

Let us assume that $\Phi_k(\cdot, h_{k-1}, \ldots, h_1)$ is not nondecreasing. Then there exist $t_{k-1} < \sigma < \tau < T$ such that $\hat{r}_k(\sigma) > \hat{r}_k(\tau)$. By (2.4) we have

$$w(\sigma) - w(\tau) = 2(-1)^{k+1} (\hat{r}_k(\sigma) - \hat{r}_k(\tau)).$$

Therefore, $MS(w)(\tau)$ contains a point $(\tau_k, \hat{r}_k(\tau_k))$ for some $\tau_k \in (t_{k-1}, \tau)$, but this contradicts the memory preservation property.

Let P be injective and let us assume $\hat{r}_k(\sigma) = \hat{r}_k(\tau)$ for some $t_{k-1} < \sigma < \tau < T$. We put $u_1(t) = u(t)$ for $t \in [0, \sigma]$, $u_1(t) = u(\sigma)$ for $t \in (\sigma, T)$, $u_2(t) = u(t)$ for $t \in [0, \tau]$, $u_2(t) = u(\tau)$ for $t \in (\tau, T)$. We see that (2.4) implies $P(u_1) = P(u_2)$, but this is a contradiction. ∎

The following concept is closely related to the "erasure of memory", cf. [1].

(2.6) **Definition.** We say that the system $\{\Phi_k: k = 1, 2, \ldots\}$ of identification functions is *reducible*, if

(i) Φ_k is continuous in $\bar{D}_k$ for $k = 1, 2, \ldots$,

(ii) for every $k \geqq 2$ and $0 \leqq h_{k-1} \leqq \ldots \leqq h_1$ we have

$$\Phi_k(0, h_{k-1}, \ldots, h_1) = 0,$$
$$\Phi_k(h_{k-1}, h_{k-1}, \ldots, h_1) = \Phi_{k-1}(h_{k-1}, \ldots, h_1),$$
$$\Phi_k(h_{k-1}, h_{k-2}, \ldots, h_1, h_1) = \Phi_{k-1}(h_{k-1}, \ldots, h_1),$$

(iii) for every $k \geqq 3$, $3 \leqq j \leqq k$, $0 \leqq h_{k+1} \leqq h_k \leqq \ldots \leqq h_1$ we have

$$\Phi_{k+1}(h_{k+1}, \ldots, h_{j+1}, h_j, h_j, h_{j-2}, \ldots, h_1) =$$
$$= \Phi_{k-1}(h_{k+1}, \ldots, h_{j+1}, h_{j-2}, \ldots, h_1).$$

(2.7) **Proposition.** *Let P: $C([0, T]) \to C([0, T])$ be a continuous memory preserving operator. Then its identification functions are reducible.*

Proof. Let us choose arbitrarily $k \geqq 1$, $(h_k, \ldots, h_1) \in \bar{D}_k$ and a sequence $\{(h_k^{(n)}, \ldots, h_1^{(n)}\} \subset D_k$ such that $h_j^{(n)} \to h_j$ as $n \to \infty$. We construct a partition $0 = t_1 < t_2 < \ldots < t_k = T$ and define functions $u^{(n)}$, u to be linear in every interval $[t_{j-1}, t_j]$, $j = 2, \ldots, k$,

$$u(t_j) = (-1)^{j+1} h_j + \sum_{i=1}^{j-1} (-1)^{i+1} (h_i - h_{i+1}),$$

$$u^{(n)}(t_j) = (-1)^{j+1} h_j^{(n)} + \sum_{i=1}^{j-1} (-1)^{i+1} (h_i^{(n)} - h_{i+1}^{(n)}), \quad j = 1, \ldots, k.$$

We have $\|u^{(n)} - u\|_{[0,T]} \leqq 2 \sum_{i=1}^{k} |h_i^{(n)} - h_i|$, hence $u^{(n)} \to u$.

Put $w = P(u)$, $w^{(n)} = P(u^{(n)})$. By hypothesis we have $\|w^{(n)} - w\|_{[0,T]} \to 0$. On the other hand, we have

$$\Phi_k(h_k^{(n)}, \ldots, h_1^{(n)}) = \tfrac{1}{2}(-1)^{k+1} (w^{(n)}(t_k) - w^{(n)}(t_{k-1})),$$

hence

$$|\Phi_k(h_k^{(n)}, \ldots, h_1^{(n)}) - \Phi_k(h_k^{(m)}, \ldots, h_1^{(m)})| \leqq \|w^{(n)} - w^{(m)}\|_{[0,T]}$$

for arbitrary m, n.

This proves the continuity of Φ_k in $\bar{D}_k$. It remains to verify (2.6) (ii), (iii). We prove for example

$$\Phi_k(h_{k-1}, h_{k-1}, \ldots, h_1) = \Phi_{k-1}(h_{k-1}, \ldots, h_1)$$

(the other relations are analogous).

We choose $0 < h_{k-1} < \ldots < h_1$ and a sequence $0 < h_k^{(n)} \nearrow h_{k-1}$. Using the partition $0 = t_1 < \ldots < t_k = T$ as before we construct functions $u, u^{(n)}$ which are linear in $[t_{j-1}, t_j]$, $j = 2, \ldots, k$ and

$$u(t_j) = u^{(n)}(t_j) = (-1)^{j+1} h_j + \sum_{i=1}^{j-1} (-1)^{i+1} (h_i - h_{i+1}),$$

$$j = 1, \ldots, k - 1,$$

$$u(T) = u(t_{k-1}) + 2(-1)^{k+1} h_{k-1} = u(t_{k-2}),$$

$$u^{(n)}(T) = u^{(n)}(t_{k-1}) + 2(-1)^{k+1} h_k^{(n)}.$$

We have indeed $\|u - u^{(n)}\|_{[0,T]} = |h_{k-1} - h_k^{(n)}| \to 0$. Put $w = P(u)$, $w^{(n)} = P(u^{(n)})$. We have

$$\begin{aligned} MS(u^{(n)})(T) &= \{(t_1, h_1), \dots, (t_{k-2}, h_{k-2}), (t_{k-1}, h_{k-1}), (T, h_k^{(n)})\}, \\ MS(w^{(n)})(T) &= \{(t_1, r_1), \dots, (t_{k-2}, r_{k-2}), (t_{k-1}, r_{k-1}), (T, r_k^{(n)})\}, \\ MS(u)(T) &= \{(t_1, h_1), \dots, (t_{k-3}, h_{k-3}), (T, h_{k-2})\}, \\ MS(w)(T) &= \{(t_1, h_1), \dots, (t_{k-3}, h_{k-3}), (T, r_{k-2})\}, \end{aligned}$$

where

$$\begin{aligned} r_j &= \Phi_j(h_j, \dots, h_1), \quad j = 1, \dots, k-1, \\ r_k^{(n)} &= \Phi_k(h_k^{(n)}, h_{k-1}, \dots, h_1). \end{aligned}$$

Using (2.4) we obtain $|w^{(n)}(T) - w(T)| = 2|r_{k-1} - r_k^{(n)}|$ and the continuity argument completes the proof. ■

(2.8) **Proposition.** *Let P be an injective continuous memory preserving operator, $\lim_{h\to\infty} \Phi_1(h) = \infty$. Let us assume $r_k \to 0 \Leftrightarrow h_k \to 0$ in (2.2). Then P is bijective and P^{-1} is a memory preserving operator with reducible identification functions.*

Proof. Let $w \in C([0, T])$ be given $MS(w)(t) = \{(t_j, r_j)\}$. Lemma (2.5) and Proposition (2.7) enable us to compute $h_1, h_2, \dots$ from (2.2) in the form

$$\begin{aligned} h_1 &= \Psi_1(r_1) \\ h_2 &= \Psi_2(r_2, r_1) \\ &\vdots \\ h_k &= \Psi_k(r_k, \dots, r_1). \end{aligned}$$

We can define an operator $Q : w \mapsto u$, where $u(t)$ is given by (1.6). The functions Ψ_k are reducible identification functions for Q. A straightforward computation shows that Q maps $C([0, T])$ into $C([0, T])$ and $P_0Q = Q_0P$ is the identity.

Notation. Let $h_0 \geqq h_1 \geqq h_2 \geqq \dots$ be a given sequence, $h_j \searrow 0$. We put

$$(2.9) \quad \begin{cases} \lambda(h, \{h_j\}) = (-1)^{k+1}(h_k - h) + \sum_{j=0}^{k-1} (-1)^{j+1}(h_j - h_{j+1}) \\ \qquad \text{for} \quad h \in [h_{k+1}, h_k), \\ \lambda(h, \{h_j\}) = 0 \quad \text{for} \quad h \geqq h_0. \end{cases}$$

For two sequences $\{h_j\}$, $\{\tilde{h}_j\}$, $h_j \searrow 0$, $\tilde{h}_j \searrow 0$ we denote

$$(2.10) \qquad d(\{h_j\}, \{\tilde{h}_j\}) = \max\{|h_0 - \tilde{h}_0|, \max_{h \geqq 0} |\lambda(h, \{h_j\}) - \lambda(h, \{\tilde{h}_j\})|\}.$$

Remark. Let $u, \tilde{u} \in C([0, T])$ be given functions and let $t \in [0, T]$ be fixed, $MS(u)(t) = \{(t_j, h_j)\}$, $MS(\tilde{u})(t) = \{(\tilde{t}_j, \tilde{h}_j)\}$. Formula (1.5) yields

$$\lambda(h, \{h_j\}) = l_h(u)(t), \quad \lambda(h, \{\tilde{h}_j\}) = l_h(\tilde{u})(t)$$

provided we put $h_0 = h_1$ if $\boldsymbol{h} = h_1$ and the same for $\tilde{h}_0$.

By (1.2) (iv) we have

$$d(\{h_j\}, \{\tilde{h}_j\}) \leqq \|u - \tilde{u}\|_{[0,t]}.$$

The converse is also true.

(2.11) **Lemma.** *Let $h_j \searrow 0$, $\tilde{h}_j \searrow 0$ be two sequences. Then there exist functions $u, \tilde{u} \in C([0, T])$ such that $\lambda(h, \{h_j\}) = l_h(u)(T)$, $\lambda(h, \{\tilde{h}_j\}) = l_h(\tilde{u})(T)$ for every $h \geqq 0$ and*

$$\|u - \tilde{u}\|_{[0,T]} \leqq d(\{h_j\}, \{\tilde{h}_j\}).$$

Proof. Let us assume for instance $h_0 = h_1 > \tilde{h}_0 = \tilde{h}_1 > 0$ (the other cases are analogous). We find $k \geqq 1$ such that $h_k \geqq \tilde{h}_1 > h_{k+1}$ and put

$$\begin{aligned}
h_j^* &= h_j \quad \text{for } j = 1, \ldots, k, \\
\tilde{h}_j^* &= \tilde{h}_1 \quad \text{for } j = 1, \ldots, k, \quad \text{if } k \text{ is odd}, \\
\tilde{h}_j^* &= \tilde{h}_1 \quad \text{for } j = 1, \ldots, k-1, \ \tilde{h}_k^* = \tilde{h}_2, \qquad \text{if } k \text{ is even}, \ \tilde{h}_2 \geqq h_{k+1}, \\
\tilde{h}_j^* &= \tilde{h}_1 \quad \text{for } j = 1, \ldots, k+1, \ h_{k+1}^* = h_{k+1}, \ \text{if } k \text{ is even}, \ \tilde{h}_2 < h_{k+1}.
\end{aligned}$$

We construct the sequences $\{h_j^*\}, \{\tilde{h}_j^*\}$ so that they satisfy the following requirements:

(a) $\lambda(h, \{h_j\}) = \lambda(h, \{h_j^*\})$, $\lambda(h, \{\tilde{h}_j\}) = \lambda(h, \{\tilde{h}_j^*\})$ for every $h \geqq 0$,

(b) the open interval $(\tilde{h}_j^*, h_j^*)$ or $(h_j^*, \tilde{h}_j^*)$ does not contain any other element of $\{h_i^*, \tilde{h}_i^*, h_i, \tilde{h}_i, i = 1, 2, \ldots\}$.

For $j \leqq k$ and $h \geqq \tilde{h}_k^*$ (a), (b) hold. For $j > k$ we proceed by induction. Let us assume that (b) holds for $j \leqq l$ and (a) holds for $h \geqq \min\{h_l^*, \tilde{h}_l^*\}$.

Let us assume for instance $h_l^* \geqq \tilde{h}_l^*$.

Let $h_m = \max\{h_i; h_i < h_l^*\}$, $\tilde{h}_n = \max\{\tilde{h}_i; \tilde{h}_i < \tilde{h}_l^*\}$. Three cases are possible:

(i) $\tilde{h}_l^* > \tilde{h}_n \geqq h_m \geqq \tilde{h}_{n+1}$ or $\tilde{h}_l^* \geqq h_m \geqq \tilde{h}_n \geqq h_{m+1}$,

(ii) $\tilde{h}_l^* > \tilde{h}_n \geqq \tilde{h}_{n+1} > h_m$,

(iii) $\tilde{h}_l^* \geqq h_m \geqq h_{m+1} > \tilde{h}_n$,

and we put respectively

(i) $\tilde{h}_{l+1}^* = \tilde{h}_n$, $h_{l+1}^* = h_m$,

(ii) $\tilde{h}_{l+1}^* = \tilde{h}_n$, $h_{l+1}^* = h_{l+2}^* = \tilde{h}_{l+2}^* = \tilde{h}_{n+1}$,

(iii) $h_{l+1}^* = h_m$, $\tilde{h}_{l+1}^* = \tilde{h}_{l+2}^* = h_{l+2}^* = h_{m+1}$.

We verify easily that (a), (b) hold for $j \leqq l + 1$ and $h \geqq \min\{h_{l+1}^*, \tilde{h}_{l+1}^*\}$.

We choose an arbitrary sequence $0 = t_1 < t_2 < \ldots \leqq T$, $t_j \nearrow T$ as $j \to \infty$. The functions $u, \tilde{u}$ are linear in each interval $[t_j, t_{j+1}]$, and

$$\left\{\begin{aligned} u(t_j) &= \lambda(h_j^*, \{h_i^*\}) + (-1)^{j+1} h_j^* \\ \tilde{u}(t_j) &= \lambda(\tilde{h}_j^*, \{\tilde{h}_i^*\}) + (-1)^{j+1} \tilde{h}_j^*. \end{aligned}\right. \tag{2.12}$$

For $1 \leqq j < k$ we have $\lambda(\tilde{h}_j^*, \{\tilde{h}_i^*\}) = 0$, hence

$$|u(t_j) - \tilde{u}(t_j)| = |\lambda(h_j, \{h_i\}) + (-1)^{j+1}(h_j - \tilde{h}_1)| \leqq$$
$$\leqq (h_1 - h_2) + (h_2 - h_3) + \ldots + (h_j - \tilde{h}_1) = h_1 - \tilde{h}_1 .$$

For $j \geqq k$ we have either $h_j^* \in [\tilde{h}_{j+1}^*, \tilde{h}_j^*]$, hence $\lambda(h_j^*, \{\tilde{h}_i^*\}) - \lambda(\tilde{h}_j^*, \{\tilde{h}_i^*\}) = = (-1)^j (h_j^* - \tilde{h}_j^*)$ and therefore

$$|u(t_j) - \tilde{u}(t_j)| = |\lambda(h_j^*, \{h_i^*\}) - \lambda(h_j^*, \{\tilde{h}_i^*\})| \leqq d(\{h_i^*\}, \{\tilde{h}_i^*\}) =$$
$$= d(\{h_i\}, \{\tilde{h}_i\}) ,$$

or $\tilde{h}_j^* \in [h_{j+1}^*, h_j^*]$ with the same conclusion.

Consequently, $\|u - \tilde{u}\|_{[0,T]} \leqq d(\{h_i\}, \{\tilde{h}_i\})$ and it remains to verify that $\lambda(h, \{h_i^*\}) = l_h(u)(T)$, $\lambda(h, \{\tilde{h}_i^*\}) = l_h(\tilde{u})(T)$. This follows from the formula

$$h_{j+1}^* = \tfrac{1}{2}(-1)^{j+1}(u(t_j) - u(t_{j+1})) \quad \tilde{h}_{j+1}^* = \tfrac{1}{2}(-1)^{j+1}(\tilde{u}(t_j) - \tilde{u}(t_{j+1})).$$

Lemma (2.11) is proved. ∎

(2.13) **Theorem.** *Let P: $C([0, T]) \to C([0, T])$ be a memory preserving operator with identification functions $\{\Phi_k\}$. Then the following conditions are equivalent:*

(i) *P is locally Lipschitz, i.e. Lipschitz on bounded sets;*

(ii) *there exists a nondecreasing function ψ: $[0, \infty) \to [0, \infty)$ such that for every pair of sequences $h_j \searrow 0$, $\tilde{h}_j \searrow 0$ we have $d(\{r_j\}, \{\tilde{r}_j\}) \leqq \leqq \psi(\max\{\boldsymbol{h}, \tilde{\boldsymbol{h}}\})\, d(\{h_j\}, \{\tilde{h}_j\})$, where $r_j, \tilde{r}_j$ are given by (2.2) for $\{h_j\}, \{\tilde{h}_j\}$, respectively;*

(iii) $|\lambda(0, \{r_j\}) - \lambda(0, \{\tilde{r}_j\})| \leqq \psi(\max\{\boldsymbol{h}, \tilde{\boldsymbol{h}}\})\, d(\{h_j\}, \{\tilde{h}_j\})$.

Proof. The implication (ii) $\Rightarrow$ (iii) is trivial. Let us assume that (iii) holds. For arbitrary functions $u, \tilde{u} \in C([0, T])$ and $t \in [0, T]$ we find $MS(u)(t) = \{(t_j, h_j)\}$, $MS(\tilde{u})(t) = \{(\tilde{t}_j, \tilde{h}_j)\}$. We have $|P(u)(t) - P(\tilde{u})(t)| = |\lambda(0, \{r_j\}) - \lambda(0, \{\tilde{r}_j\})| \leqq \leqq \psi(\max\{\|u\|_{[0,t]}, \|\tilde{u}\|_{[0,t]}\})\, \|u - \tilde{u}\|_{[0,t]}$, hence (i) is verified.

Finally, assume that (i) holds and let $h_j \searrow 0$, $\tilde{h}_j \searrow 0$ be given sequences. Following Lemma (2.11) we construct the functions $u, \tilde{u}$ such that $\|u - \tilde{u}\|_{[0,T]} = d(\{h_j\}, \{\tilde{h}_j\})$. Putting $w = P(u)$, $\tilde{w} = P(\tilde{u})$ we have

$$d(\{r_j\}, \{\tilde{r}_j\}) \leqq \|w - \tilde{w}\|_{[0,T]} \quad \text{and (ii) follows easily.}$$

∎

(2.14) **Corollary.** *Let P be a memory preserving operator with identification functions $\{\Phi_k\}$ such that $\lim_{h\to\infty} \Phi_1(h) = \infty$ and there exist a nondecreasing function ψ and a positive nonincreasing function φ such that for every pair of sequences $h_j \searrow 0$ $\tilde{h}_j \searrow 0$ we have*

$$\varphi(\max\{\boldsymbol{h}, \tilde{\boldsymbol{h}}\})\, d(\{h_j\}, \{\tilde{h}_j\}) \leqq d(\{r_j\}, \{\tilde{r}_j\}) \leqq$$
$$\leqq \psi(\max\{\boldsymbol{h}, \tilde{\boldsymbol{h}}\})\, d(\{h_j\}, \{\tilde{h}_j\}) .$$

Then P, P^{-1} are locally Lipschitz memory preserving operators.

Remark. Corollary (2.14) follows immediately from (2.8) and (2.13). We see that the system of memory preserving operators satisfying the hypotheses of (2.14) is a group with respect to superposition. The unit element is the identity with identification functions $\Phi_k(h_k, \ldots, h_1) = h_k$.

The forth coming sections are devoted to the study of important particular cases.

3. ISHLINSKII OPERATOR

Let us consider a continuous memory preserving operator P which is *shift invariant*, i.e. the form of the hysteresis loop is independent of its starting point in the (u, w)-plane. In terms of identification functions this means that Φ_k depends only on h_k. Proposition (2.7) yields $\Phi_k = \Phi_{k-1} = \ldots = \Phi_1$. In particular, the form of the hysteresis loops is independent of the memory level.

Let us assume that Φ_1 is an increasing absolutely continuous function and Φ_1' has locally bounded variation. Let $u \in C([0, T])$, $t \in [0, T]$ be given, $MS(u)(t) = \{(t_j, h_j)\}$. By (2.4) we have

$$P(u)(t) = \sum_{j=0/1}^{\infty} (-1)^{j+1} (\Phi_1(h_j) - \Phi_1(h_{j+1})) =$$

$$= \sum_{j=0/1}^{\infty} \int_{h_{j+1}}^{h_j} (-1)^{j+1} \Phi_1'(h) \, \mathrm{d}h = - \int_0^{\infty} \frac{\mathrm{d}\lambda}{\mathrm{d}h} (h, \{h_j\}) \, \Phi_1'(h) \, \mathrm{d}h =$$

$$= \Phi_1'(0) \, \lambda(0, \{h_j\}) + \int_0^{\infty} \lambda(h, \{h_j\}) \, \mathrm{d}\Phi_1'(h) \,,$$

hence

$$P(u)(t) = \Phi_1'(0) \, u(t) + \textstyle\int_0^\infty l_h(u)(t) \, \mathrm{d}\Phi_1'(h) \,, \tag{3.1}$$

which is the classical definition of the Ishlinskii operator (cf. [5], [6]). We conclude that the following theorem holds:

(3.2) **Theorem.** *Let Φ, Φ^{-1} be increasing functions in $[0, \infty)$, $\Phi(0) = 0$, whose derivatives have locally bounded variation. Let P be the memory preserving operator with identification function $r_k = \Phi(h_k)$. Then* **P**, **P**$^{-1}$ *are locally Lipschitz continuous Ishlinskii operators with generating functions Φ, Φ^{-1}, respectively.*

Example. Putting $\Phi(x) = x$ for $x \in [0, h]$, $\Phi(x) = c(x - h) + h$ for $x > h$ we obtain

$$(I + (c - 1) \, l_h)^{-1} = \left(I + \left(\frac{1}{c} - 1\right) l_h\right)$$

where I is the identity and $c > 0$ is a constant.

Let us investigate the case where the form of the hysteresis loops in the (u, w)-plane depends only on the u-coordinate of its starting point (congruency of hysteresis loops, cf. [1], [9]). We say that the memory preserving operator is *vertical-shift invariant.*

Let $u \in C([0, T])$ be given, $t \in [0, T]$, $MS(u)(t) = \{(t_j, h_j)\}$ and let us assume for instance $\boldsymbol{h} = h_1$. We denote

$$u_j = u(t_j) = (-1)^{j+1} h_j + \sum_{i=1}^{j-1} (-1)^{i+1} (h_i - h_{i+1}) .$$

The vertical-shift invariance means that the value of the identification function $r_k = \Phi_k(h_k, \ldots, h_1)$ depends only on h_k and u_{k-1}. In order words,

$$\Phi_k(h_k, h_{k-1}, \ldots, h_1) = \Phi_k(h_k, h_{k-1} - \xi_{k-1}, \ldots, h_1 - \xi_1)$$

for every $k \geqq 3$, $h_1 \geqq h_2 \geqq \ldots \geqq h_k$ and $\xi_1, \ldots, \xi_{k-1}$ such that $h_1 - \xi_1 \geqq \ldots \ldots \geqq h_{k-1} - \xi_{k-1} \geqq h_k$ satisfying the condition that u_{k-1} is independent of $\{\xi_i\}$, i.e.

$$(-1)^k \xi_{k-1} + \sum_{i=1}^{k-2} (-1)^{i+1} (\xi_i - \xi_{i+1}) = 0 .$$

We put the last relation into the form

$$\xi_1 = 2 \sum_{j=2}^{k-1} (-1)^j \xi_j . \tag{4.1}$$

Let us choose $\xi_j = h_j - h_k$ for $j = 3, \ldots, k - 1$.

Using the identity

$$2 \sum_{j=2}^{k-1} (-1)^j h_j = h_1 + (-1)^{k+1} h_k - \lambda(h_k, \{h_j\})$$

we obtain from (4.1)

$$h_2 - \xi_2 = \tfrac{1}{2}(h_1 - \xi_1 + h_k - \lambda(h_k, \{h_j\})) .$$

The reducibility of Φ_k implies

$$r_k = \Phi_3(h_k, \tfrac{1}{2}(h_1 - \xi_1 + h_k - \lambda(h_k, \{h_j\})), h_1 - \xi_1) , \tag{4.2}$$

where ξ_1 is to be chosen.

Putting $\xi_1 = h_1 - h_k - |\lambda(h_k, \{h_j\})|$ we can transform (4.2) into

$$r_k = \Phi_2(h_k, h_k + |\lambda(h_k, \{h_j\})|) . \tag{4.3}$$

Let us introduce the function

$$S(h, \varrho) = \int_0^\varrho \Phi_2(h, h + |\xi|) \, d\xi , \quad h \geqq 0 , \quad \varrho \in R^1 , \tag{4.4}$$

and let us first assume that S is sufficiently smooth.

Putting the relation

$$r_k = \frac{\partial S}{\partial \varrho} (h_k, \lambda(h_k, \{h_j\})) \tag{4.5}$$

into (2.4) we proceed as in Section 3 obtaining

$$P(u)(t) = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}h}\lambda(h,\{h_j\})\frac{\mathrm{d}}{\mathrm{d}h}\left(\frac{\partial S}{\partial \varrho}(h,\lambda(h,\{h_j\}))\right)\Bigg]\,\mathrm{d}h =$$

$$= \int_0^\infty -\frac{\mathrm{d}}{\mathrm{d}h}\left[\frac{\partial S}{\partial h}(h,\lambda(h,\{h_j\}))\right] + \left(\frac{\partial^2 S}{\partial h^2} - \frac{\partial^2 S}{\partial \varrho^2}\right)(h,\lambda(h,\{h_j\}))\,\mathrm{d}h\,.$$

We use the fact that $\lambda(h,\{h_j\}) = 0$ for $h \geqq \boldsymbol{h}$ and $|(\mathrm{d}/\mathrm{d}h)\,\lambda(h,\{h_j\})| = 1$ for a.e. $h \in (0, \boldsymbol{h})$.

Let us denote

$$(4.6) \qquad \left(\frac{\partial^2 S}{\partial h^2} - \frac{\partial^2 S}{\partial \varrho^2}\right)(h,\varrho) = \mu(h,\varrho)\,,$$

$$\frac{\partial S}{\partial h}(0,\varrho) = g(\varrho)\,.$$

Notice that we have $S(0,\varrho) = 0$.

This yields

$$(4.7) \qquad P(u)(t) = g(u(t)) + \textstyle\int_0^\infty \mu(h, l_h(u)(t))\,\mathrm{d}h\,,$$

which is the equivalent definition of the Preisach operator (cf. [7]). We may conclude:

(4.8) **Theorem.** (i) *Every vertical-shift invariant continuous memory preserving operator with smooth identification functions is a Preisach operator.*

(ii) *Let P be a Preisach operator (4.7) and let S be the solution of the wave equation (4.6) with Cauchy data $S(0,\varrho) = 0$, $(\partial S/\partial h)(0,\varrho) = g(\varrho)$. Let us assume that g, μ are odd and continuously differentiable with respect to ϱ,*

$$g'(\varrho \pm h) + \int_0^h \frac{\partial \mu}{\partial \varrho}(\sigma, \varrho \pm (h-\sigma))\,\mathrm{d}\sigma > 0$$

for every $\varrho \in R^1$, $h \geqq 0$. Then P is a memory preserving operator with identification functions

$$\Phi_k(h_k,\ldots,h_1) = \frac{\partial S}{\partial \varrho}(h_k, \lambda(h_k,\{h_j\}))\,.$$

Proof. It remains to prove (ii). It suffices to show that the function $h \mapsto (\partial S/\partial \varrho)(h,\lambda(h,\{h_j\}))$ is increasing for every $h_n \downarrow 0$. We have

$$(4.9) \qquad S(h,\varrho) = \frac{1}{2}\int_{\varrho-h}^{\varrho+h} g(\xi)\,\mathrm{d}\xi + \frac{1}{2}\int_0^h \int_{\varrho-h+\sigma}^{\varrho+h-\sigma} \mu(\sigma,\xi)\,\mathrm{d}\xi\,\mathrm{d}\sigma\,,$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}h}\left(\frac{\partial S}{\partial \varrho}(h,\lambda(h,\{h_j\})\right) > 0\,. \qquad \blacksquare$$

(4.10) **Remarks.** (i) In the Cauchy problem (4.6) the variable h plays the role of "time". We observe some kind of duality between the time variable t and the memory variable h in the elementary hysteresis operator $l_h(u)(t)$.

It is shown in [7] that the graph of the function $h \mapsto l_h(u)(t)$ for a fixed t in the Preisach (ϱ, h) – halfplane is the interface between the positive and negative domain. It consists of characteristics of the equation (4.6).

(ii) Taking into account (4.9) we see that the requirement of smoothness is not necessary. In fact, it suffices to assume that g', $\partial\mu/\partial\varrho$ are integrable functions or measures.

(4.11) **Theorem.** *Let P be the Preisach operator* (4.7) *and let $\mu(h, \varrho)$, $g(\varrho)$ be given functions which are odd with respect to ϱ and such that g' is locally bounded, $\partial\mu/\partial\varrho$ is locally integrable. Let us assume that $\lim_{h\to\infty} g(h) + \int_0^h \mu(\sigma, h - \sigma)\,d\sigma = \infty$ and that there exist functions $\eta \in L^1(0, \infty)$, $\xi \in L^1_{\mathrm{loc}}(0, \infty)$ such that $\xi(h) \geqq (\partial\mu/\partial\varrho)(h, \varrho) \geqq -\eta(h)$, $\int_0^r |\eta(h)|\,dh < \inf\{g'(s);\, s \in R^1\}$ for every $r \geqq 0$.*

Then P, P^{-1} are locally Lipschitz memory preserving operators. Moreover, if $\int_0^\infty |\eta(h)|\,dh < \inf\{g'(s);\, s \in R^1\}$, then P^{-1} is Lipschitz.

Remark. This theorem was motivated by Theorem 5.14 of [2]. Here, the assumption of positivity of the Preisach measure is replaced by a weaker one. Roughly speaking, we assume that the Preisach measure is bounded from below by an Ishlinskii measure.

Proof. The local Lipschitz continuity follows immediately from (4.7). We make use of the fact that $l_h(u)(t) = 0$ for $h \geqq \|u\|_{[0,t]}$. We have $\lim_{h\to\infty} \Phi_1(h) = +\infty$, hence by (2.8) it suffices to prove that P is injective and P^{-1} is locally Lipschitz.

Let $u, v \in C([0, T])$ be given, $w = P(u)$, $z = P(v)$. We find $t \in [0, T]$ such that $|u(t) - v(t)| = \|u - v\|_{[0,T]}$ and construct $MS(u)(t) = \{(t_j, h_j)\}$, $MS(v)(t) = \{(\tilde{t}_j, \tilde{h}_j)\}$, $MS(w)(t) = \{(t_j, r_j)\}$, $MS(z)(t) = \{(\tilde{t}_j, \tilde{r}_j)\}$.

We have to prove that there exists a positive function ψ such that

$$(4.12) \qquad |\lambda(0, \{h_j\}) - \lambda(0, \{\tilde{h}_j\})| \leqq \psi(\max\{\boldsymbol{h}, \tilde{\boldsymbol{h}}\})\, d(\{r_j\}, \{\tilde{r}_j\}) .$$

The case $\lambda(0, \{h_j\}) = \lambda(0, \{\tilde{h}_j\})$ is trivial. Let us assume for instance $\lambda(0, \{h_j\}) > \lambda(0, \{\tilde{h}_j\})$ and put $\hat{h} = \min\{h \geqq 0;\, \lambda(h, \{h_j\}) = \lambda(h, \{\tilde{h}_j\})\}$. Indeed, we have $\hat{h} \leqq \max\{\boldsymbol{h}, \tilde{\boldsymbol{h}}\}$.

The function $h \mapsto (\partial S/\partial\varrho)(h, \lambda(h, \{h_j\}))$ is increasing (see the proof of (4.8)) and tends to ∞ as $h \to \infty$. Therefore, for each $r \geqq 0$ there exists a unique $h \geqq 0$ such that

$$(4.13) \qquad r = \frac{\partial S}{\partial \varrho}(h, \lambda(h, \{h_j\})) .$$

This enables us to compute $\lambda(r, \{r_j\})$ from (2.9). Repeating the proof of (4.7) we obtain

$$(4.14) \qquad \lambda(r, \{r_j\}) = \frac{\partial S}{\partial h}(h, \lambda(h, \{h_j\}) + \int_h^\infty \mu(\sigma, \lambda(\sigma, \{h_j\}) \, d\sigma \,,$$

where h is given by (4.13).

Using (4.14) for $r = \hat{r} = (\partial S/\partial \varrho)(\hat{h}, \lambda(\hat{h}, \{h_j\})) = (\partial S/\partial \varrho)(\hat{h}, \lambda(\hat{h}, \{\tilde{h}_j\})$ and for $r = 0$ we obtain

$$\begin{aligned} &g(\lambda(0, \{h_j\})) - g(\lambda(0, \{\tilde{h}_j\})) = \\ &= (\lambda(0, \{r_j\}) - \lambda(0, \{\tilde{r}_j\})) - (\lambda(\hat{r}, \{r_j\}) - \lambda(\hat{r}, \{\tilde{r}_j\})) - \\ &- \int_0^{\hat{h}} \int_{\lambda(\sigma,\{\tilde{h}_j\})}^{\lambda(\sigma,\{h_j\})} \frac{\partial \mu}{\partial \varrho}(\sigma, \varrho) \, d\varrho \, d\sigma \,. \end{aligned}$$

Let us denote $\alpha = \inf\{g'(\xi); \xi \in R^1\} > 0$. We obtain

$$\begin{aligned} (4.15) \qquad &\alpha(\lambda(0, \{h_j\}) - \lambda(0, \{\tilde{h}_j\})) \leqq \\ &\leqq 2d(\{r_j\}, \{\tilde{r}_j\}) + \textstyle\int_0^{\hat{h}} [\lambda(\sigma, \{h_j\}) - \lambda(\sigma, \{\tilde{h}_j\})] \, \eta(\sigma) \, d\sigma \,. \end{aligned}$$

By (1.2)(iv) we have $|\lambda(\sigma, \{h_j\}) - \lambda(\sigma, \{\tilde{h}_j\})| \leqq \lambda(0, \{h_j\}) - \lambda(0, \{\tilde{h}_j\})$. Putting $\psi(x) = 2(\alpha - \int_0^x |\eta(h)| \, dh)^{-1}$ we get (4.12) directly from (4.15).

Indeed, if $\int_0^\infty |\eta(h)| \, dh < \alpha$, then ψ is bounded and P^{-1} is Lipschitz. ∎

Let $g: R^1 \to R^1$ be an odd increasing function. The Nemytskii operator $G: C([0, T]) \to C([0, T])$ is defined by the formula $G(u)(t) = g(u(t))$. Let us denote by $\mathcal{G}$ the set of such operators.

Similarly, we denote by $\mathcal{W}$, $\mathcal{F}$ the set of memory preserving Preisach and Ishlinskii operators, respectively. We obviously have $\mathcal{G} \subset \mathcal{W}$, $\mathcal{F} \subset \mathcal{W}$, $\mathcal{F} \cap \mathcal{G} = \{cI\}$, where I is the identity and $c > 0$ is a constant.

If we rewrite the formulas (4.14), (4.13) in the form

$$(4.16) \qquad l_r(P(u))(t) = \frac{\partial S}{\partial h}(h, l_h(u)(t)) + \int_h^\infty \mu(\sigma, l_\sigma(u)(t)) \, d\sigma \,,$$

where $r = (\partial S/\partial \varrho)(h, l_h(u)(t))$, we obtain a superposition formula for Preisach operators.

(4.17) **Theorem.** (i) *Let $G \in \mathcal{G}$, $W \in \mathcal{W}$ be given. Then $W \circ G \in \mathcal{W}$.*

(ii) *Let $F \in \mathcal{F}$, $W \in \mathcal{W}$ be given. Then $F \circ W \in \mathcal{W}$.*

(iii) *Let $W \in \mathcal{W}$ be given by* (4.7) *with* $(\partial\mu/\partial\varrho)(h, \varrho) > 0$ (or <0) *for every $h > 0$, $\varrho \in R^1$. Let us assume $W^{-1} \in \mathcal{W}$. Then $W, W^{-1} \in \mathcal{F}$.*

Remarks. (i) We can give a natural interpretation of (4.17) (iii). A "horizontal-shift invariant" operator which is vertical-shift invariant, is shift invariant.

(ii) As a consequence of (4.17) we see that if $F \in \mathscr{F}$ is given by (3.1) with $\Phi_1'(0) > 0$, $\Phi_1'' > 0$ a.e. and $G \in \mathscr{G}$ is arbitrary, then $G \circ F \in \mathscr{W}$ implies $G = cI$.

Indeed, we see from (4.5), (4.9) that the identification function for G has the form

$$(4.18) \qquad r_k = \tfrac{1}{2}(g(\lambda(h_k, \{h_j\}) + h_k) - g(\lambda(h_k, \{h_j\}) - h_k)).$$

The superposition of memory preserving operators corresponds to the superposition of identification functions. Applying (4.18) with $k = 2$ and (4.17) (iii) we conclude that there exists a function ψ such that

$$\tfrac{1}{2}(g(\Phi_1(h_1)) - g(\Phi_1(h_1) - 2\Phi_1(h_2))) = \psi(h_2)$$

holds for every $h_2 \leqq h_1$. This implies $g' = \text{const.}$

Proof of (4.17). (i) Let us assume (4.18) and

$$s_k = \frac{\partial S}{\partial \varrho}(r_k, \lambda(r_k, \{r_j\})).$$

For proving (i) it suffices to find a function S_1 such that

$$(4.19) \qquad s_k = \frac{\partial S_1}{\partial \varrho}(h_k, \lambda(h_k, \{h_j\})).$$

For the sake of simplicity we denote $\lambda_k = \lambda(h_k, \{h_j\})$. We have $\lambda_{k+1} - \lambda_k = (-1)^{k+1}(h_k - h_{k+1})$, hence

$$\lambda_{k+1} - (-1)^k h_{k+1} = \lambda_k - (-1)^k h_k.$$

An elementary computation yields

$$\lambda(r_k, \{r_j\}) = \sum_{j=1}^{k-1}(-1)^{j+1}(r_j - r_{j+1}) = \tfrac{1}{2}\sum_{\substack{j=1\\ j \text{ even}}}^{k-1}(g(\lambda_{j+1} + h_{j+1}) - g(\lambda_j + h_j)) + \tfrac{1}{2}\sum_{\substack{j=1\\ j \text{ odd}}}^{k-1}(g(\lambda_{j+1} - h_{j+1}) - g(\lambda_j - h_j)) =$$

$$= \tfrac{1}{2}(g(\lambda_k + h_k) + g(\lambda_k - h_k)),$$

hence (4.19) holds provided we put

$$S_1(h, \varrho) = \int_0^\varrho \frac{\partial S}{\partial \varrho}(\tfrac{1}{2}(g(\xi + h) - g(\xi - h)), \tfrac{1}{2}(g(\xi + h) + g(\xi - h)))\,\mathrm{d}\xi.$$

(ii) Let us assume S_k as above and $h_k = \Phi(s_k)$. We immediately obtain

$$h_k = \frac{\partial S_2}{\partial \varrho}(r_k, \lambda(r_k, \{r_j\})),$$

where

$$S_2(h, \varrho) = \int_0^\varrho \Phi\left(\frac{\partial S}{\partial \varrho}(h, \xi)\right)\mathrm{d}\xi, \quad \text{hence} \quad F \circ W \in \mathscr{W}.$$

(iii) Let Φ_k, Ψ_k be the identification functions corresponding to W, W^{-1} respectively, $(\partial S/\partial \varrho)(h, \varrho) = \Phi_2(h, h + |\varrho|)$.

Let $r_1 > r_2$ be arbitrarily given; we find $h_1 > h_2$ such that $r_1 = \Phi_1(h_1)$, $r_2 = \Phi_2(h_2, h_1)$.

For $\sigma \geqq 0$ we denote

$$\hat{r}_1(\sigma) = \Phi_1(h_1 + 2\sigma),$$
$$\hat{r}_2(\sigma) = \Phi_2(h_2 + \sigma, h_1 + 2\sigma),$$
$$\hat{r}_3(\sigma) = \Phi_3(h_2, h_2 + \sigma, h_1 + 2\sigma) = \Phi_2(h_2, h_1) = r_2 .$$

For σ sufficiently small we have $\lambda(\hat{r}_3(\sigma), \{\hat{r}_j(\sigma)\}) > 0$, hence

$$h_2 = \Psi_3(\hat{r}_3(\sigma), \hat{r}_2(\sigma), \hat{r}_1(\sigma)) =$$
$$= \Psi_2(r_2, \Phi_1(h_1 + 2\sigma) - 2\Phi_2(h_2 + \sigma, h_1 + 2\sigma) + 2r_2) .$$

Differentiating the last expression with respect to σ we obtain for $\sigma = 0$

$$0 = \frac{\partial \Psi_2}{\partial r_1}(r_2, r_1) f'(0) , \tag{4.20}$$

where

$$f(\sigma) = \Phi_1(h_1 + 2\sigma) - 2\Phi_2(h_2 + \sigma, h_1 + 2\sigma) =$$
$$= \frac{\partial S}{\partial \varrho}(h_1 + 2\sigma, 0) - 2\frac{\partial S}{\partial \varrho}(h_2 + \sigma, h_1 - h_2 + \sigma) .$$

Formula (4.9) yields

$$f'(0) = 2\int_{h_2}^{h_1} \frac{\partial \mu}{\partial \varrho}(\eta, h_1 - \eta)\, \mathrm{d}\mu \neq 0 .$$

The identity (4.20) implies that Ψ_2 is independent of r_1. Consequently W^{-1}, $W \in \mathscr{F}$ and (4.17) is proved. ∎

5. OTHER EXAMPLES

A. Moving model (see [3]).

Let W be a Preisach operator and $\varepsilon > 0$ a (small) constant. Let $u \in C([0, T])$ be a given input function. The output v of the moving model is given by the implicit formula

$$v = W(u + \varepsilon v) . \tag{5.1}$$

We include this model into the framework of memory preserving operators.

(5.2) **Proposition.** *Let W be a Preisach operator* (4.7) *such that there exist* $\gamma > 0$ *and* $\eta \in L^1(0, \infty)$, $0 < g'(\varrho) \leqq \gamma$, $0 \leqq (\partial\mu/\partial\varrho)(h, \varrho) \leqq \eta(h)$ *for every* $\varrho \in R^1$, $h > 0$.

Put $\varepsilon_0 = (\gamma + \int_0^\infty \eta(h)\,dh)^{-1}$. *Then for every* $\varepsilon \in (0, \varepsilon_0)$ *the moving model* (5.1) *can be represented by the memory preserving operator* $P_\varepsilon = W(I - \varepsilon W)^{-1}$, *which is Lipschitz continuous. In other words,* (5.1) *is equivalent to the identity* $v = P_\varepsilon(u)$.

Proof. Putting $z = u + \varepsilon v$ we see that we have $v = W(z)$, $u = z - \varepsilon W(z)$, hence (5.1) is formally equivalent to $v = P_\varepsilon(u)$.

Theorem (4.11) says that $(I - \varepsilon W)^{-1}$ is Lipschitz, hence P_ε is a superposition of Lipschitz continuous memory preserving operators.

B. Modified Ishlinskii and Preisach models

When we take into account for instance the effect of hardening (softening) in one-dimensional elasto-plasticity, we are led to consider the identification functions in the form (cf. (4.2) for $\xi_1 = 0$)

$$r_k = \Phi_2(h_k, h_1)\,, \tag{5.3}$$

or

$$r_k = \Phi_3(h_k, \tfrac{1}{2}(h_1 + h_k - \lambda(h_k, \{h_j\})), h_1)\,. \tag{5.4}$$

Such models correspond again to memory preserving operators which generalize the Ishlinskii and Preisach operators, respectively. Let us note that we always have

$$h_k + |\lambda(h_k, \{h_j\})| \leqq h_1\,.$$

Using the procedure of Sections 3 and 4 we obtain the following expressions for operators (5.3), (5.4), respectively:

$$P(u)(t) = \frac{\partial \Phi_2}{\partial h_2}(0, \|u\|_{[0,t]})\,u(t) + \int_0^\infty l_h(u)(t)\,\frac{\partial^2 \Phi_2}{\partial h_2^2}(h, \|u\|_{[0,t]})\,dh\,, \tag{5.3'}$$

$$P(u)(t) = \frac{\partial S}{\partial h}(0, u(t), \|u\|_{[0,t]}) + \int_0^\infty \mu(h, l_h(u)(t), \|u\|_{[0,t]})\,dh\,, \tag{5.4'}$$

where

$$\frac{\partial S}{\partial \varrho}(h, \varrho, \xi) = \Phi_3(h, \tfrac{1}{2}(\xi + h - \varrho), \xi)$$

and

$$\mu(h, \varrho, \xi) = \left(\frac{\partial^2 S}{\partial h^2} - \frac{\partial^2 S}{\partial \varrho^2}\right)(h, \varrho, \xi)\,.$$

It is easy to state continuity and inversion theorems for these operators analogous to those which are proved in Sections 3 and 4.

C. Hysterons of the first kind or mh-hysterons.

Let $f_l > f_r$ be given increasing continuously differentiable functions $R^1 \to R^1$, $\lim_{r \to +\infty} f_r(x) = \lim_{x \to -\infty} (-f_l(x)) = +\infty$.

For every piecewise monotone function $u \in C([0, T])$ we define

$$(5.5) \qquad P(u)(t) = \begin{cases} \max\{P(u)(t_0), f_r(u(t))\}, & t \in (t_0, t_1], \\ \quad \text{if } u \text{ is nondecreasing in } [t_0, t_1], \\ \min\{P(u)(t_0), f_l(u(t))\}, & t \in (t_0, t_1], \\ \quad \text{if } u \text{ is nonincreasing in } [t_0, t_1], \end{cases}$$

$$P(u)(0) = \begin{cases} \max\{v_0, f_r(u(0))\}, & \text{if } u_0(0) \geqq \frac{1}{2}(f_r^{-1}(v_0) + f_l^{-1}(v_0)), \\ \min\{v_0, f_l(u(0))\}, & \text{if } u(0) < \frac{1}{2}(f_r^{-1}(v_0) + f_l^{-1}(v_0)), \end{cases}$$

where $v_0 \in R^1$ is fixed.

It can be shown (cf. [5]) that the value of $P(u)$ can be defined for arbitrary continuous inputs and that the operator P: $C([0, T]) \to C([0, T])$ is locally Lipschitz. Notice that the assumptions here are more restrictive than in [5] or [4], but this is not substantial.

The operator P is called the hysteron of the first kind ([5]) or the mh-hysteron ([4]).

(5.6) **Theorem.** *There exist increasing bijections φ, ψ: $R^1 \to R^1$ of class C^1 such that for every $u \in C([0, T])$ we have*

$$(5.7) \qquad P(u)(t) = \varphi^{-1}(l_1(\psi(u))(t)),$$

where l_1 is the operator (1.1) for $h = 1$.

Proof. Put $u_0 = \frac{1}{2}(f_r^{-1}(v_0) + f_l^{-1}(v_0))$ and by induction

$$u_k = f_r^{-1}(f_l(u_{k-1})) \quad \text{for} \quad k > 0,$$

$$u_k = f_l^{-1}(f_r(u_{k+1})) \quad \text{for} \quad k < 0.$$

We choose an arbitrary increasing continuously differentiable function φ_0: $[f_r(u_0), f_l(u_0)] \to R^1$ satisfying

$$\varphi_0(f_r(u_0)) = -1, \quad \varphi_0(v_0) = 0, \quad \varphi_0(f_l(u_0)) = 1,$$

$$\varphi_0'(f_r(u_0)+)\, f_r'(u_0) = \varphi_0'(f_l(u_0)-)\, f_l'(u_0).$$

By definition we have $f_r(u_{k+1}) = f_l(u_k)$ for every integer k, and $u_k \to \pm\infty$ as $k \to \pm\infty$, hence $R^1 = \bigcup_{k=-\infty}^{\infty} [f_r(u_k), f_l(u_k)]$.

For $x \in [f_r(u_k), f_l(u_k)]$ we define

$$\varphi_k(x) = 2 + \varphi_{k-1}(f_r(f(_l^{-1}x))) \quad \text{for} \quad k > 0,$$

$$\varphi_k(x) = -2 + \varphi_{k+1}(f_l(f_r^{-1}(x))) \quad \text{for} \quad k < 0,$$

$$\varphi(x) = \varphi_k(x).$$

We easily check that φ is a well defined increasing continuously differentiable bijection $R^1 \to R^1$ and for every $x \in R^1$ we have $\varphi(f_l(x)) - \varphi(f_r(x)) = 2$.

Finally, we put $\psi(x) = \frac{1}{2}(\varphi(f_l(x)) + \varphi(f_r(x)))$.

It suffices to verify (5.7) for an arbitrary continuous piecewise monotone function. Comparing (1.1) to (5.5) we see immediately that for $v(t) = \psi(u(t))$ we have $l_1(v)(t) = \varphi(P(u)(t))$, hence (5.7) holds. ∎

(5.8) **Corollary.** *Let us assume $f_l(x) = -f_r(-x)$ for every $x \in R^1$, $v_0 = 0$. Let g be a given continuous increasing function. Then the operator P_g given by the formula $P_g(u) = g(u) + P(u)$, where P is the mh-hysteron (5.5), is a Preisach operator.*

Proof. The function φ_0 in the proof of (5.6) can be chosen to be odd, hence φ, ψ are odd increasing functions. The operator $P_1(v) = g(\psi^{-1}(v)) + \varphi^{-1}(l_1(v))$ has the form (4.7), with $\mu(h, \varrho) = \varphi^{-1}(\varrho)\,\delta(h - 1)$, where δ is the delta-function. The corresponding identification function can be computed from (4.9), hence it is a Preisach operator (cf. Remark (4.10 (ii)). By Theorem (4.17) the operator $P_g = P_1 \circ \psi$ is again Preisach.

Remark. Krasnoselskii and Pokrovskii ([5]) have proved the existence of a function f of two variables such that

$$P(u)(t) = f(u(t), l_1(u)(t)),$$

but this representation does not emphasize the memory preservation property.

6. DISSIPATION OF ENERGY

We first investigate the properties of memory preserving operators with respect to differentiable functions.

(6.1) **Lemma.** *Let P be a locally Lipschitz memory preserving operator with identification functions $\Phi_k(h_k, \ldots, h_1)$, and let us assume that $\partial\Phi_k/\partial h_k$ is continuous. Then for every $u \in W^{2,1}(0, T)$ we have $P(u) \in W^{1,\infty}(0, T)$ and for every $t \in [0, T]$ the limits $(P(u))'(t+)$, $(P(u))'(t-)$ exist.*

Proof. Let $u \in W^{2,1}(0, T)$ be given. The local Lipschitz continuity yields for every $t > s$

$$|P(u)(t) - P(u)(s)| \leqq \psi(\|u\|_{[0,T]})\,\|u(\cdot) - u(s)\|_{[s,t]}, \tag{6.2}$$

hence $P(u) \in W^{1,\infty}(0, T)$.

The set $M = \{t \in (0, T);\ u'(t) \neq 0\}$ is open, $M = \bigcup_{i=1}^{\infty} (\alpha_i, \beta_i)$. For $t \in (0, T) \setminus M$ the inequality (6.2) implies $(P(u))'(t) = 0$.

The sign of u' is constant in (α_i, β_i). We find $MS(u)(\alpha_i) = \{(t_j, h_j)\}$. Let us assume for instance $t = t_1$ and $u'(t) > 0$ for $t \in (\alpha_i, \beta_i)$.

The interval (α_i, β_i) may contain points τ_{2k+1} such that $\tau_{2k+1} < \tau_{2k-1}$ and $u(\tau_{2k+1}) = u(t_{2k+1})$. If the set $\{\tau_{2k+1}\} \subset (\alpha_i, \beta_i)$ is infinite, then $\tau_{2k+1} \searrow \alpha_i$ as $k \to \infty$.

For $\tau \in (\tau_{2k+1}, \tau_{2k-1})$, $k \geqq 1$ we have

$$MS(u)(\tau) = \{(t_1, h_1), \ldots, (t_{2k}, h_{2k}), (\tau, \tfrac{1}{2}(u(\tau) - u(t_{2k})))\},$$

hence

$$MS(P(u))(\tau) = \{(t_1, r_1), \ldots, (t_{2k}, r_{2k}), (\tau, \Phi_{2k+1}(\tfrac{1}{2}(u(\tau) - u(t_{2k})), h_{2k}, \ldots, h_1))\}.$$

Using (2.4) we obtain

$$P(u)'(\tau) = u'(\tau) \frac{\partial \Phi_{2k+1}}{\partial h_{2k+1}} (\tfrac{1}{2}(u(\tau) - u(t_{2k})), h_{2k}, \ldots, h_1). \tag{6.3}$$

For $\tau > \tau_1$ we have simply $P(u)(\tau) = \Phi_1(u(\tau))$ and Lemma (6.1) follows easily. ∎

(6.4) **Assumption.** P is a locally Lipschitz memory preserving operator. The functions $h \mapsto \Phi_k(h, h_{k-1}, \ldots, h_1)$ are of class C^2 in $[0, h_{k-1}]$ for every $h_j \searrow 0$.

We assume that there exist constants $H > 0$, $\gamma \geqq 0$ such that for every $h_j \searrow 0$, $h_1 \leqq H$ the function $h \mapsto \varphi(h, \{h_j\})$ defined by the formula

$$\varphi(h, \{h_j\}) = \begin{cases} \Phi_k(h, h_{k-1}, \ldots, h_1) & \text{for} \quad h \in [h_k, h_{k-1}), \\ \Phi_1(h) & \text{for} \quad h \geqq h_1 \end{cases}$$

is concave and increasing in $[0, H]$,

$$\frac{d^2\varphi}{dh^2}(h, \{h_j\}) \leqq -\gamma \quad \text{for every} \quad h \in (0, H) \setminus \{h_j\}.$$

(6.5) **Proposition.** (i) *Let Φ_1: $[0, \infty) \to [0, \infty)$ be of class C^2, $\Phi_1''(0) < 0$. Then the Ishlinskii operator* (3.1) *satisfies* (6.4).

(ii) *Let $g(u) = \varepsilon u$, $\varepsilon > 0$ a constant, and let μ: $[0, \infty) \times R^1 \to R^1$ be of class C^2, $(\partial\mu/\partial\varrho)(0, 0) < 0$. Let us assume that there exists $\eta \in L^1(0, \infty)$ such that $(\partial\mu/\partial\varrho)(h, \varrho) \geqq -\eta(h)$ for every $(h, \varrho) \in (0, \infty) \times R^1$ and $\int_0^r |\eta(h)| \, dh < \varepsilon$ for every $r \geqq 0$. Then the Preisach operator* (4.7) *satisfies* (6.4).

(iii) *Let $g(u) = \varepsilon u$, $\varepsilon > 0$ a constant, and let μ: $[0, \infty) \times R^1 \to R^1$ be of class C^2, $(\partial\mu/\partial\varrho)(0, 0) > 0$, $(\partial u/\partial\varrho)(h, \varrho) \geqq 0$ for every $(h, \varrho) \in (0, \infty) \times R^1$. Let P be the Preisach operator* (4.7). *Then P^{-1} satisfies* (6.4).

Proof. The case (i) is trivial, since we have simply $\varphi(h, \{h_j\}) = \Phi_1(h)$.

(ii) We have $\varphi(h, \{h_j\}) = (\partial S/\partial\varrho)(h, \lambda(h, \{h_j\}))$, where S is given by (4.9). A straightforward computation yields

$$\frac{d^2\varphi}{dh^2}(h, \{h_j\}) = \frac{\partial\mu}{\partial\varrho}(h, \lambda(h, \{h_j\})) +$$
$$+ (1 + |\varkappa|) \int_0^h \frac{\partial^2\mu}{\partial\varrho^2}(\sigma, h - \sigma + \varkappa\lambda(h, \{h_j\})) \, d\sigma,$$

where

$$\varkappa = \operatorname{sign}\left(\frac{d\lambda}{dh}(h, \{h_j\})\right).$$

Since $|\lambda(h, \{h_j\}) \leqq h_1$, we see that (6.4) holds provided H is sufficiently small.

(iii) We have $r_k = \Phi_k(h_k, \ldots, h_1) \Leftrightarrow h_k = (\partial S/\partial \varrho)(r_k, \lambda(r_k, \{r_j\}))$, where S is given by (4.9).

Denoting $\psi(r, \{r_j\}) = (\partial S/\partial \varrho)(r, \lambda(r, \{r_j\}))$ we obtain $\psi(\varphi(h, \{h_j\}), \{r_j\}) = h$. After differentiation we have

$$\frac{d^2\varphi}{dh^2}(h, \{h_j\}) = -\frac{d^2\psi}{dr^2}(r, \{r_j\})\left(\frac{d\psi}{dr}(r, \{r_j\})\right)^{-3}, \quad r = \varphi(h, \{h_j\}).$$

Repeating the argument of (ii) we see that for r, r_1 sufficiently small we have

$$\varepsilon \leqq \frac{d\psi}{dr} \leqq 2\varepsilon, \quad \frac{d^2\psi}{dr^2} \geqq \hat{\gamma} > 0,$$

hence (6.4) follows easily. ■

Inequalities of the following type play a crucial role in the theory of partial differential equations with hysteresis (see [6], [7]).

(6.6) **Theorem.** *Let P be a memory preserving operator satisfying (6.4) and le $u \in W^{2,1}(0, T)$ be given, $\|u\|_{[0,T]} \leqq H$. For $t \in [0, T]$ put*

$$E(t) = \tfrac{1}{2}(P(u))'(t)\, u'(t).$$

Then for every $0 \leqq s < t \leqq T$ we have

$$E(t-) - E(s+) \leqq \int_s^t (P(u))'(\tau)\, u''(\tau)\, d\tau - \frac{\gamma}{4}\int_s^t |u'(\tau)|^3\, d\tau.$$

Proof. Let $0 \leqq s < t \leqq T$ be given. The set $\{\tau \in (s, t);\, u'(\tau) \neq 0\}$ is open, $u'(\tau) = 0$ implies $(P(u))'(\tau) = 0$ (see the proof of Lemma (6.1)). Therefore, it suffices to consider the case when u is strictly monotone in (s, t). Since P is odd, the cases $u' > 0$ and $u' < 0$ are symmetric. Let us assume for instance $u'(\tau) > 0$ for $\tau \in (s, t)$ and let $MS(u)(s) = \{(t_j, h_j)\}$, $t = t_1$ (the argument is similar for $t = t_0$).

We construct (see the proof of (6.1)) the sequence $\{\tau_{2k+1}\} \subset (s, t)$ such that $u(\tau_{2k+1}) = u(t_{2k+1})$, $\tau_{2k+1} < \tau_{2k-1}$. If (s, t) contains infinitely many points τ_{2k+1}, then $\tau_{2k+1} \searrow s$ as $k \to \infty$ and $u'(s) = (P(u))'(s) = 0$.

For $\tau \in (\tau_{2k+1}, \tau_{2k-1})$ we have (6.3), hence

$$\int_{\tau_{2k+1}}^{\tau_{2k-1}} (P(u))'(\tau)\, u''(\tau)\, d\tau = E(\tau_{2k-1}-) - E(\tau_{2k+1}+) -$$

$$- \int_{\tau_{2k+1}}^{\tau_{2k-1}} \frac{1}{4} (u'(\tau))^3 \frac{\partial^2 \Phi_{2k+1}}{\partial h_{2k+1}^2} \left(\frac{1}{2} (u(\tau) - u(t_{2n})),\, h_{2k}, \ldots, h_1 \right) d\tau \geqq$$

$$\geqq E(\tau_{2k-1}\,-) - E(\tau_{2k+1}\,+) + \frac{\gamma}{4} \int_{\tau_{2k+1}}^{\tau_{2k-1}} |u'(\tau)|^3\, d\tau\,.$$

In the case $u(t) > u(t_1)$ for $\tau \in (\tau_1, t)$ we have $P(u)(\tau) = \Phi_1(u(\tau))$, hence

$$\int_{\tau_1}^{t} (P(u))'(\tau)\, u''(\tau)\, d\tau \geqq E(t-) - E(\tau_1+) + \frac{\gamma}{2} \int_{\tau_1}^{t} |u'(\tau)|^3\, d\tau\,.$$

It remains to prove

(6.7) $$E(\tau_{2k-1}\,+) \leqq E(\tau_{2k-1}\,-)$$

for every $k = 1, 2, \ldots$ such that $\tau_{2k-1} \in (s, t)$.

For $k \geqq 2$ (6.3) yields

$$E(\tau_{2k-1}\,+) - E(\tau_{2k-1}\,-) = \tfrac{1}{2}(u'(\tau_{2k-1}))^2 \left[\frac{\partial \Phi_{2k-1}}{\partial h_{2k-1}} (h_{2k-1}, \ldots, h_1) - \right.$$

$$\left. - \frac{\partial \Phi_{2k+1}}{\partial h_{2k+1}} (h_{2k}, h_{2k}, \ldots, h_1) \right] =$$

$$= \tfrac{1}{2}(u'(\tau_{2k-1}))^2 \left[\frac{d\varphi}{dh} (h_{2k-1}\,+, \{h_j\}) - \frac{d\varphi}{dh} (h_{2k}\,-, \{h_j\}) \right] \leqq 0\,,$$

since φ is concave. The same inequality holds for $k = 1$ and Theorem (6.6) is proved. ■

The proof of (6.6) contains also the following result.

(6.8) **Corollary.** *Let the hypotheses of Theorem (6.6) be satisfied. Then for every $t \in (0, T)$ we have*

$$E(t+) \leqq E(t-)\,.$$

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Souhrn

OPERÁTORY ZACHOVÁVAJÍCÍ HYSTEREZNÍ PAMĚŤ

PAVEL KREJČÍ

Rozvoj matematických metod vyšetřování problémů s hysterezí v poslední době ukázal, že podstatnou roli hraje struktura hysterezní paměti. V této práci jsou charakterizovány hysterezní operátory, které mají paměť Preisachova typu (operátory zachovávající paměť). Jsou vyšetřeny jejich vlastnosti (spojitost, invertibilita) a jsou nalezeny vztahy mezi speciálními třídami těchto operátorů (Preisachovy, Išlinského a Němyckého operátory). Pro obecný operátor zachovávající paměť je odvozena energetická nerovnost.

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