Martin Anděl

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NON-NEGATIVE LINEAR PROCESSES

MARTIN ANDĚL

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Summary. Conditions under which the linear process is non-negative are investigated in the paper. In the definition of the linear process a strict white noise is used. Explicit results are presented also for the models AR(1) and AR(2).

Keywords: autoregressive model, linear process, non-negative process, strict white noise.

AMS subject classification: 62M10.

1. INTRODUCTION

In this paper we assume that $\{e_t\}$ is a strict white noise with a finite second moment, i.e. a series of i.i.d. random variables such that $\mathsf{E}e_t^2 < \infty$. Let F be the distribution function of e_t.

The process

$$(1.1) \qquad X_t = \sum_{k=0}^{\infty} c_k e_{t-k}$$

such that $c_0 = 1$ and $\sum |c_k| < \infty$, is called linear. The condition $\sum |c_k| < \infty$ ensures that (1.1) converges in the quadratic mean even if $\mathsf{E}e_t \neq 0$. We shall investigate only linear processes with real coefficients c_k.

The process X_t is called non-negative, if $X_t \geqq 0$ with probability one for all t. Such processes occur frequently in practice (e.g. annual discharge of a river, precipitation, air and water pollution etc.).

If $c_k \geqq 0$ and $e_t \geqq 0$ for all k and t, then $X_t \geqq 0$ for all t. We prove that the condition $c_k \geqq 0$ is also necessary to ensure non-negativity of X_t (when $e_t \geqq 0$) while if $c_k \geqq 0$ the condition $e_t \geqq 0$ is not necessary for $X_t \geqq 0$ (see Theorems 3.1, 3.2). Explicit results are presented for the models AR(1) and AR(2).

2. PRELIMINARIES

Lemma 2.1. *Let $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ be two sequences of random events. Assume that the following conditions are fulfilled:*

a) *Events* $A_1, A_2, \ldots$ *are independent.*

b) *Events* A_n, B_n *are independent for each* n.

c) $\lim_{i \to \infty} \mathsf{P}(B_i) = 1$.

d) $\sum_{i=1}^{\infty} \mathsf{P}(A_i) = \infty$.

Then infinitely many events $A_i \cap B_i$ *occur with probability one.*

Proof. See [3].

Lemma 2.2. *Define*

$$Z_m = \sum_{k=m}^{\infty} c_k e_{t-k} .$$

Then for arbitrary $c > 0$ *we have*

$$\mathsf{P}(|Z_m| \geqq c) \to 0 \quad as \quad m \to \infty .$$

Proof. We assume that $\sum |c_k| < \infty$. If $\mathsf{E}e_t = \mu$ and $\operatorname{var} e_t = \sigma^2$, then

$$\mathsf{E}Z_m = \mu \sum_{k=m}^{\infty} c_k \to 0 , \quad \operatorname{var} Z_m = \sigma^2 \sum_{k=m}^{\infty} c_k^2 \to 0 .$$

Thus

$$\mathsf{E}Z_m^2 = \operatorname{var} Z_m + (\mathsf{E}Z_m)^2 \to 0$$

and

$$\mathsf{P}(|Z_m| \geqq c) = \int_{|Z_m| \geqq c} \mathrm{d}P \leqq c^{-2} \int Z_m^2 \, \mathrm{d}P = c^{-2} \mathsf{E}Z_m^2 \to 0 .$$

3. NON-NEGATIVE LINEAR PROCESSES

Theorem 3.1. *Let* $c_k \geqq 0$ *for all* k *and* $\sum c_k < \infty$. *If there exist* $c > 0$ *and* $q \in (0, 1]$ *such that*

$$\mathsf{P}(e_t < -c) = q ,$$

then with probability 1 *there exist infinitely many indices* t *such that* $X_t < 0$.

Proof. Let j_m be the smallest integer such that $j_m q^m \geqq 1$ $(m = 1, 2, \ldots)$. Define sets $S_1, S_2, \ldots$ of positive integers as follows. Let

$$S_1 = \{1, \ldots, j_1\} .$$

Let S_2 contain elements of j_2 couples

$$(j_1 + 1, j_1 + 2), \ldots, (j_1 + 2j_2 - 1, j_1 + 2j_2) ,$$

let S_3 contain elements of j_3 triples starting with

$$(j_1 + 2j_2 + 1, j_1 + 2j_2 + 2, j_1 + 2j_2 + 3) ,$$

and so on. We denote by $n_1, n_2, \ldots$ successively the numbers $1, \ldots, j_1$, then the last members of the couples, triples, etc. If $n_i \in S_m$, we put

$$X_{n_i} = U_{n_i} + Z_{n_i}$$

where

$$U_{n_i} = \sum_{k=0}^{m-1} c_k e_{n_i-k}\,, \quad Z_{n_i} = \sum_{k=m}^{\infty} c_k e_{n_i-k}\,.$$

Introduce events

$$A_i = \{U_{n_i} < -c\}\,, \quad B_i = \{Z_{n_i} < c\}\,, \quad i = 1, 2, \ldots.$$

We have chosen indices in such a way that $A_1, A_2, \ldots$ are independent. It is clear that A_i and B_i are also independent. For $n_i \in S_m$ we have

$$\mathsf{P}(A_i) = \mathsf{P}\Big(\sum_{k=0}^{m-1} c_k e_{n_i-k} < -c\Big) \geqq \mathsf{P}(e_{n_i-k} < -c,\ k = 0, \ldots, m-1) = q^m\,.$$

Thus

$$\sum_{i=1}^{\infty} \mathsf{P}(A_i) = \sum_{m=1}^{\infty} j_m q^m = \infty\,.$$

Lemma 2.2 yields

$$\mathsf{P}(B_i) \geqq \mathsf{P}(|Z_{n_i}| < c) \to 1\,.$$

Now, Theorem 3.1 follows from Lemma 2.1.

We have proved that if $c_k \geqq 0$, then $e_t \geqq 0$ is a necessary condition for $X_t \geqq 0$. Our proof follows the ideas of the proof of Lemma 10.2 in [2] where the non-negativity of the AR(1) model is considered.

Theorem 3.2. *Let* $e_t \geqq 0$, $\operatorname{var} e_t > 0$ *and* $\sum |c_k| < \infty$. *Assume that* $F(d) - F(c) < 1$ *for all* $0 < c < d < \infty$. *If there exists an index* k_0 *such that* $c_{k_0} < 0$, *then with probability* 1 *we have* $X_t < 0$ *for infinitely many indices* t.

Proof. Denote $M = \max |c_i|$, $c = |c_{k_0}|$. Since $F(d) - F(c) < 1$ for all $0 < c < d < \infty$, at least one of the following cases must occur:

a) The variables $e_t \geqq 0$ can be arbitrarily small, i.e. for every $\varepsilon > 0$ there exists $\gamma > 0$ such that $\mathsf{P}(e_t < \varepsilon) > \gamma$.

b) The variable e_t can be arbitrarily large, i.e. for every $\varepsilon > 0$ there exists $\gamma > 0$ such that $\mathsf{P}(e_t > \varepsilon) > \gamma$.

First, consider the case a). Since $e_t \geqq 0$, $\operatorname{var} e_t > 0$, there exists $\gamma > 0$ such that $\mathsf{P}(e_t > 2\gamma) = \delta > 0$. Further,

$$\beta_n = \mathsf{P}\left(e_t < \frac{c\gamma}{nM}\right) > 0\,, \quad n = 1, 2, \ldots.$$

Thus for $n > k_0$ we have

$$\mathsf{P}(\sum_{k=0}^{n-1} c_k e_{t-k} < -c\gamma) \geqq$$

$$\geqq \mathsf{P}\left(e_{t-k_0} > 2\gamma,\ c_k e_{t-k} < \frac{c\gamma}{n} \quad \text{for} \quad k = 0, \ldots, n-1;\ k \neq k_0\right) \geqq$$

$$\geqq \mathsf{P}\left(e_{t-k_0} > 2\gamma,\ e_{t-k} < \frac{c\gamma}{nM} \quad \text{for} \quad k = 0, \ldots, n-1;\ k \neq k_0\right) =$$

$$= \beta_n^{n-1}\delta > 0\,.$$

Let j_m be the smallest integer such that

$$j_m \beta_m^{m-1}\delta \geqq 1\,, \quad m > k_0\,.$$

The remaining part of the proof is now analogous to the proof of Theorem 3.1.

Consider the case b). It is clear that there exists $\gamma > 0$ such that $\mathsf{P}(e_t < \gamma/M) = \delta > 0$. Further,

$$\beta_n = \mathsf{P}(e_t > \gamma n/c) > 0 \quad \text{for} \quad n > k_0\,.$$

If $n > k_0$ then

$$\mathsf{P}(\sum_{k=0}^{n-1} c_k e_{t-k} < -\gamma) \geqq$$

$$\geqq \mathsf{P}\left(e_{t-k_0} > \frac{\gamma n}{c},\ c_k e_{t-k} < \gamma \quad \text{for} \quad k = 0, \ldots, n-1;\ k \neq k_0\right) \geqq$$

$$\geqq \mathsf{P}\left(e_{t-k_0} > \frac{\gamma n}{c},\ e_{t-k} < \frac{\gamma}{M} \quad \text{for} \quad k = 0, \ldots, n-1;\ k \neq k_0\right) =$$

$$= \beta_n \delta^{n-1} > 0\,.$$

Let j_m be the smallest integer such that

$$j_m \beta_m \delta^{m-1} \geqq 1\,, \quad m > k_0\,.$$

Again, the proof can be completed in the same way as that of Theorem 3.1.

Remark 3.3. The assumption $F(d) - F(c) < 1$ for all $0 < c < d < \infty$ in Theorem 3.2 cannot be omitted. Define $c_0 = 1$, $c_1 = -1$, $c_2 = 1$, $c_k = 0$ for $k \geqq 3$. Let e_t have the rectangular distribution on the interval $(2, 3)$. Then

$$X_t = e_t - e_{t-1} + e_{t-2} > 1$$

and so X_t is non-negative, although $e_t \geqq 0$ and $c_1 < 0$.

The AR(1) process X_t is a linear process satisfying

$$X_t = bX_{t-1} + e_t .$$

This process exists if and only if $b \in (-1, 1)$. Since

$$X_t = \sum_{n=0}^{\infty} b^n e_{t-n} ,$$

the conditions of non-negativity of X_t follow from theorems introduced in Section 3.

Theorem 4.1. *Let $b \in [0, 1)$. If there exist $c > 0$ and $q \in (0, 1]$ such that $\mathsf{P}(e_t < -c) = q$, then with probability one $X_t < 0$ for infinitely many indices t.*

Proof follows from Theorem 3.1.

Theorem 4.2. *Let $F(d) - F(c) < 1$ for all $0 < c < d < \infty$. Let $e_t \geqq 0$, $\operatorname{var} e_t > 0$. If $b \in (-1, 0)$, then with probability one $X_t < 0$ for infinitely many indices t.*

Proof follows from Theorem 3.2.

Now, consider an AR(2) process

$$X_t = b_1 X_{t-1} + b_2 X_{t-2} + e_t .$$

Let z_1, z_2 be the roots of $z^2 - b_1 z - b_2 = 0$. It is known that X_t exists if and only if $|z_1| < 1$, $|z_2| < 1$. This condition is satisfied if and only if (b_1, b_2) belong to the triangle Δ with vertices $(-2, -1)$, $(2, -1)$, $(0, 1)$. See Fig. 1.

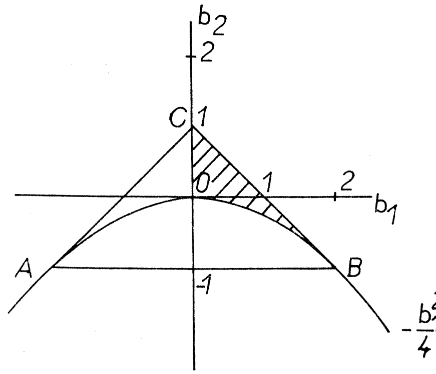

Fig. 1

Theorem 4.3. *Let $F(d) - F(c) < 1$ for all $0 < c < d < \infty$. Let $e_t \geqq 0$. If X_t is an AR(2) process, then $X_t \geqq 0$ for all t if and only if $(b_1, b_2) \in \Delta$, $b_2 \geqq -b_1^2/4$, $b_1 \geqq 0$.*

Proof. The process AR(2) is a linear process with coefficients c_k which coincide with the coefficients in the expansion of the function

$$c(z) = (1 - b_1 z - b_2 z^2)^{-1}$$

in the neighbourhood of zero (see [1]). Let α_1, α_2 be the roots of $1/c(z)$. Then

$$1 - b_1 z - b_2 z^2 = -b_2(z - \alpha_1)(z - \alpha_2).$$

Assume that $\alpha_1 \neq \alpha_2$. Then

$$c(z) = -b_2^{-1}[A_1(z - \alpha_1)^{-1} + A_2(z - \alpha_2)^{-1}],$$

where

$$A_1 = (\alpha_1 - \alpha_2)^{-1}, \quad A_2 = -(\alpha_1 - \alpha_2)^{-1}.$$

Thus

$$c(z) = b_2^{-1}(\alpha_1 - \alpha_2)^{-1} \sum_{k=0}^{\infty} (\alpha_1^{-k-1} - \alpha_2^{-k-1})\, z^k.$$

This implies

$$c_k = b_2^{-1}(\alpha_1 - \alpha_2)^{-1}(\alpha_1^{-k-1} - \alpha_2^{-k-1}).$$

Since

$$\alpha_{1,2} = [-b_1 \pm (b_1^2 + 4b_2)^{1/2}]/(2b_2),$$

we get

$$\alpha_1\alpha_2 = -1/b_2, \quad \alpha_1 - \alpha_2 = (b_1^2 + 4b_2)^{1/2}/b_2.$$

Then

$$c_k = \left(-\frac{1}{2}\right)^{k+1} \{[-b_1 - (b_1^2 + 4b_2)^{1/2}]^{k+1} -$$

$$- [-b_1 + (b_1^2 + 4b_2)^{1/2}]^{k+1}\}(b_1^2 + 4b_2)^{-1/2} =$$

$$= 2^{-k} b_1^k \sum_{j=0}^{[k/2]} \binom{k+1}{2j+1}\left(1 + \frac{4b_2}{b_1^2}\right)^j.$$

If $b_1 > 0$, $b_2 > -b_1^2/4$, then $c_k \geqq 0$ for all k. Since $c_1 = b_1$, we have $c_1 < 0$ if $b_1 < 0$. In the case $b_2 = -b_1^2/4$ we can derive

$$c(z) = \sum_{k=0}^{\infty} 2^{-k}(k+1)\, b_1^k z^k, \quad c_k = 2^{-k}(k+1)\, b_1^k.$$

If $b_2 = -b_1^2/4$ and $b_1 > 0$, then $c_k \geqq 0$ for all k. The case $b_2 = 0$ is trivial.

If $b_2 < -b_1^2/4$, then

$$\alpha_{1,2} = r(\cos\varphi \pm \mathrm{i}\sin\varphi),$$

where $r > 0$, $\varphi \in (0, \pi)$. After a computation we obtain

$$c_k = r^{-k}\frac{\sin(k+1)\varphi}{\sin\varphi}.$$

Thus $c_k < 0$ for infinitely many indices k.

The set of the vectors (b_1, b_2) which correspond to a non-negative AR(2) process X_t is depicted in Fig. 1 as OBC.

References

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Souhrn

NEZÁPORNÉ LINEÁRNÍ PROCESY

MARTIN ANDĚL

V článku se vyšetřují podmínky, za nichž je lineární proces nezáporný. V definici lineárního procesu se přitom užívá striktní bílý šum. Explicitních výsledků je dosaženo rovněž pro modely AR(1) a AR(2).

Author's address: Martin Anděl, matematicko-fyzikální fakulta Univerzity Karlovy, Sokolovská 83, 186 00 Praha 8.