Applications of Mathematics

Štefan Varga

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Applications of Mathematics, Vol. 36 (1991), No. 2, 134–144

Persistent URL: http://dml.cz/dmlcz/104450

QUADRATIC ESTIMATIONS IN MIXED LINEAR MODELS

ŠTEFAN VARGA

(Received September 22, 1989)

Summary. In the paper four types of estimations of the linear function of the variance components are presented for the mixed linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$ with expectation $E(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta}$ and covariance matrix $D(\mathbf{Y}) = \theta_1\mathbf{V}_1 + \ldots + \theta_m\mathbf{V}_m$.

Keywords: Mixed linear model, minimum norm quadratic estimation, variance components.

AMS classification: 62J99.

INTRODUCTION

The usual mixed linear model is

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}\,, \tag{1}$$

where $\mathbf{Y}$ is the n-vector random variable, $\mathbf{X}$ is a given $n \times p$-matrix, $\boldsymbol{\beta}$ an unknown p-vector of parameters and $\mathbf{e}$ a random n-vector of errors with expectation zero and a covariance matrix

$$D(\mathbf{e}) = D(\mathbf{Y}) = \theta_1\mathbf{V}_1 + \ldots + \theta_m\mathbf{V}_m = \mathbf{V}_\theta\,. \tag{2}$$

The matrices $\mathbf{V}_i$ $(i = 1, 2, \ldots, m)$ are known symmetric $n \times n$ – matrices and θ_i $(i = 1, 2, \ldots, m)$ are unknown variance components.

The minimum norm quadratic estimators (MINQUE) of the function of variance components

$$q = \sum_1^m f_i\theta_i = \mathbf{f}'\boldsymbol{\theta} \tag{3}$$

are given in the papers [2] and [4]. These estimators are based on the vector $\mathbf{Y}$, the matrix $\mathbf{V} = \mathbf{V}_1 + \ldots + \mathbf{V}_m$ and prior values $(\alpha_1, \ldots, \alpha_m)' = \boldsymbol{\alpha}$ of the variance components $(\theta_1, \ldots, \theta_m)' = \boldsymbol{\theta}$, and they are in the form

$$\tilde{q}(\mathbf{Y}, \mathbf{V}, \boldsymbol{\alpha}) = \mathbf{Y}'\mathbf{A}(\mathbf{V}, \boldsymbol{\alpha})\,\mathbf{Y} \tag{4}$$

(the matrix $\mathbf{A}$ in (4) depends on the matrix $\mathbf{V}$ and the vector $\boldsymbol{\alpha}$).

The minimum norm quadratic estimation of the function (3) which is based on the vector $\mathbf{Y}$, the matrix $\mathbf{V}$ and the matrix $\mathbf{S}$ of prior values of the elements of the covariance matrix $\mathbf{V}_\theta = \theta_1 \mathbf{V}_1 + \ldots + \theta_m \mathbf{V}_m$ (MINQUE(S))

$$\hat{q}(\mathbf{Y}, \mathbf{V}, \mathbf{S}) = \mathbf{Y}'\mathbf{A}(\mathbf{V}, \mathbf{S})\,\mathbf{Y} \tag{5}$$

is defined in the paper [5].

In the present paper four estimations of the type of (5) are given:

(a) without restrictions: MINQE(S)

(b) invariant for translation in $\boldsymbol{\beta}$: MINQE(I, S)

(c) unbiased: MINQE(U, S)

(d) satisfying both (b) and (c): MINQE(U, I, S).

Further, the relationship between these estimation and the corresponding estimations of the type of (4) which are given in the papers [2] and [4] is established.

The estimation MINQE(U, I, S) is studied in the paper [5] and the estimations MINQE(I, S) and MINQE(U, S) in the paper [6].

1. NATURAL ESTIMATION AND S-ESTIMATION

We assume that the vector of all variance components $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)'$ ($'$ denotes transposition) is an element of the set $\mathcal{O}$ of all $\boldsymbol{\theta} \in \mathcal{R}^m$ ($\mathcal{R}^m$ is the m-dimensional real linear space) such that $\mathbf{V}_\theta$ defined in (2) becomes positive definite (p.d.). Further we assume that the matrix $\mathbf{V} = \mathbf{V}_1 + \ldots + \mathbf{V}_m$ and the matrix $\mathbf{S}$ of prior values of the elements of the covariance matrix $\mathbf{V}_\theta$ are positive definite too.

Let $\mathcal{A}$ be a set of symmetric $n \times n$ – matrices and $(\mathcal{A}, \langle \cdot, \cdot \rangle)$ a Hilbert space where $\langle \cdot, \cdot \rangle$ denotes the inner product of elements $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ given by $\langle \mathbf{A}, \mathbf{B} \rangle = \operatorname{tr} \mathbf{AB}$ ($\operatorname{tr} \mathbf{C}$ denotes the trace of the matrix $\mathbf{C}$).

The natural estimator of the function (3) in the mixed linear model (1) is defined by the expression

$$\mathbf{e}_*' \sum_1^m \lambda_i \mathbf{V}^{-1/2} \mathbf{V}_i \mathbf{V}^{-1/2} \mathbf{e}_* \tag{6}$$

(see (5.4.3) in [4]), where $\mathbf{e}_* = \mathbf{V}^{-1/2}\mathbf{e}$ and the vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)'$ is a solution of the linear system

$$\mathbf{M}\boldsymbol{\lambda} = \mathbf{f}\,. \tag{7}$$

The (i, j)-th element of the matrix $\mathbf{M}$ is $\mathbf{M}_{i,j} = \operatorname{tr} \mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{V}_j$ and $\mathbf{f} = (f_1, \ldots, f_m)'$.

The transformation $\mathbf{e} = \mathbf{S}^{1/2}\boldsymbol{\varepsilon}$ ($\boldsymbol{\varepsilon} = \mathbf{S}^{-1/2}\mathbf{e}$) in the linear model (1) yields the natural estimator (6) of $\mathbf{f}'\boldsymbol{\theta}$ in the form

$$\boldsymbol{\varepsilon}'\mathbf{N}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}' \sum_1^m \varkappa_i \mathbf{S}^{1/2}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}^{1/2}\boldsymbol{\varepsilon}\,, \tag{8}$$

where the vector $\boldsymbol{\varkappa} = (\varkappa_1, \ldots, \varkappa_m)'$ is a solution of the linear system

$$\mathbf{M}\boldsymbol{\varkappa} = \mathbf{f}\,. \tag{9}$$

The matrix $\mathbf{M}$ is defined as in (7).

The quadratic estimator (5) with respect to the transformation $\mathbf{e} = \mathbf{S}^{1/2}\varepsilon$ has the form

(10) $$\hat{q}(\mathbf{Y}, \mathbf{V}, \mathbf{S}) = \mathbf{Y}'\mathbf{A}\mathbf{Y} = (\mathbf{X}\beta + \mathbf{S}^{1/2}\varepsilon)'\mathbf{A}(\mathbf{X}\beta + \mathbf{S}^{1/2}\varepsilon) = $$
$$= (\varepsilon', \beta') \begin{pmatrix} \mathbf{S}^{1/2}\mathbf{A}\mathbf{S}^{1/2} & \mathbf{S}^{1/2}\mathbf{A}\mathbf{X} \\ \mathbf{X}'\mathbf{A}\mathbf{S}^{1/2} & \mathbf{X}'\mathbf{A}\mathbf{X} \end{pmatrix} \begin{pmatrix} \varepsilon \\ \beta \end{pmatrix}.$$

The difference between the estimator (10) and the natural estimator (8) of the function $\mathbf{f}'\mathbf{\theta}$ is

(11) $$\mathbf{Y}'\mathbf{A}\mathbf{Y} - \varepsilon'\mathbf{N}\varepsilon =$$
$$= (\varepsilon', \beta') \begin{pmatrix} \mathbf{S}^{1/2}\mathbf{A}\mathbf{S}^{1/2} - \mathbf{N} & \mathbf{S}^{1/2}\mathbf{A}\mathbf{X} \\ \mathbf{X}'\mathbf{A}\mathbf{S}^{1/2} & \mathbf{X}'\mathbf{A}\mathbf{X} \end{pmatrix} \begin{pmatrix} \varepsilon \\ \beta \end{pmatrix}.$$

The minimum norm quadratic estimation which is a function of the matrix $\mathbf{S}$ (MINQE(S)) is obtained by minimizing the Euclidean norm of the matrix $\mathbf{H}$ of the quadratic form (11) defined by

(12) $$\mathbf{H} = \begin{pmatrix} \mathbf{S}^{1/2}\mathbf{A}\mathbf{S}^{1/2} - \mathbf{N} & \mathbf{S}^{1/2}\mathbf{A}\mathbf{X} \\ \mathbf{X}'\mathbf{A}\mathbf{S}^{1/2} & \mathbf{X}'\mathbf{A}\mathbf{X} \end{pmatrix}.$$

The suqare of the Euclidean norm of the matrix $\mathbf{H}$ is

(13) $$\|\mathbf{H}\|^2 = \operatorname{tr}(\mathbf{S}^{1/2}\mathbf{A}\mathbf{S}^{1/2} - \mathbf{N})^2 + 2\operatorname{tr}\mathbf{X}'\mathbf{A}\mathbf{S}\mathbf{A}\mathbf{X} + \operatorname{tr}(\mathbf{X}'\mathbf{A}\mathbf{X})^2.$$

It is shown in the paper [4] that a quadratic estimation $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ of the function $\mathbf{f}'\mathbf{\theta}$ is invariant with respect to translation in β if $\mathbf{A} \in \mathscr{A}_1$, unbiased if $\mathbf{A} \in \mathscr{A}_2$, invariant and unbiased if $\mathbf{A} \in \mathscr{A}_3$, where

(14) $$\mathscr{A}_1 = \{\mathbf{A} \in \mathscr{A}\colon\ \mathbf{A}\mathbf{X} = \mathbf{0}\},$$

(15) $$\mathscr{A}_2 = \{\mathbf{A} \in \mathscr{A}\colon\ \mathbf{X}'\mathbf{A}\mathbf{X} = \mathbf{0};\ \operatorname{tr}\mathbf{A}\mathbf{V}_j = \mathbf{f}_j,\ j = 1, \ldots, m\},$$

(16) $$\mathscr{A}_3 = \{\mathbf{A} \in \mathscr{A}\colon\ \mathbf{A}\mathbf{X} = \mathbf{0};\ \operatorname{tr}\mathbf{A}\mathbf{V}_j = \mathbf{f}_j,\ j = 1, \ldots, m\}.$$

Definition 1.1. *A quadratic estimator* $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ *of the function* $\mathbf{f}'\mathbf{\theta}$ *is*

(a) MINQE(S) *if the matrix* $\mathbf{A}$ *minimizes the expression* (13) *in the class* $\mathscr{A}$;

(b) MINQE(I, S) *if the matrix* $\mathbf{A}$ *minimizes the expression* (13) *in the class* $\mathscr{A}_1$;

(c) MINQE(U, S) *if the matrix* $\mathbf{A}$ *minimizes the expression* (13) *in the class* $\mathscr{A}_2$;

(d) MINQE(U, I, S) *if the matrix* $\mathbf{A}$ *minimizes the expression* (13) *in the class* $\mathscr{A}_3$.

Theorem 1.2. a) *The* MINQE(S) *of the function* $\mathbf{f}'\mathbf{\theta}$ *in the model* (1) *exists iff*

$$\mathbf{f} \in \mathscr{M}(\mathbf{M}),$$

where the matrix $\mathbf{M}$ *is defined as in* (7) *(the* (i, j)*-th element of the matrix* $\mathbf{M}$ *is* $\mathbf{M}_{i,j} = \operatorname{tr}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{V}_j$) *and* $\mathscr{M}(\mathbf{M})$ *denotes the vector space generated by the columns of* $\mathbf{M}$.

b) *If* $\mathbf{f} \in \mathscr{M}(\mathbf{M})$, *then the* MINQE(S) *of the function* $\mathbf{f}'\mathbf{\theta}$ *is the statistic* $\mathbf{Y}'\mathbf{A}_1\mathbf{Y}$, *where*

$$(17)\qquad \mathbf{A}_1 = \sum_1^m \varkappa_i \mathbf{T}^{-1}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{T}^{-1}\,,$$

$\mathbf{T} = \mathbf{S} + \mathbf{X}\mathbf{X}'$ *and the vector* $\boldsymbol{\varkappa} = (\varkappa_1, \ldots, \varkappa_m)'$ *is a solution of the linear system* (9).

Proof. a) The matrix $\mathbf{A}_1$ in $\mathbf{Y}'\mathbf{A}_1\mathbf{Y}$ exists iff the linear system (9) is consistent. This system is consistent iff $\mathbf{f} \in \mathscr{M}(\mathbf{M})$.

b) The matrix $\mathbf{A}_1$ is symmetric, therefore it suffices to prove that it minimizes the expression (13) for which

$$(18)\qquad \begin{aligned} &\operatorname{tr}(\mathbf{S}^{1/2}\mathbf{A}\mathbf{S}^{1/2} - \mathbf{N})^2 + 2\operatorname{tr}\mathbf{X}'\mathbf{A}\mathbf{S}\mathbf{A}\mathbf{X} + \operatorname{tr}(\mathbf{X}'\mathbf{A}\mathbf{X})^2 = \\ &= \operatorname{tr}\mathbf{A}\mathbf{S}\mathbf{A}\mathbf{S} + 2\operatorname{tr}\mathbf{X}'\mathbf{A}\mathbf{S}\mathbf{A}\mathbf{X} + \operatorname{tr}(\mathbf{X}'\mathbf{A}\mathbf{X})^2 - \\ &- 2\sum_1^m \varkappa_i \operatorname{tr}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{A} + \operatorname{tr}\mathbf{N}^2 = \\ &= \operatorname{tr}\mathbf{A}(\mathbf{S} + \mathbf{X}\mathbf{X}')\,\mathbf{A}(\mathbf{S} + \mathbf{X}\mathbf{X}') - 2\sum_1^m \varkappa_i \operatorname{tr}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{A} + \operatorname{tr}\mathbf{N}^2 = \\ &= \operatorname{tr}\mathbf{A}\mathbf{T}\mathbf{A}\mathbf{T} - 2\sum_1^m \varkappa_i \operatorname{tr}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{A} + \operatorname{tr}\mathbf{N}^2 \end{aligned}$$

is satisfied.

Because $\operatorname{tr}\mathbf{N}^2$ is indepedent of the matrix $\mathbf{A}$, the matrix $\mathbf{A}_1$ minimizes the expression (13) or (18) in the class $\mathscr{A}$ if

$$\begin{aligned} &\operatorname{tr}(\mathbf{A}_1 + \mathbf{D})\,\mathbf{T}(\mathbf{A}_1 + \mathbf{D})\,\mathbf{T} - \\ &- 2\sum_1^m \varkappa_i \operatorname{tr}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}(\mathbf{A}_1 + \mathbf{D}) \geqq \\ &\geqq \operatorname{tr}\mathbf{A}_1\mathbf{T}\mathbf{A}_1\mathbf{T} - 2\sum_1^m \varkappa_i \operatorname{tr}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{A}_1 \end{aligned}$$

holds for each symmetric matrix $\mathbf{D}$.

$$\begin{aligned} &\operatorname{tr}(\mathbf{A}_1 + \mathbf{D})\,\mathbf{T}(\mathbf{A}_1 + \mathbf{D})\,\mathbf{T} - 2\sum_1^m \varkappa_i \operatorname{tr}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}(\mathbf{A}_1 + \mathbf{D}) = \\ &= \operatorname{tr}\mathbf{A}_1\mathbf{T}\mathbf{A}_1\mathbf{T} - 2\sum_1^m \varkappa_i \operatorname{tr}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{A}_1 + \operatorname{tr}\mathbf{D}\mathbf{T}\mathbf{D}\mathbf{T} + \\ &+ 2\operatorname{tr}\mathbf{A}_1\mathbf{T}\mathbf{D}\mathbf{T} - 2\sum_1^m \varkappa_i \operatorname{tr}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{D} \end{aligned}$$

With regard to the fact that the expression $\operatorname{tr}\mathbf{D}\mathbf{T}\mathbf{D}\mathbf{T}$ is nonnegative it suffices to prove that

$$\operatorname{tr}\mathbf{A}_1\mathbf{T}\mathbf{D}\mathbf{T} = \sum_1^m \varkappa_i \operatorname{tr}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{D}\,.$$

$$\begin{aligned} &\operatorname{tr}\mathbf{A}_1\mathbf{T}\mathbf{D}\mathbf{T} = \operatorname{tr}\sum_1^m \varkappa_i\mathbf{T}^{-1}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{T}^{-1}\mathbf{T}\mathbf{D}\mathbf{T} = \\ &= \sum_1^m \varkappa_i \operatorname{tr}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{D} \end{aligned}$$

Corollary 1.3. *One choice of the* MINQE(S) *of the vector of unknown variance* *components* $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)'$ *is*

(19) $$\hat{\boldsymbol{\theta}} = \mathbf{M}^{-}\mathbf{m}$$

provided the MINQE(S) *exists for all components of the vector* $\boldsymbol{\theta}$. *The matrix* $\mathbf{M}$ *is defined as in* (7) *and the i-th element of the vector* $\mathbf{m}$ *is* $\mathbf{m}_i = \mathbf{Y}'\mathbf{T}^{-1}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{T}^{-1}\mathbf{Y}$ ($\mathbf{M}^{-}$ *is a g-inverse of the matrix* $\mathbf{M}$ *defined by* $\mathbf{M}\mathbf{M}^{-}\mathbf{M} = \mathbf{M}$).

Proof. The MINQE(S) of the function $\mathbf{f}'\boldsymbol{\theta}$ is

$$\widehat{\mathbf{f}'\boldsymbol{\theta}} = \mathbf{Y}'\mathbf{A}_1\mathbf{Y} = \mathbf{Y}'\sum_1^m \varkappa_i\mathbf{T}^{-1}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{T}^{-1}\mathbf{Y} =$$
$$= \sum_1^m \varkappa_i\mathbf{Y}'\mathbf{T}^{-1}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{T}^{-1}\mathbf{Y} = \boldsymbol{\varkappa}'\mathbf{m} = \mathbf{f}'\mathbf{M}^{-}\mathbf{m}$$

because $\boldsymbol{\varkappa} = \mathbf{M}^{-}\mathbf{f}$ is a solution of the linear system $\mathbf{M}\boldsymbol{\varkappa} = \mathbf{f}$.

Theorem 1.4. a) *The* MINQE(I, S) *of the function* $\mathbf{f}'\boldsymbol{\theta}$ *in the model* (1) *exists iff*

$$\mathbf{f} \in \mathscr{M}(\mathbf{M}),$$

where the matrix $\mathbf{M}$ *is defined as in* (7).

b) *If* $\mathbf{f} \in \mathscr{M}(\mathbf{M})$ *then the* MINQE(I, S) *of the function* $\mathbf{f}'\boldsymbol{\theta}$ *is the statistic* $\mathbf{Y}'\mathbf{A}_2\mathbf{Y}$, *where*

(20) $$\mathbf{A}_2 = \sum_1^m \varkappa_i\mathbf{Q}_S'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_S,$$

where $\mathbf{Q}_S = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{S}^{-1}$ ($\mathbf{I}$ *is the unit matrix*) *and* $\boldsymbol{\varkappa} = (\varkappa_1, \ldots, \varkappa_m)'$ *is a solution of the linear system* (9).

Proof. a) The matrix $\mathbf{A}_2$ in $\mathbf{Y}'\mathbf{A}_2\mathbf{Y}$ exists iff the linear system (9) is consistent. This system is consistent iff $\mathbf{f} \in \mathscr{M}(\mathbf{M})$.

b) It is obvious that the matrix $\mathbf{A}_2$ is symmetric. The equation $\mathbf{A}_2\mathbf{X} = \mathbf{0}$ is satisfied because of $\mathbf{Q}_S\mathbf{X} = \mathbf{0}$. It suffices to prove that the matrix $\mathbf{A}_2$ minimizes the expression (13) in the class $\mathscr{A}_1$ for which

$$\operatorname{tr}(\mathbf{S}^{1/2}\mathbf{A}\mathbf{S}^{1/2} - \mathbf{N})^2 + 2\operatorname{tr}\mathbf{X}'\mathbf{A}\mathbf{S}\mathbf{A}\mathbf{X} + \operatorname{tr}(\mathbf{X}'\mathbf{A}\mathbf{X})^2 =$$
$$= \operatorname{tr}\mathbf{A}\mathbf{S}\mathbf{A}\mathbf{S} - 2\sum_1^m \varkappa_i \operatorname{tr}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{A} + \operatorname{tr}\mathbf{N}^2$$

is satisfied because of $\mathbf{A}\mathbf{X} = \mathbf{0}$.

The matrix $\mathbf{A}_2$ minimizes the expression (13) in the class $\mathscr{A}_1$ if for each symmetric matrix $\mathbf{D}$ which satisfies the condition $\mathbf{D}\mathbf{X} = \mathbf{0}$ the inequality

$$\operatorname{tr}(\mathbf{A}_2 + \mathbf{D})\mathbf{S}(\mathbf{A}_2 + \mathbf{D})\mathbf{S} - 2\sum_1^m \varkappa_i \operatorname{tr}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}(\mathbf{A}_2 + \mathbf{D}) \geqq$$
$$\geqq \operatorname{tr}\mathbf{A}_2\mathbf{S}\mathbf{A}_2\mathbf{S} - 2\sum_1^m \varkappa_i \operatorname{tr}\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{A}_2$$

holds.

$$\operatorname{tr}(\mathbf{A}_2 + \mathbf{D})\,\mathbf{S}(\mathbf{A}_2 + \mathbf{D})\,\mathbf{S} - 2\sum_1^m \varkappa_i \operatorname{tr} \mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}(\mathbf{A}_2 + \mathbf{D}) =$$

$$= \operatorname{tr} \mathbf{A}_2\mathbf{S}\mathbf{A}_2\mathbf{S} - 2\sum_1^m \varkappa_i \operatorname{tr} \mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{A}_2 + \operatorname{tr} \mathbf{DSDS} +$$

$$+ 2\operatorname{tr} \mathbf{A}_2\mathbf{SDS} - 2\sum_1^m \varkappa_i \operatorname{tr} \mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SD}\,.$$

With regard to the fact that the expression tr **DSDS** is nonnegative it suffices to prove that

$$\operatorname{tr} \mathbf{A}_2\mathbf{SDS} = \sum_1^m \varkappa_i \operatorname{tr} \mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SD}\,.$$

$$\operatorname{tr} \mathbf{A}_2\mathbf{SDS} = \sum_1^m \varkappa_i \operatorname{tr} \mathbf{S}\mathbf{Q}_s'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_s\mathbf{SD} =$$

$$= \sum_1^m \varkappa_i \operatorname{tr}\left[\mathbf{S} - \mathbf{X}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^- \mathbf{X}'\right] \mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\,.$$

$$.\left[\mathbf{S} - \mathbf{X}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^- \mathbf{X}'\right] \mathbf{D} = \sum_1^m \varkappa_i \operatorname{tr} \mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SD}\,.$$

Corollary 1.5. *One choice of the* MINQE(I, S) *of the vector of unknown variance components* $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)'$ *is*

(21) $$\hat{\boldsymbol{\theta}} = \mathbf{M}^- \mathbf{u}$$

provided the MINQE(I, S) *exists for all components of the vector* $\boldsymbol{\theta}$. *The matrix* $\mathbf{M}$ *is defined as in* (7) *and the i-th element of the vector* $\mathbf{u}$ *is* $\mathbf{u}_i = \mathbf{Y}'\mathbf{Q}_s'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_s\mathbf{Y}$.

Proof. The MINQE(I, S) of the function $\mathbf{f}'\boldsymbol{\theta}$ is

$$\widehat{\mathbf{f}'\theta} = \mathbf{Y}'\mathbf{A}_2\mathbf{Y} = \mathbf{Y}'\sum_1^m \varkappa_i \mathbf{Q}_s'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_s\mathbf{Y} =$$

$$= \sum_1^m \varkappa_i \mathbf{Y}'\mathbf{Q}_s'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_s\mathbf{Y} = \boldsymbol{\varkappa}'\mathbf{u} = \mathbf{f}'\mathbf{M}^-\mathbf{u}\,.$$

Theorem 1.6. a) *The* MINQE(U, S) *of the function* $\mathbf{f}'\boldsymbol{\theta}$ *in the model* (1) *exists iff*

$$\mathbf{f} \in \mathscr{M}(\mathbf{M}) \quad \text{and} \quad \mathbf{C}\boldsymbol{\varkappa} - \mathbf{f} \in \mathscr{M}(\mathbf{B})\,,$$

where the matrix $\mathbf{M}$ *is defined as in* (7), *the* (i, j)*-th element of the matrix* $\mathbf{B}$ *is* $\mathbf{B}_{i,j} = \operatorname{tr} \mathbf{T}^{-1}(\mathbf{V}_i - \mathbf{P}_T\mathbf{V}_i\mathbf{P}_T')\,\mathbf{T}^{-1}\mathbf{V}_j$, *the* (i, j)*-th element of the matrix* $\mathbf{C}$ *is* $\mathbf{C}_{i,j} = \operatorname{tr} \mathbf{T}^{-1}(\mathbf{W}_i - \mathbf{P}_T\mathbf{W}_i\mathbf{P}_T')\,\mathbf{T}^{-1}\mathbf{V}_j$ *and* $\mathbf{T} = \mathbf{S} + \mathbf{XX}'$, $\mathbf{W}_i = \mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}$, $\mathbf{P}_T = \mathbf{X}(\mathbf{X}'\mathbf{T}^{-1}\mathbf{X})^- \mathbf{X}'\mathbf{T}^{-1}$.

b) *If* $\mathbf{f} \in \mathscr{M}(\mathbf{M})$ *and* $\mathbf{C}\boldsymbol{\varkappa} - \mathbf{f} \in \mathscr{M}(\mathbf{B})$ *then the* MINQE(U, S) *of the function* $\mathbf{f}'\boldsymbol{\theta}$ *is the statistic* $\mathbf{Y}'\mathbf{A}_3\mathbf{Y}$, *where*

(22) $$\mathbf{A}_3 = \sum_1^m \varkappa_i \mathbf{T}^{-1}(\mathbf{W}_i - \mathbf{P}_T\mathbf{W}_i\mathbf{P}_T')\,\mathbf{T}^{-1} - \sum_1^m \lambda_i \mathbf{T}^{-1}(\mathbf{V}_i - \mathbf{P}_T\mathbf{V}_i\mathbf{P}_T')\,\mathbf{T}^{-1}\,,$$

where $\boldsymbol{\varkappa} = (\varkappa_1, \ldots, \varkappa_m)'$ *is a solution of the linear system* (9) *ynd* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)'$

is a solution of the linear system

$$(23) \qquad \mathbf{C}\varkappa + \mathbf{B}\lambda = \mathbf{f}\,.$$

Proof. a) The matrix $\mathbf{A}_3$ in $\mathbf{Y}'\mathbf{A}_3\mathbf{Y}$ exists iff the linear system (9) and the linear system (23) are consistent. The linear system (9) is consistent iff $\mathbf{f}' \in \mathcal{M}(\mathbf{M})$. The system (23) is consistent iff the system $\mathbf{C}\varkappa - \mathbf{f} = \mathbf{B}\lambda$ is consistent and this is true iff $\mathbf{C}\varkappa - \mathbf{f} \in \mathcal{M}(\mathbf{B})$.

b) The symmetric matrix $\mathbf{A}_3$ defined in (22) satisfies the condition $\operatorname{tr} \mathbf{A}_3\mathbf{V}_j = \mathbf{f}_j$ $(j = 1, \ldots, m)$ because the equation (23) holds. The equation $\mathbf{X}'\mathbf{A}_3\mathbf{X} = \mathbf{0}$ is satisfied because of $\mathbf{X}'\mathbf{T}^{-1}\mathbf{P}_T = \mathbf{X}'\mathbf{T}^{-1}$ (See Lemma 2.2.6 of the paper [3]). It suffices to prove that the matrix $\mathbf{A}_3$ minimizes the expression (13) in the class $\mathcal{A}_2$.

We can write the expression (13) $(\mathbf{X}'\mathbf{A}\mathbf{X} = \mathbf{0}, \mathbf{T} = \mathbf{S} + \mathbf{X}\mathbf{X}')$ in the form of

$$(24) \qquad \operatorname{tr} \mathbf{ATAT} - 2 \sum_{1}^{m} \varkappa_i \operatorname{tr} \mathbf{SV}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SA} + \operatorname{tr} \mathbf{N}^2\,.$$

Let $\mathbf{D}$ be a matrix for which

$$(25) \qquad \mathbf{D}' = \mathbf{D}\,, \quad \mathbf{X}'\mathbf{D}\mathbf{X} = \mathbf{0}\,, \quad \operatorname{tr} \mathbf{DV}_i = \mathbf{0} \quad (i = 1, \ldots, m)$$

holds. The matrix $\mathbf{A}_3$ minimizes the expression (24) in the class $\mathcal{A}_2$ if for each matrix $\mathbf{D}$ which satisfies the conditions (25) the inequality

$$\operatorname{tr} (\mathbf{A}_3 + \mathbf{D})\,\mathbf{T}(\mathbf{A}_3 + \mathbf{D})\,\mathbf{T} - 2 \sum \varkappa_i \operatorname{tr} \mathbf{SV}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}(\mathbf{A}_3 + \mathbf{D}) \geqq$$
$$\geqq \operatorname{tr} \mathbf{A}_3\mathbf{T}\mathbf{A}_3\mathbf{T} - 2 \sum \varkappa_i \operatorname{tr} \mathbf{SV}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SA}_3$$

holds.

$$\operatorname{tr} (\mathbf{A}_3 + \mathbf{D})\,\mathbf{T}(\mathbf{A}_3 + \mathbf{D})\,\mathbf{T} - 2 \sum_{1}^{m} \varkappa_i \operatorname{tr} \mathbf{SV}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}(\mathbf{A}_3 + \mathbf{D}) =$$
$$= \operatorname{tr} \mathbf{A}_3\mathbf{T}\mathbf{A}_3\mathbf{T} - 2 \sum_{1}^{m} \varkappa_i \operatorname{tr} \mathbf{SV}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SA}_3 + \operatorname{tr} \mathbf{DTDT} +$$
$$+ 2 \operatorname{tr} \mathbf{A}_3\mathbf{TDT} - 2 \sum_{1}^{m} \varkappa_i \operatorname{tr} \mathbf{SV}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SD}\,.$$

With regard to the fact that the expression $\operatorname{tr} \mathbf{DTDT}$ is nonnegative it suffices to prove that $\operatorname{tr} \mathbf{A}_3\mathbf{TDT} = \sum \varkappa_i \operatorname{tr} \mathbf{SV}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SD}$. The matrix $\mathbf{D}$ satisfies the conditions (25) and therefore $\mathbf{P}_T'\mathbf{D}\mathbf{P}_T = \mathbf{0}$, $\operatorname{tr} \mathbf{DV}_i = \mathbf{0}$ $(i = 1, \ldots, m)$ and

$$\operatorname{tr} \mathbf{A}_3\mathbf{TDT} = \sum_{1}^{m} \varkappa_i \operatorname{tr} (\mathbf{SV}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SD} - \mathbf{SV}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SP}_T'\mathbf{DP}_T) -$$
$$- \sum_{1}^{m} \lambda_i \operatorname{tr} (\mathbf{DV}_i - \mathbf{V}_i\mathbf{P}_T'\mathbf{DP}_T) = \sum_{1}^{m} \varkappa_i \operatorname{tr} \mathbf{SV}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SD}\,.$$

Corollary 1.7. *One choice of the* MINQE(U, S) *of the vector of unknown variance components* $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)'$ *is*

$$(26) \qquad \hat{\boldsymbol{\theta}} = \mathbf{M}^-\mathbf{u} - \mathbf{M}^-\mathbf{CB}^-\mathbf{v} + \mathbf{B}^-\mathbf{v}$$

provided the MINQ(U, S) *exists for all components of the vector* $\boldsymbol{\theta}$. *The matrices*

$\mathbf{M}, \mathbf{B}, \mathbf{C}$ *are defined as in Theorem* 1.6, *the i-th element of the vector* $\mathbf{u}$ *is* $\mathbf{u}_i = \mathbf{Y}'\mathbf{T}^{-1}(\mathbf{W}_i - \mathbf{P}_T\mathbf{W}_i\mathbf{P}_T')\,\mathbf{T}^{-1}\mathbf{Y}$, *the i-th element of the vector* $\mathbf{v}$ *is* $\mathbf{v}_i = \mathbf{Y}'\mathbf{T}^{-1}(\mathbf{V}_i - \mathbf{P}_T\mathbf{V}_i\mathbf{P}_T')\,\mathbf{T}^{-1}\mathbf{Y}$.

Proof. The MINQE(U, S) of the function $\mathbf{f}'\boldsymbol{\theta}$ is

$$\widehat{\mathbf{f}'\boldsymbol{\theta}} = \mathbf{Y}'\mathbf{A}_3\mathbf{Y} = \sum_1^m \varkappa_i \mathbf{Y}'\mathbf{T}^{-1}(\mathbf{W}_i - \mathbf{P}_T\mathbf{W}_i\mathbf{P}_T')\,\mathbf{T}^{-1}\mathbf{Y} -$$

$$- \sum_1^m \lambda_i \mathbf{Y}'\mathbf{T}^{-1}(\mathbf{V}_i - \mathbf{P}_T\mathbf{V}_i\mathbf{P}_T')\,\mathbf{T}^{-1}\mathbf{Y} = \boldsymbol{\varkappa}'\mathbf{u} - \boldsymbol{\lambda}'\mathbf{v} =$$

$$= \mathbf{f}'(\mathbf{M}^-\mathbf{u} - \mathbf{M}^-\mathbf{C}\mathbf{B}^-\mathbf{v} + \mathbf{B}^-\mathbf{v})$$

because $\boldsymbol{\varkappa}' = \mathbf{f}'\mathbf{M}^-$ is a solution of the linear system (9) and $\boldsymbol{\lambda}' = \mathbf{f}'\mathbf{M}^-\mathbf{C}\mathbf{B}^- - \mathbf{f}'\mathbf{B}^-$ is a solution of the linear system (23).

Theorem 1.8. a) *The* MINQE(U, I, S) *of the function* $\mathbf{f}'\boldsymbol{\theta}$ *in the model* (1) *exists iff*

$$\mathbf{f} \in \mathscr{M}(\mathbf{M}) \quad \text{and} \quad \mathbf{L}\boldsymbol{\varkappa} - \mathbf{f} \in \mathscr{M}(\mathbf{K}),$$

where the matrix $\mathbf{M}$ *is defined as in* (7), *the* (i, j)*-th element of the matrix* $\mathbf{K}$ *is* $\mathbf{K}_{i,j} = \operatorname{tr} \mathbf{V}_j\mathbf{Q}_S'\mathbf{S}^{-1}\mathbf{V}_i\mathbf{S}^{-1}\mathbf{Q}_S$, *the* (i, j)*-th element of the matrix* $\mathbf{L}$ *is* $\mathbf{L}_{i,j} = \operatorname{tr} \mathbf{V}_j\mathbf{Q}_S'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_S$ *and* $\mathbf{Q}_S = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^-\mathbf{X}'\mathbf{S}^{-1}$.

b) *If* $\mathbf{f} \in \mathscr{M}(\mathbf{M})$ *and* $\mathbf{L}\boldsymbol{\varkappa} - \mathbf{f} \in \mathscr{M}(\mathbf{K})$ *then the* MINQE(U, I, S) *of the function* $\mathbf{f}'\boldsymbol{\theta}$ *is the statistic* $\mathbf{Y}'\mathbf{A}_4\mathbf{Y}$, *where*

$$\text{(27)} \qquad \mathbf{A}_4 = \sum_1^m \varkappa_i \mathbf{Q}_S'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_S - \sum_1^m \gamma_i \mathbf{Q}_S'\mathbf{S}^{-1}\mathbf{V}_i\mathbf{S}^{-1}\mathbf{Q}_S,$$

where $\boldsymbol{\varkappa} = (\varkappa_1, \ldots, \varkappa_m)'$ *is a solution of the linear system* (9) *and* $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m)'$ *is a solution of the linear system*

$$\text{(28)} \qquad \mathbf{L}\boldsymbol{\varkappa} - \mathbf{K}\boldsymbol{\gamma} = \mathbf{f}.$$

Proof. See [5], Theorem 2.3.

Corollary 1.9. *One choice of the* MINQE(U, I, S) *of the vector of unknown variance components* $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)'$ *is*

$$\text{(29)} \qquad \hat{\boldsymbol{\theta}} = \mathbf{M}^-\mathbf{m} - \mathbf{M}^-\mathbf{L}\mathbf{K}^-\mathbf{n} + \mathbf{K}^-\mathbf{n}$$

provided the MINQE(U, I, S) *exists for all components of the vector* $\boldsymbol{\theta}$. *The matrices* $\mathbf{M}, \mathbf{K}, \mathbf{L}$ *are defined as in Theorem* 1.8, *the i-th element of the vector* $\mathbf{m}$ *is* $\mathbf{m}_i = \mathbf{Y}'\mathbf{Q}_S'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_S\mathbf{Y}$ *and the i-th element of the vector* $\mathbf{n}$ *is* $\mathbf{n}_i = \mathbf{Y}'\mathbf{Q}_S'\mathbf{S}^{-1}\mathbf{V}_i\mathbf{S}^{-1}\mathbf{Q}_S\mathbf{Y}$.

Proof. See [5], Corollary 2.4.

2. A COMPARISON OF MINQE (S) AND MINQE

The estimations of the function $\mathbf{f}'\boldsymbol{\theta}$ obtained in this paper (MINQE(S)) are quadratic estimations of the type of $\mathbf{Y}'\mathbf{A}(\mathbf{S})\mathbf{Y}$, where the matrix $\mathbf{A}(\mathbf{S})$ is a function of the known matrix $\mathbf{S}$ which contains prior values of the elements of the covariance matrix $\mathbf{V}_\theta$ in the model (1).

In the papers [2] and [4] quadratic estimations of the function $\mathbf{f}'\boldsymbol{\theta}$ obtained by Rao (MINQE) are defined which are of the type of $\mathbf{Y}'\mathbf{A}(\mathbf{V})\mathbf{Y}$. The matrix $\mathbf{A}(\mathbf{V})$ is a function of the known matrix $\mathbf{V} = \alpha_1\mathbf{V}_1 + \ldots + \alpha_m\mathbf{V}_m$, where $\alpha_1, \ldots, \alpha_m$ are prior values of the variance components $\theta_1, \ldots, \theta_m$ in the model (1).

It is shown in Theorem 2.1 that the MINQE(S) is equal to the MINQE if the matrix $\mathbf{S}$ does not contribute to the estimated situation by new information ($\mathbf{S} = \alpha_1\mathbf{V}_1 + \ldots + \alpha_m\mathbf{V}_m = \mathbf{V}$).

Theorem 2.1. *If* $\mathbf{S} = \mathbf{V}$, *then the* MINQE(I, S) *of the function* $\mathbf{f}'\boldsymbol{\theta}$ *is equal to the* MINQE(I), *the* MINQE(U, S) *of the* $\mathbf{f}'\boldsymbol{\theta}$ *is equal to the* MINQE(U) *and the* MINQE(U, I, S) *of the* $\mathbf{f}'\boldsymbol{\theta}$ *is equal to the* MINQE(U, I).

Proof. It is shown that the MINQE(U, I, S) is equal to the MINQE(U, I) in Theorem 2.7 of the paper [5].

If $\mathbf{S} = \mathbf{V}$ then the MINQE(I, S) of the $\mathbf{f}'\boldsymbol{\theta}$ is (see (20))

$$\widehat{\mathbf{f}'\boldsymbol{\theta}} = \mathbf{Y}' \sum_1^m \varkappa_i \mathbf{Q}_V' \mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_V\mathbf{Y} = \mathbf{Y}'\mathbf{A}_*\mathbf{Y},$$

where

$$\mathbf{A}_* = \sum_1^m \varkappa_i \mathbf{Q}_V' \mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_V =$$
$$= \sum_1^m [\mathbf{I} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^-\mathbf{X}'] \,.$$
$$.\, \mathbf{V}^{-1}\varkappa_i\mathbf{V}_i\mathbf{V}^{-1}[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^-\mathbf{X}'\mathbf{V}^{-1}] =$$
$$= \sum_1^m \mathbf{V}^{-1}[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^-\mathbf{X}'\mathbf{V}^{-1}] \,.$$
$$.\, \varkappa_i\mathbf{V}_i[\mathbf{I} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^-\mathbf{X}']\,\mathbf{V}^{-1} \,.$$

If $\sum \varkappa_i\mathbf{V}_i = \mathbf{W}$ and $\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^-\mathbf{X}'\mathbf{V}^{-1} = \mathbf{I} - \mathbf{P}$ then we have

$$\mathbf{A}_* = \mathbf{V}^{-1}(\mathbf{I} - \mathbf{P})\,\mathbf{W}(\mathbf{I} - \mathbf{P}')\,\mathbf{V}^{-1}$$

and $\mathbf{Y}'\mathbf{A}_*\mathbf{Y}$ is the MINQE(I) of the $\mathbf{f}'\boldsymbol{\theta}$ defined by the formula (5.4.11) in the paper [4].

If $\mathbf{S} = \mathbf{V}$ then the MINQE(U, S) of the function $\mathbf{f}'\boldsymbol{\theta}$ is (see (22))

$$\widehat{\mathbf{f}'\boldsymbol{\theta}} = \mathbf{Y}'(\sum_1^m \varkappa_i\mathbf{T}^{-1}(\mathbf{W}_i - \mathbf{P}_T\mathbf{W}_i\mathbf{P}_T')\,\mathbf{T}^{-1} -$$
$$- \sum_1^m \lambda_i\mathbf{T}^{-1}(\mathbf{V}_i - \mathbf{P}_T\mathbf{V}_i\mathbf{P}_T')\,\mathbf{T}^{-1})\,\mathbf{Y} =$$
$$= \mathbf{Y}'(\sum_1^m \varkappa_i\mathbf{T}^{-1}(\mathbf{V}_i - \mathbf{P}_T\mathbf{V}_i\mathbf{P}_T')\,\mathbf{T}^{-1} -$$

$$-\sum_1^m \lambda_i \mathbf{T}^{-1}(\mathbf{V}_i - \mathbf{P}_T\mathbf{V}_i\mathbf{P}'_T)\,\mathbf{T}^{-1})\,\mathbf{Y} =$$

$$= \mathbf{Y}'(\sum_1^m (\varkappa_i - \lambda_i)\,\mathbf{T}^{-1}(\mathbf{V}_i - \mathbf{P}_T\mathbf{V}_i\mathbf{P}'_T)\,\mathbf{T}^{-1}\mathbf{Y} = \sum_1^m \delta_i \mathbf{Y}'\mathbf{A}_i\mathbf{Y}\,,$$

where $\mathbf{A}_i = \mathbf{T}^{-1}(\mathbf{V}_i - \mathbf{P}_T\mathbf{V}_i\mathbf{P}'_T)\,\mathbf{T}^{-1}$ and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_m)'$ is a solution of the linear system $\mathbf{G}\boldsymbol{\delta} = \mathbf{f}$ (the (i, j)-th element of the matrix $\mathbf{G}$ is $\mathbf{G}_{i,j} = \operatorname{tr} \mathbf{A}_i\mathbf{V}_j$). This result is equal to the MINQE(U) of the $\mathbf{f}'\boldsymbol{\theta}$ which is defined by the formula (5.2.2) in the paper [4].

3. EXAMPLE

We consider a very simple situation when we have two independent measurements y_1, y_2 of the unknown parameter $\boldsymbol{\beta}$ with different variances ($V(y_1) = \theta_1$ and $V(y_2) = = \theta_2$). The mixed linear model (1) is

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}\,,$$

where $\mathbf{Y} = (y_1, y_2)'$, $\mathbf{X} = (1, 1)'$, $\mathbf{e} = (e_1, e_2)'$ and

$$D(\mathbf{e}) = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} = \theta_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \theta_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \theta_1\mathbf{V}_1 + \theta_2\mathbf{V}_2\,.$$

Let

$$\mathbf{S} = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$$

be a matrix which contains prior values of the elements of the covariance matrix $D(\mathbf{e})$. We will show four estimators of some functions of the unknown variance components θ_1, θ_2.

a) The MINQE(S) of θ_1 and θ_2 are (see (17) or (19))

$$\hat{\theta}_1 = \frac{s_1^2}{(s_1s_2 + s_1 + s_2)^2}\,[y_1^2(s_2 + 1)^2 - 2y_1y_2(s_2 + 1) + y_2^2]$$

$$\hat{\theta}_2 = \frac{s_2^2}{(s_1s_2 + s_1 + s_2)^2}\,[y_2^2(s_1 + 1)^2 - 2y_1y_2(s_1 + 1) + y_1^2]$$

If $\mathbf{S} = \mathbf{V}$ ($s_1 = s_2 = 1$) then MINQE(S) of θ_1 and θ_2 are

$$\hat{\theta}_1 = \tfrac{1}{9}(2y_1 - y_2)^2\,, \quad \hat{\theta}_2 = \tfrac{1}{9}(2y_2 - y_1)^2\,.$$

b) The MINQE(I, S) of θ_2 is (see (20) or (21))

$$\hat{\theta}_2 = \frac{s_2^2}{(s_1 + s_2)^2}\,(y_1 - y_2)^2$$

If $\mathbf{S} = \mathbf{V}$ ($s_1 = s_2 = 1$) then the MINQE(I, S) of θ_2 is

$$\hat{\theta}_2 = \tfrac{1}{4}(y_1 - y_2)^2$$

This estimator is equal to the MINQE(I).

c) If $s_1 = 2$ and $s_2 = 1$ then the MINQE(U, S) of θ_1 is (see (22) or (26))

$$\hat{\theta}_1 = y_1^2 - y_1 y_2$$

and this estimator is equal to the MINQE(U) for $\alpha_1 = 2$ and $\alpha_2 = 1$.

d) If $s_1 = s_2 = 1$ then the MINQE(U, I, S) of θ_1 and θ_2 do not exist but for example the MINQE(U, I, S) of the function $\theta_1 + \theta_2$ is independent of the matrix **S** (see (27)) and is equal to the MINQE(U, I)

$$\theta_1 \hat{+} \theta_2 = (y_1 - y_2)^2 .$$

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Súhrn

KVADRATICKÉ ODHADY V ZMIEŠANÝCH LINEÁRNYCH MODELOCH

ŠTEFAN VARGA

V práci sú uvedené nutné a postačujúce podmienky existencie a explicitné vzťahy štyroch typov odhadov lineárnej funkcie variančných komponentov v zmiešanom lineárnom modeli $\mathbf{Y} = \mathbf{X}\beta + \mathbf{e}$ so strednou hodnotou $E(\mathbf{Y}) = \mathbf{X}\beta$ a s kovariančnou maticou $D(\mathbf{Y}) = \theta_1 \mathbf{V}_1 + \ldots$ $\ldots + \theta_m \mathbf{V}_m$.

Author's address: RNDr. *Štefan Varga,* CSc., Chemickotechnologická fakulta SVŠT, Katedra matematiky, Radlinského 9, 812 37 Bratislava.