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ON SEMIREGULAR FAMILIES OF TRIANGULATIONS AND LINEAR INTERPOLATION

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Summary. We consider triangulations formed by triangular elements. For the standard linear interpolation operator π_h we prove the interpolation order to be $\|v - \pi_h v\|_{1,p} \leqq Ch|v|_{2,p}$ for $p > 1$ provided the corresponding family of triangulations is only semiregular. In such a case the well-known Zlámal's condition upon the minimum angle need not be satisfied.

Keywords: finite elements, linear interpolation, maximum angle condition.

AMS Classification: 65N30.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with a Lipschitz boundary $\partial\Omega$. Let T_h denote the standard triangulation of $\bar{\Omega}$ into (closed) triangles K. This means that the union of all $K \in T_h$ is $\bar{\Omega}$, the interiors of all $K \in T_h$ are mutually disjoint, and any side of any $K \in T_h$ is either a side of another element from T_h, or a subset of the boundary $\partial\Omega$. As usual, we set $h_K = \operatorname{diam} K$ and the discretization parameter h will be the maximum of h_K over all $K \in T_h$.

A set of triangulations $\mathscr{F} = \{T_h\}$ of $\bar{\Omega}$ is called a family of triangulations if for every $\varepsilon > 0$ there exists $T_h \in \mathscr{F}$ with $h < \varepsilon$.

Next we introduce two definitions. The first is standard (see e.g. [3, p. 124]) while second is new.

Definition 1.1. *A family of triangulations $\mathscr{F}$ is said to be regular if there exists a constant $m > 0$ such that for any $T_h \in \mathscr{F}$ and any $K \in T_h$ there exists a ball b_K of radius r_K such that $b_K \subset K$ and*

$$(1.1) \qquad m h_K \leqq r_K .$$

Definition 1.2. *A family of triangulations $\mathscr{F}$ is said to be semiregular if there exists a constant $M > 0$ such that for any $T_h \in \mathscr{F}$ and any $K \in T_h$ we have*

$$(1.2) \qquad M h_K \geqq R_K ,$$

where R_K is the radius of the circumscribed ball $\mathscr{B}_K$ of K (i.e., all vertices of K belong to $\partial\mathscr{B}_K$).

Note that we always have $M \geqq \frac{1}{2}$, since $h_K \leqq 2R_K$. Moreover,

$$(1.3) \qquad r_K \leqq \varrho_K < h_K\,,$$

where

$$(1.4) \qquad \varrho_K = \frac{2\,\mathrm{meas}_2\,(K)}{\mathrm{meas}_1\,(\partial K)}$$

is the radius of the inscribed ball of K. In Section 2 we show that any regular family of triangulations is semiregular, but the converse implication is not valid. To see this, take for example $\Omega = (0, 1) \times (0, 1)$ and consider the sequence (family) of triangulations $\mathscr{F} = \{T^i\}_{i=0}^{\infty}$ as sketched in Figure 1. Sides parallel to the axes x_1 and x_2 have the lengths 2^{-i} and 2^{-2i}, respectively.

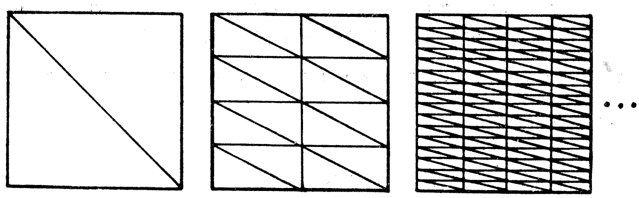

Fig. 1

The condition (1.2) is clearly satisfied for $M = \frac{1}{2}$, since $R_K = \frac{1}{2}h_K$. However, we see that the triangles in T^i degenerate if $i \to \infty$ and for $K \in T^i$ we have, by (1.3) and (1.4),

$$\frac{r_K}{h_K} \leqq \frac{\varrho_K}{h_K} = \frac{2\,\mathrm{meas}_2\,(K)}{h_K\,\mathrm{meas}_1\,(\partial K)} < \frac{h_K\,2^{-2i}}{h_K\,2^{-i}} = 2^{-i} \to 0 \quad \text{as} \quad i \to \infty\,,$$

i.e., the condition (1.1) does not hold.

Let us denote by $W_p^k(\Omega)$, $k \in \{0, 1, \ldots\}$, $p \geqq 1$, the Sobolev space with the standard norm $\|\cdot\|_{k,p} = \|\cdot\|_{k,p,\Omega}$ and the seminorm $|\cdot|_{k,p} = |\cdot|_{k,p,\Omega}$. By $\partial_i v$ and $\partial_{ij} v$ we shall mean the derivatives $\partial v/\partial x_i$ and $\partial^2 v/\partial x_i\,\partial x_j$, respectively. The symbol $P_k(\Omega)$ stands for the space of polynomials of degree at most k defined on Ω.

With any triangulation T_h we associate the finite element space

$$V_h = \{v \in C(\overline{\Omega})|\ v|_K \in P_1(K)\ \forall K \in T_h\}$$

and the interpolation operator π_h: $C(\overline{\Omega}) \to V_h$ which is uniquely determined by the relation

$$(1.5) \qquad \pi_h\, v(x) = v(x)$$

for all vertices x of all $K \in T_h$. Recall that $\pi_h v$ is well-defined also for $v \in W_p^2(\Omega)$, $p > 1$, due to the Sobolev imbedding (see [9, p. 72])

$$(1.6) \qquad W_p^2(\Omega) \subset C(\bar{\Omega}),$$

which is valid for all bounded domains having a Lipschitz boundary.

The main aim of this paper is to generalize the well-known approximation properties of the interpolation operator π_h (see e.g. [3, 5, 10]). In Section 2 we will prove the following theorem.

Theorem 1.3. *Let $\mathscr{F}$ be a semiregular family of triangulations of $\bar{\Omega}$ and let $p > 1$. Then there exists a constant $C > 0$ such that for any $T_h \in \mathscr{F}$ with $h \leqq 1$ we have*

$$(1.7) \qquad \|v - \pi_h v\|_{1,p} \leqq Ch|v|_{2,p} \quad \forall v \in W_p^2(\Omega).$$

The approximation properties of π_h are thus preserved when the condition (1.1) for a ball contained in $K \in T_h$ is replaced by the weaker condition (1.2) for the circumscribed ball. In [1, p. 223], [10, p. 138] or [12, p. 365] there are examples for which (1.2) do not hold and the linear triangular elements loose their usual approximation properties (1.7).

Note that the proof of convergence (rate of convergence) of the finite element method of many problems is usually transformed just to the investigation of approximation properties of π_h (see e.g. Céa's lemma in [3, p. 104] for linear elliptic problems, or [8, p. 207] for some nonlinear problems).

To prove Theorem 1.3 we shall employ the standard technique using transformations of $K \in T_h$ onto a reference triangle. However, all estimates are done more finely than usual. Our technique differs from those presented in [1, 2, 4, 5, 6, 11], where (1.7) is obtained for $p \geqq 2$ (cf. Remark 2.2). For the three-dimensional case we refer to [5, 7].

2. INTERPOLATION ORDER FOR SEMIREGULAR FAMILIES OF TRIANGULATIONS

First of all we prove the following theorem.

Theorem 2.1. *Any regular family of triangulations of a polygon is semiregular.*

Proof. Let $\mathscr{F}$ be a regular family of triangulations of a polygon, let $T_h \in \mathscr{F}$ and $K \in T_h$ be arbitrary. Denote by α_K the minimum angle of K (see Figure 2). Then by (1.1) and (1.3)

$$mh_K \leqq r_K \leqq \varrho_K < h_K \operatorname{tg} \frac{\alpha_K}{2},$$

and therefore,

(2.1) $\alpha_K \geqq \alpha_0 \equiv 2 \operatorname{arctg} m > 0$ (Zlámal's condition [13, p. 397]).

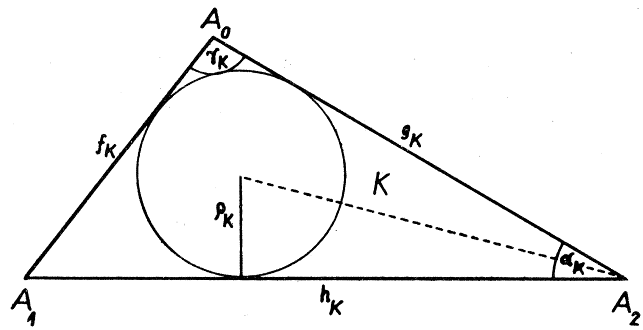

Fig. 2

This implies that

(2.2) $\gamma_K \leqq \gamma_0 \equiv \pi - \alpha_0$,

where γ_K is the maximum angle of K. We immediately see that $\alpha_0 \leqq \pi/3$ and $\gamma_K \in \in [\alpha_0, \pi - \alpha_0]$. Hence,

(2.3) $\sin \gamma_K \geqq \sin \alpha_0 = \sin (\pi - \alpha_0)$.

Setting now $M = (2 \sin \alpha_0)^{-1}$, we find by (2.3) and the Sine Theorem that

$$Mh_K = \frac{h_K}{2 \sin \alpha_0} \geqq \frac{h_K}{2 \sin \gamma_K} = R_K .$$

Thus (1.2) holds. ∎

Remark 2.2. By (1.2) and the Sine Theorem we have

$$M \geqq \frac{1}{2 \sin \gamma_K}$$

which immediately implies (2.2), i.e., the maximum angle condition is equivalent to the semiregularity of a family of triangulations. Recall that the minimum angle condition (2.1) is equivalent to the regularity of a family of triangulations. We see that (2.1) implies the condition (2.2) upon the maximum angle, but not conversely (cf. Figure 1). Synge (see [11, p. 211]) was probably the first who suggested that the greatest angle should be bounded away from π as $h \to 0$ to obtain $\|v - \pi_h v\|_{1,p} = = \mathcal{O}(h)$ for $p = \infty$. An analogous result for $p = 2$ has been obtained by Babuška and Aziz [1], Barnhill, Gregory [2] and Gregory [4]. We prove (1.7) for $p \in (1, \infty)$. For the case $p \in (2, \infty]$ the estimate (1.7) may be proved by a technique presented in Jamet [5, p. 55]. We see that it is better to pay attention to angles that tend to π rather than those that tend to zero. Note that we can construct a family of triangulations $\mathscr{F}$ such that

$$\inf_{K \in T_h \in \mathscr{F}} \alpha_K = 0$$

and (2.2) hold. Then (1.7) still remains valid due to Theorem 1.3. This fact may be useful for developing FE-software for adaptive mesh refinement as we need not prescribe any lower positive bound upon the minimum angle. Thus we may employ triangular elements which are almost "flat" (almost degenerate). This can be advisable for covering thin slots or strips of different materials (e.g. in magnetic head, transformer thins, insulation of wires) to save computer memory.

Let $\hat{K}$ be the closed reference triangle with the vertices $\hat{A}_0 = (0, 0)^{\mathsf{T}}$, $\hat{A}_1 = (1, 0)^{\mathsf{T}}$, $\hat{A}_2 = (0, 1)^{\mathsf{T}}$. Analogously to (1.5) we define $\hat{\pi}\hat{v} \in P_1(\hat{K})$ by

$$(\hat{\pi}\hat{v})(\hat{A}_i) = \hat{v}(\hat{A}_i), \quad i = 0, 1, 2, \quad \hat{v} \in C(\hat{K}).$$

Lemma 2.3. *For any $p \in (1, \infty)$ there exists a constant $\hat{C} > 0$ such that*

$$\|\hat{\partial}_i(\hat{v} - \hat{\pi}\hat{v})\|_{0,p,\hat{K}} \leqq \hat{C}|\hat{\partial}_i\hat{v}|_{1,p,\hat{K}} \ \forall \hat{v} \in W_p^2(\hat{K}), \quad i = 1, 2, \tag{2.4}$$

where $\hat{\partial}_i = \partial/\partial\hat{x}_i$.

Proof. For simplicity we omit the symbol ^ throughout the whole proof. Define the operator

$$Q\colon W_p^1(K) \to P_0(K) \tag{2.5}$$

by

$$Qz = \int_0^1 z(s, 0)\, ds. \tag{2.6}$$

According to the Trace Theorem (see [9, p. 86]), we have $z|_{\partial K} \in L^p(\partial K)$ for $z \in W_p^1(K)$, i.e., Q is well-defined. Since Qz is constant, we get by the Hölder inequality and the Trace Theorem that

$$\begin{aligned} \|Qz\|_{0,p,K}^p &= \text{meas}_2(K)\, |\textstyle\int_0^1 z(s, 0)\, ds|^p \leqq \\ &\leqq \tfrac{1}{2}\textstyle\int_0^1 |z(s, 0)|^p\, ds \leqq C_1\|z\|_{1,p,K}^p \ \forall z \in W_p^1(K), \end{aligned} \tag{2.7}$$

where C_1 depends upon p. Therefore, Q is continuous.

By (2.6), we immediately see that

$$Qz = z \quad \forall z \in P_0(K). \tag{2.8}$$

Now due to Theorem 3.1.4 from [3, p. 121] there exists $C > 0$ such that

$$\|z - Qz\|_{0,p,K} \leqq C|z|_{1,p,K} \quad \forall z \in W_p^1(K). \tag{2.9}$$

Let $v \in W_p^2(K)$ be arbitrary now. Setting $z = \partial_1 v \in W_p^1(K)$, we find that

$$Qz = \int_0^1 \partial_1 v(s, 0)\, ds = v(1, 0) - v(0, 0) = \partial_1(\pi v),$$

and from (2.9) we get (2.4) for $i = 1$. The case $i = 2$ can be proved similarly. ∎

Consider now an arbitrary triangle K with vertices A_i, $i = 0, 1, 2$ (A_i – column vectors). Assume that the greatest angle γ_K of K is at the vertex A_0 (see Figure 2). Let f_K and g_K denote the lengths of the sides A_0A_1 and A_0A_2, respectively. Define an affine one-to-one mapping $F_K: \hat{K} \to K$ by

$$F_K(\hat{x}) = B_K\hat{x} + A_0 , \quad \hat{x} = (\hat{x}_1, \hat{x}_2)^\mathrm{T} \in \hat{K} , \tag{2.10}$$

where $B_K = (B_{ij})_{i,j=1}^2 = (A_1 - A_0, A_2 - A_0)$ is a nonsingular 2×2 matrix as $F_K(\hat{A}_i) = A_i$, $i = 0, 1, 2$. From the expression for B_K we arrive at

$$\begin{aligned} &B_{11}^2 + B_{21}^2 = f_K^2 , \quad B_{12}^2 + B_{22}^2 = g_K^2 , \\ &|B_{11}| \leqq f_K , \quad |B_{21}| \leqq f_K , \\ &|B_{12}| \leqq g_K , \quad |B_{22}| \leqq g_K , \\ &\max(|B_{11}|, |B_{12}|, |B_{21}|, |B_{22}|) \leqq h_K . \end{aligned} \tag{2.11}$$

Note that f_K and g_K may considerably differ (see e.g. Figure 1). For every $v \in L^p(K)$ and almost every $\hat{x} \in \hat{K}$ let us set

$$\hat{v}(\hat{x}) = v(x) , \tag{2.12}$$

where $x = F_K(\hat{x})$. Thus we have a one-to-one correspondence between $\hat{v}$ and v. From (2.10) and (2.12) we can directly derive the next two relations. If $\hat{v} \in W_p^1(\hat{K})$ and $v \in W_p^1(K)$ then

$$(\hat{\partial}_1\hat{v}(\hat{x}), \hat{\partial}_2\hat{v}(\hat{x}))^\mathrm{T} = B_K^\mathrm{T}(\partial_1 v(x), \partial_2 v(x))^\mathrm{T} \tag{2.13}$$

for almost every $\hat{x} \in \hat{K}$ and the corresponding $x \in K$. If $\hat{v} \in W_p^2(\hat{K})$ and $v \in W_p^2(K)$ then similarly for the second derivatives we have

$$\begin{pmatrix} \hat{\partial}_{11}\hat{v} & \hat{\partial}_{12}\hat{v} \\ \hat{\partial}_{12}\hat{v} & \hat{\partial}_{22}\hat{v} \end{pmatrix} = B_K^\mathrm{T} \begin{pmatrix} \partial_{11}v & \partial_{12}v \\ \partial_{12}v & \partial_{22}v \end{pmatrix} B_K . \tag{2.14}$$

Lemma 2.4. *Let $p \geqq 1$. Then for any $v \in W_p^1(K)$ and $\hat{v} \in W_p^1(\hat{K})$ satisfying* (2.12) *we have*

$$|v|_{1,p,K} \leqq 2|\det B_K|^{(1/p)-1} \left(g_K\|\hat{\partial}_1\hat{v}\|_{0,p,\hat{K}} + f_K\|\hat{\partial}_2\hat{v}\|_{0,p,\hat{K}}\right) . \tag{2.15}$$

Proof. Let us denote the entries of B_K^{-1} by C_{ij}, that is

$$B_K^{-1} = \frac{1}{\det B_K}\begin{pmatrix} B_{22} & -B_{12} \\ -B_{21} & B_{11} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} . \tag{2.16}$$

Recall a special case of the Jensen inequality

$$(|y_1| + |y_2|)^{1/p} \leqq |y_1|^{1/p} + |y_2|^{1/p} , \quad p \geqq 1 , \quad y_i \in \mathbb{R}^1 . \tag{2.17}$$

Using now (2.17), the substitution $x = F_K(\hat{x})$, (2.13), the Minkowski inequality and (2.16), we get that

$$|v|_{1,p,K} = (\sum_{i=1}^{2} \int_K |\partial_i v|^p \, dx)^{1/p} \leqq \|\partial_1 v\|_{0,p,K} + \|\partial_2 v\|_{0,p,K} =$$
$$= |\det B_K|^{1/p} ((\int_{\hat{K}} |C_{11}\hat{\partial}_1 \hat{v} + C_{21}\hat{\partial}_2 \hat{v}|^p \, d\hat{x})^{1/p} +$$
$$+ (\int_{\hat{K}} |C_{12}\hat{\partial}_1 \hat{v} + C_{22}\hat{\partial}_2 \hat{v}|^p) \, d\hat{x})^{1/p}) \leqq$$
$$\leqq |\det B_K|^{1/p} (|C_{11}| \, \|\hat{\partial}_1 \hat{v}\|_{0,p,\hat{K}} + |C_{21}| \, \|\hat{\partial}_2 \hat{v}\|_{0,p,\hat{K}} +$$
$$+ |C_{12}| \, \|\hat{\partial}_1 \hat{v}\|_{0,p,\hat{K}} + |C_{22}| \, \|\hat{\partial}_2 \hat{v}\|_{0,p,\hat{K}}) =$$
$$= |\det B_K|^{(1/p)-1} ((|B_{22}| + |B_{12}|) \, \|\hat{\partial}_1 \hat{v}\|_{0,p,\hat{K}} +$$
$$+ (|B_{21}| + |B_{11}|) \, \|\hat{\partial}_2 \hat{v}\|_{0,p,\hat{K}}) \, .$$

Thus (2.15) follows from (2.11). ∎

Lemma 2.5. *Let* $p \geqq 1$. *Then for any* $\hat{v} \in W_p^2(\hat{K})$ *and* $v \in W_p^2(K)$ *satisfying* (2.12) *we have*

$$|\hat{\partial}_1 \hat{v}|_{1,p,\hat{K}} \leqq 8 |\det B_K|^{-1/p} f_K h_K |v|_{2,p,K} \, , \tag{2.18}$$
$$|\hat{\partial}_2 \hat{v}|_{1,p,\hat{K}} \leqq 8 |\det B_K|^{-1/p} g_K h_K |v|_{2,p,K} \, .$$

Proof. Using the substitution $\hat{x} = F_K^{-1}(x) = B_K^{-1}x - B_K^{-1}A_0$, (2.14), (2.17), the Minkowski inequality and (2.11), we find that

$$|\hat{\partial}_1 \hat{v}|_{1,p,\hat{K}} = (\sum_{j=1}^{2} \int_{\hat{K}} |\hat{\partial}_{1j} \hat{v}|^p \, d\hat{x})^{1/p} =$$
$$= (|\det B_K^{-1}| \sum_{j=1}^{2} \int_K | \sum_{k,l=1}^{2} B_{k1} B_{lj} \partial_{kl} v|^p \, dx)^{1/p} \leqq$$
$$\leqq |\det B_K^{-1}|^{1/p} \sum_{j=1}^{2} (\int_K | \sum_{k,l=1}^{2} B_{k1} B_{lj} \partial_{kl} v|^p \, dx)^{1/p} \leqq$$
$$\leqq |\det B_K|^{-1/p} \sum_{j=1}^{2} \sum_{k,l=1}^{2} |B_{k1} B_{lj}| \, (\int_K |\partial_{kl} v|^p \, dx)^{1/p} \leqq$$
$$\leqq 2 |\det B_K|^{-1/p} f_K h_K \sum_{k,l=1}^{2} \|\partial_{kl} v\|_{0,p,K} \, . \tag{2.19}$$

From here we obtain the first relation of (2.18). The second can be proved analogously. ∎

For any $K \in T_h$ and $v \in C(\bar{\Omega})$ we set

$$\pi_K v = (\pi_h v)|_K \, . \tag{2.20}$$

Using the transformation (2.10), we easily deduce (see e.g. [4, p. 124]) that

$$(\pi_K v)^\wedge = \hat{\pi}(v|_K)^\wedge \, . \tag{2.21}$$

Proof of Theorem 1.3. Let $\mathscr{F}$ be a semiregular family of triangulations of $\bar{\Omega}$, let $T_h \in \mathscr{F}$ and $K \in T_h$ be arbitrary (see Figure 2). According to well-known relations from planar geometry and (1.2) we have

$$|\det B_K| = 2 \operatorname{meas}_2 (K) = \frac{f_K g_K h_K}{2R_K} \geqq \frac{f_K g_K}{2M} . \tag{2.22}$$

Now let $v \in W_2^2(\Omega)$ be arbitrary. Writing $v - \pi_K v$ in (2.15) instead of v, we get by (2.21), (2.4), (2.18) and (2.22) that

$$\begin{aligned}
&|v - \pi_K v|_{1,p,K} \leqq \\
&\leqq 2|\det B_K|^{(1/p)-1} (g_K \|\hat{\partial}_1(\hat{v} - (\pi_K v)^\wedge)\|_{0,p,\hat{K}} + f_K \|\hat{\partial}_2(\hat{v} - (\pi_K v)^\wedge)\|_{0,p,\hat{K}}) \leqq \\
&\leqq 2\hat{C}|\det B_K|^{(1/p)-1} (g_K|\hat{\partial}_1 \hat{v}|_{1,p,\hat{K}} + f_K|\hat{\partial}_2 \hat{v}|_{1,p,\hat{K}}) \leqq \\
&32\hat{C}|\det B_K|^{-1} f_K g_K h_K |v|_{2,p,K} \leqq C_1 h_K |v|_{2,p,K} ,
\end{aligned}$$

and thus

$$|v - \pi_K v|_{1,p,K}^p \leqq C_1^p h_K^p |v|_{2,p,K}^p .$$

Summing this inequality over all triangles $K \in T_h$, we find that

$$|v - \pi_h v|_{1,p} \leqq C_1 h |v|_{2,p} \quad \forall v \in W_p^2(\Omega) , \tag{2.23}$$

where C_1 is independent of h. By [3, p. 118, 120]

$$\begin{aligned}
&\|v - \pi_K v\|_{0,p,K} = |\det B_K|^{1/p} \|\hat{v} - (\pi_K v)^\wedge\|_{0,p,\hat{K}} \leqq \\
&\leqq C_2 |\det B_K|^{1/p} |\hat{v}|_{2,p,\hat{K}} \leqq \\
&\leqq C_3 |\det B_K|^{1/p} \|B_K\|^2 |\det B_K|^{-1/p} |v|_{2,p,K} \leqq \\
&\leqq C_0 h_K^2 |v|_{2,p,K} \quad \forall v \in W_p^2(K) ,
\end{aligned}$$

where $\|\cdot\|$ stands for the spectral norm. Hence, we have

$$\|v - \pi_h v\|_{0,p} \leqq C_0 h^2 |v|_{2,p} \quad \forall v \in W_p^2(\Omega) \tag{2.24}$$

without any regularity assumptions upon the family of triangulations. Finally, from (2.17), (2.24) and (2.23) we have

$$\|v - \pi_h v\|_{1,p} \leqq \|v - \pi_h v\|_{0,p} + |v - \pi_h v|_{1,p} \leqq (C_0 h^2 + C_1 h) |v|_{2,p} , \tag{2.25}$$

and the desired estimate (1.7) follows for $h \leqq 1$. ∎

3. NUMERICAL EXAMPLE

Assume that $\Omega = (0, 1) \times (0, 1)$,

$$v(x_1, x_2) = 2x_1^2 - x_1 x_2 - 3x_2^2 ,$$

and consider the sequence of triangulations $\{T^i\}_{i=0}^\infty$ of Figure 1. Let $\|\cdot\|_1 = \|\cdot\|_{1,2}$ and $|\cdot|_2 = |\cdot|_{2,2}$. To evaluate the norms $\|v - \pi_h v\|_1$, we have used a numerical integration formula which is exact for all quintic polynomials on each triangle (see [8, p. 58]). Thus the norms $\|v - \pi_h v\|_1$ were computed exactly (except the rounding errors). The next table confirms the theoretical rate of convergence $\mathcal{O}(h)$ as stated in Theorem 1.3.

i	h	$\|v - \pi_h v\|_1$	$h^{-1}\|v - \pi_h v\|_1/\|v\|_2$
0	1·414214	2·100264	0·203996
1	0·559017	0·732818	0·180067
2	0·257694	0·318461	0·169751
3	0·125973	0·153600	0·167485
4	0·062622	0·076344	0·167462

The numbers of the last column tend to 0·167 … . If each T^i consisted of right-angled isosceles triangles then the numbers of the last column would tend to 0·202 … for the same v.

Finally, note that the approximation of the Laplacian with homogeneous Dirichlet boundary conditions over the triangulations of Figure 1 by Courant's basis functions yields the standard sparse Gram matrices whose main diagonals contain only 4's and there are at most four -1's in each row. The condition number of these matrices behaves as $\mathcal{O}(h^{-2})$ when h tends to zero.

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Souhrn

O SEMIREGULÁRNÍCH SYSTÉMECH TRIANGULACÍ A LINEÁRNÍ INTERPOLACI

MICHAL KŘÍŽEK

V článku se uvažují triangulace tvořené trojúhelníkovými prvky. Pro standardní lineární interpolační operátor π_h se dokazuje, že řád interpolace je $\|v - \pi_h v\|_{1,p} \leqq Ch|v|_{2,p}$ pro $p > 1$ za předpokladu, že odpovídající systém triangulací je pouze semiregulární. V tomto případě známá Zlámalova podmínka na minimální úhel nemusí být splněna.

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