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Applications of Mathematics, Vol. 36 (1991), No. 2, 81–95

Persistent URL: http://dml.cz/dmlcz/104447

SPECIAL EXACT CURVED FINITE ELEMENTS

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(Received February 8, 1989)

Summary. Special exact curved finite elements useful for solving contact problems of the second order in domains boundaries of which consist of a finite number of circular arcs and a finite number of line segments are introduced and the interpolation estimates are proved.

Keywords: finite elements, special exact curved finite elements, interpolation estimates.

AMS classification: 65L60.

1. INTRODUCTION

Curved triangle elements were introduced by Zlámal in [8]. The present paper describes an other type of curved elements suitable for solving contact problems of the second order in domains Ω whose boundaries Γ consist of a finite number of circular arcs and a finite number of line segments — the case very frequent in electrical engineering and mechanical engineering. The suggested curved elements in this paper — so called special exact curved finite elements — can be used not only along the boundary Γ but also in the interior of Ω and are the natural generalization of linear elements in the case of boundaries Γ described above. Special exact curved finite elements were successfully used to solve contact problems of the electromagnetic field, see [1].

2. DEFINITION AND BASIC PROPERTIES OF A MAPPING F

Let $P_0 = [x_0, y_0]$ be in the (x, y)-plane E_2, $r > 0$ a constant, suppose $x_0^2 + y_0^2 \neq r^2$. Let us denote $K = \{[x, y] \in E_2,\ x^2 + y^2 < r^2\}$, $r_0 = \sqrt{(x_0^2 + y_0^2)}$. We define a mapping $F\colon E_2 \to E_2$ by

$$(1)\qquad F \equiv \begin{cases} x = x_0 + v(r\cos u - x_0) \\ y = y_0 + v(r\sin u - y_0)\,. \end{cases}$$

Then F is a continuous mapping and its partial derivatives

$$(2)\qquad \frac{\partial x}{\partial u} = -vr \sin u\,, \quad \frac{\partial y}{\partial u} = vr \cos u\,,$$

$$\frac{\partial x}{\partial v} = r \cos u - x_0\,, \quad \frac{\partial y}{\partial v} = r \sin u - y_0$$

are finite and continuous in E_2. For the Jacobian J of the mapping F we have

$$J = vr[r_0 \cos (u - u_0) - r]\,.$$

The basic properties of F are given in the two following propositions.

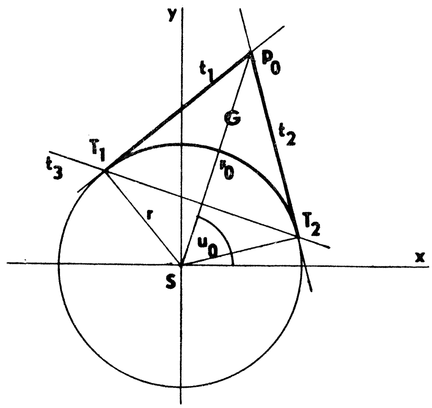

Fig. 1

Proposition 1. *If $r > r_0$ then*

(1) *F maps $\langle 0, 2\pi) \times \langle 0, 1\rangle$ on $\bar{K}$;*

(2) *F is a bijective mapping of $\langle 0, 2\pi) \times (0, 1\rangle$ onto $\bar{K} \setminus \{P_0\}$;*

(3) *F is a bijective regular mapping of $(0, 2\pi) \times (0, 1)$ onto $K \setminus P_0P$, where P_0P is the segment with endpoints $P_0 = [x_0, y_0]$, $P = [r, 0]$.*

Proof. Let $[x, y] \in \bar{K}$. If $[x, y] = [x_0, y_0]$ then the corresponding $[u, v]$ is of the form $[u, 0]$, $u \in \langle 0, 2\pi)$. For $[x, y[\neq [x_0, y_0]$ we get from (1) an equation for v

$$v^2(r_0^2 - r^2) + 2v(x_0x + y_0y - r_0^2) + (x - x_0)^2 + (y - y_0)^2 = 0\,.$$

Using the inequalities $x_0^2 + y_0^2 < r^2$, $x^2 + y^2 \leqq r^2$ we can easily prove that the discriminant D of the equation,

$$D = 4\{(x_0x + y_0y - r_0^2)^2 - (r_0^2 - r^2)\,[(x - x_0)^2 + (y - y_0)^2]\}$$

is nonnegative. We can also prove that the condition $0 < v \leqq 1$ is satisfied only for one of the roots of the equation. Further, $J \neq 0$ if and only if $v \neq 0$.

If $r_0 > r$ let t_1, t_2 be the tangents to the circle $k \equiv x^2 + y^2 = r^2$ passing through the point P_0, $T_1 = t_1 \cap k$, $T_2 = t_2 \cap k$, let t_3 be the straight line passing through T_1, T_2. Let $S = [0, 0]$, let $\varrho_i \equiv t_i S$, $i = 1, 2$ be the half-plane defined by t_i and S, $\varrho_3 \equiv t_3 P_0$. Let us denote $G = \varrho_1 \cap \varrho_2 \cap \varrho_3 \cap (E_2 \setminus K)$, $u_1 = u_0$-arccos r/r_0, $u_2 = u_0 + \arccos r/r_0$ (see Figure 1).

Proposition 2. *If $r_0 > r$ then*

(1) *F maps $\langle u_1, u_2 \rangle \times \langle 0, 1 \rangle$ onto G;*

(2) *F is a bijective mapping of $\langle u_1, u_2 \rangle \times (0, 1\rangle$ onto $G \setminus P_0$;*

(3) *F is a bijective regular mapping of $(u_1, u_2) \times (0, 1)$ onto the interior of G.*

Proof. Let $[x, y] \in G$. If $[x, y] = [x_0, y_0]$, then the corresponding $[u, v]$ is of the form $[u, 0]$, $u \in \langle u_1, u_2 \rangle$. If $[x, y] \neq [x_0, y_0]$ we get again a quadratic equation for v with the discriminant

$$D = 4[x^2(r^2 - y_0^2) + y^2(r^2 - x_0^2) + \\ + 2x_0 y_0 \, xy - 2r^2 \, x_0 x - 2r^2 \, y_0 y + r^2 r_0^2] .$$

Using the equation of the tangents $t_1 \equiv R_1 = 0$ and $t_2 \equiv P_2 = 0$ we can see that

$$\begin{aligned} R_1 R_2 &= (r^2 - x_0^2)\, D/4 &&\text{for} \quad |x_0| \neq r , \\ &= D &&\text{for} \quad x_0 = r , \\ &= -D &&\text{for} \quad x_0 = -r . \end{aligned}$$

Therefore $D \geqq 0$ for $[x, y] \in G \setminus P_0$.

From the properties of the tangents it follows that

$$\begin{aligned} x_0 x + y_0 y - r_0^2 < 0 &\quad \text{for} \quad [x, y] \in G \setminus P_0 , \\ x_0 x + y_0 y \geqq r^2 &\quad \text{for} \quad [x, y] \in G . \end{aligned}$$

Using these inequalities we can conclude that for both roots of the quadratic equation in question we have $0 < v \leqq 1$. However only for one of them the corresponding u lies in the closed interval $\langle u_0 - \arccos r/r_0, \ u_0 + \arccos r/r_0 \rangle$. For $u \in (u_1, u_2)$ we have $\cos(u - u_0) > r/r_0$, therefore $J = vr[r_0 \cos(u - u_0) - r] \neq 0$ in $(u_1, u_2) \times (0, 1)$.

3. TRIANGULATION OF A DOMAIN Ω

Let Ω be a nonempty bounded domain in the plane with a lipschitzian boundary Γ consisting of a finite number of circular arcs and of a finite number of line segments. We divide Ω into a finite number of triangles T which are either curved or straight sided triangles. Curved triangles have two straight sides, the third side is a circular arc. As usual we shall suppose that any two triangles are either disjoint or have a common vertex or a common side. Such a division will be called a triangulation of Ω.

For a curved triangle T we denote by P_0, P_1, P_2 its vertices supposing that P_1, P_2 lie on the circular arc. By the angle of sides of a curved triangle we understand the angle of the tangent lines constructed at the common vertex to these sides. With every curved triangle T we associate the straight sided triangle T' which has the same vertices P_0, P_1, P_2. We shall deal with curved triangles of two types, namely

a) convex triangles, and

b) nonconvex triangles, see Figure 2.

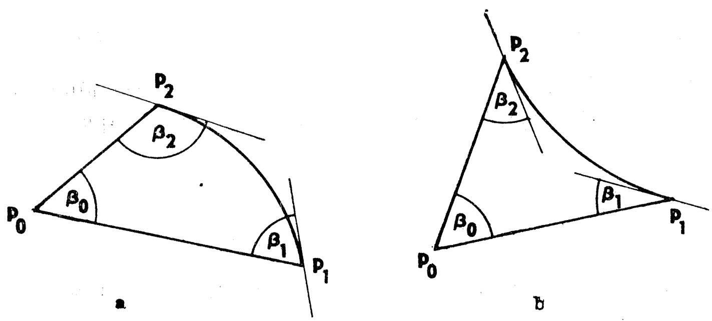

Fig. 2

With every triangulation of Ω we associate two positive constants α, R so that:

(1) All angles of straight sided triangles are greater than or equal to α.

(2) For every convex curved triangle:

(a) the vertex P_0 is inside the circle whose part is the curved side;

(b) all angles of the associated triangle T' are greater than or equal to α;

(c) $\beta_0 + \beta_1 \leqq \pi$, $\beta_0 + \beta_2 \leqq \pi$.

(3) All angles of a nonconvex curved triangle T are greater than or equal to α.

(4) The radii of the curved sides of all triangles are greater than or equal to R.

Under these conditions 1–4, we can prove by means of elementary geometry the following properties of curved triangles:

Proposition 3. *Suppose that a curved triangle T fulfils conditions 1–4, let l be the length of its curved side, d the maximal distance of vertices of T, and p the diameter of T. Then*

(a) $d \sin \alpha < l < Hd$, *where* $H = 2\pi\sqrt{(3)}/9$;

(b) $p < 2d/\sqrt{3}$.

Proposition 4. *Let T be a curved triangle fulfilling conditions* 1–4, *let P_0, P_1, P_2 be the vertices of T, r the radius of the curved side, d the greatest distance of vertices of T. Suppose that $P_0 = [u_0, r_0]$, $P_1 = [u_1, r]$, $P_2 = [u_2, r]$ are the polar coordinates of the vertices, $u_1 < u_2$. Then*

$$|r - r_0 \cos(u - u_0)| \geqq (d/2) \sin \alpha \quad \textit{for all} \quad u \in \langle u_1, u_2 \rangle .$$

4. CONSTRUCTION OF INTERPOLATION FUNCTIONS

Let Ω be a domain as in Sec. 3. We triangulate it supposing that conditions 1–4, are fulfilled. Let us denote by P_m, $m = 1, 2, \ldots, q$ the vertices of triangles of the given triangulation. For each m we shall construct a function φ_m defined in Ω,

$$\varphi_m(P_m) = 0 \quad \text{for} \quad n \neq m \,, \quad \varphi_m(P_m) = 1 \,.$$

Let T be an arbitrary triangle from the triangulation of Ω. If T is a curved triangle, let r be the radius of its curved side, P_i, P_j, P_k its vertices, P_j, P_k are on the curved side, $P_i = [x_0, y_0]$ being the intersection of the straight line sides. We suppose that the mapping

$$F \equiv \begin{cases} x = x_0 + v(r \cos u - x_0) \\ y = y_0 + v(r \sin u - y_0) \end{cases}$$

maps $\langle u_1, u_2 \rangle \times \langle 0, 1 \rangle$ onto T, $F(u_1, 1) = P_j$, $F(u_2, 1) = P_k$. Now we define functions Φ_i, φ_i:

$$\Phi_i(u, v) = 1 - v \,, \quad u \in \langle u_1, u_2 \rangle \,, \quad v \in \langle 0, 1 \rangle \,,$$
$$\varphi_i(x, y) = \Phi_i(F^{-1}(x, y)) \,, \quad [x, y] \in T \,, \quad [x, y] \neq P_i \,,$$
$$\varphi_i(P_i) = 1 \,.$$

Since $\Phi_i(u, 0) = 1$ for all $u \in \langle u_1, u_2 \rangle$ the function φ_i is well defined and continuous in T.

Similarly we define

$$\Phi_j(u, v) = v \frac{u - u_2}{u_1 - u_2} \,, \quad u \in \langle u_1, u_2 \rangle \,, \quad v \in \langle 0, 1 \rangle \,,$$
$$\varphi_j(x, y) = \Phi_j(F^{-1}(x, y)) \,, \quad [x, y] \in T \,, \quad [x, y] \neq P_i \,,$$
$$\varphi_j(P_i) = 0 \,,$$
$$\Phi_k(u, v) = v \frac{u - u_1}{u_2 - u_1} \,, \quad u \in \langle u_1, u_2 \rangle \,, \quad v \in \langle 0, 1 \rangle \,,$$
$$\varphi_k(x, y) = \Phi_k(F^{-1}(x, y)) \,, \quad [x, y] \in T \,, \quad [x, y] \neq P_i \,,$$
$$\varphi_k(P_i) = 0 \,.$$

The functions φ_j, φ_k are again well defined and continuous in T. The partial derivatives of the functions φ_i, φ_j, φ_k are

$$\frac{\partial \varphi_i}{\partial x} = \frac{\cos u}{r_0 \cos(u - u_0) - r}, \quad \frac{\partial \varphi_i}{\partial y} = \frac{\sin u}{r_0 \cos(u - u_0) - r},$$

$$\frac{\partial \varphi_j}{\partial x} = \frac{r \sin u - y_0 - (u - u_2) r \cos u}{r(u_1 - u_2)[r_0 \cos(u - u_0) - r]},$$

$$\frac{\partial \varphi_j}{\partial y} = -\frac{r \cos u - x_0 + r(u - u_2) \sin u}{r(u_1 - u_2)[r_0 \cos(u - u_0) - r]},$$

$$\frac{\partial \varphi_k}{\partial x} = \frac{r \sin u - y_0 - (u - u_1) r \cos u}{r(u_2 - u_1)[r_0 \cos(u - u_0) - r]},$$

$$\frac{\partial \varphi_k}{\partial y} = -\frac{r \cos u - x_0 + r(u - u_1) \sin u}{r(u_2 - u_1)[r_0 \cos(u - u_0) - r]}.$$

They are continuous and bounded (by Proposition 4) inside each triangle of the triangulation.

Let P_m be an arbitrary vertex in the triangulation. Let us consider all triangles T with the common vertex P_m. If T is a straight sided triangle then φ_m is a linear polynomial such that $\varphi_m(P_m) = 1$, and φ_m vanishes at the other two vertices of T. If T is a curved triangle then φ_m is one of the functions φ_i, φ_j, φ_k defined above depending on the position of P_m (i.e., relative to whether P_m lies on a curved side or not). We define $\varphi_m = 0$ on each triangle T, $P_m \in T$. From the definition of φ_m we see that $\varphi_m(P_m) = 1$, $\varphi_m(P_n) = 0$ for $n \neq m$, $n = 1, 2, \ldots, q$. It is easy to verify that φ_m is continuous in Ω and $\varphi_m \in W_2^{(1)}(\Omega)$.

Definition 1. *Let f be a function defined in Ω. The function $\varphi \in W_2^{(1)}(\Omega)$, $\varphi = \sum_{m=1}^{q} f(P_m)\, \varphi_m$ will be called the interpolation function associated with the function f.*

Another type of the interpolation function is useful in applications, especially in solving contact problems of the electromagnetic field. For details, see [1].

Again we suppose that the given triangulation of Ω fulfils conditions

(1)–(4), but we add one more condition:

(5) in each curved triangle $u_2 - u_1 \leqq \pi/2$ is valid, u_1, u_2 being defined by the mapping F.

For a curved triangle T we define similarly as before

$$\Psi_i(u, v) = 1 - v, \quad u \in \langle u_1, u_2 \rangle, \quad v \in \langle 0, 1 \rangle,$$
$$\psi_i(x, y) = \Psi_i(F^{-1}(x, y)), \quad [x, y] \in T, \quad [x, y] \neq P_i,$$
$$\psi_i(P_i) = 1,$$

$$\Psi_j(u, v) = v \frac{\sin(u - u_2)}{\sin(u_1 - u_2)}, \quad u \in \langle u_1, u_2 \rangle, \quad v \in \langle 0, 1 \rangle,$$

$$\psi_j(x, y) = \Psi_j(F^{-1}(x, y)), \quad [x, y] \in T, \quad [x, y] \neq P_i,$$

$$\psi_j(P_i) = 0,$$

$$\Psi_k(u, v) = v \frac{\sin(u - u_1)}{\sin(u_2 - u_1)}, \quad u \in \langle u_1, u_2 \rangle, \quad v \in \langle 0, 1 \rangle,$$

$$\psi_k(x, y) = \Psi_k(F^{-1}(x, y)), \quad [x, y] \in T, \quad [x, y] \neq P_i,$$

$$\psi_k(P_i) = 0.$$

The functions ψ_i, ψ_j, ψ_k are obviously continuous in T, their partial derivatives are continuous and bounded in the interior of T, (we use Proposition 4).

Let P_m be an arbitrary vertex in the given triangulation. Using the same procedure as above we define a function ψ_m with the help of functions ψ_i, ψ_j, ψ_k and linear polynomials so that $\psi_m(P_m) = 1$, $\psi_m(P_n) = 0$ for $n \neq m$. It is easy to verify that ψ_m is continuous in Ω, $\psi_m \in W_2^{(1)}(\Omega)$.

Definition 2. *Let f be a function defined in Ω. The function $\psi \in W_2^{(1)}(\Omega)$, $\psi = \sum_{m=1}^{q} f(P_m)\, \psi_m$ will be called the sine-interpolation function associated with f.*

5. INTERPOLATION ESTIMATES

Throughout this section we suppose that Ω is a bounded domain in the plane with a lipschitzian boundary Γ consisting of a finite number of circular arcs and a finite number of line segments. For functions defined in Ω we shall use the usual notation for partial derivatives

$$D^i h(x, y) = \frac{\partial^{|i|} h}{\partial x^{i_1}\, \partial y^{i_2}}, \quad i = (i_1, i_2), \quad |i| = i_1 + i_2.$$

First we shall prove some auxiliary lemmas. Lemma 2 (b) is a special case of Theorem 2 in [10], but the proof presented in this paper gives better estimates, 2 (a) is proved in [7].

Lemma 1. *Let g be a function defined and continuous in $\langle a, b \rangle$, $|g'(s)| \leqq k_1$, $s \in (a, b)$. Let $g(\varepsilon) = k_0$, $\varepsilon \in (a, b)$. Then $|g(s)| \leqq |k_0| + k_1(b - a)$.*

Lemma 2. *Let g be a function continuous in $\langle a, b \rangle$, $g(a) = \varepsilon_1$, $g(b) = \varepsilon_2$, $|g''(s)| \leqq k_2$ in (a, b).*

Then

(a) $|g(s)| \leqq \max |\varepsilon_i| + \frac{1}{8} k_2 (b - a)^2$;

(b) $|g'(s)| \leqq 2/(b - a) \max |\varepsilon_i| + \frac{1}{2} k_2 (b - a)$.

Proof. (b) Let us choose $s_0 \in (a, b)$. Then for any $s \in (a, b)$ we have

$$g'(s) = g'(s_0) + \int_{s_0}^{s} g''(\xi)\, d\xi ,$$

$$g(b) = g(a) + \int_a^b g'(s)\, ds = g(a) + \int_a^b g'(s_0)\, ds + \int_a^b \left(\int_{s_0}^{s} g''(\xi)\, d\xi\right) ds .$$

Therefore

$$|g'(s_0)| \leqq 1/(b-a)\left[|g(b) - g(a)| + \left|\int_a^b\left(\int_{s_0}^{s} g''(\xi)\, d\xi\right) ds\right|\right]$$

and

$$\left|\int_a^b\left(\int_{s_0}^{s} g''(\xi)\, d\xi\right) ds\right| \leqq \int_a^{s_0} \left|\int_{s_0}^{s} g''(\xi)\, d\xi\right| ds + \int_{s_0}^{b} \left|\int_{s_0}^{s} g''(\xi)\, d\xi\right| ds \leqq$$

$$\leqq k_2 \int_a^{s_0}\left(\int_s^{s_0} d\xi\right) ds + k_2 \int_{s_0}^{b}\left(\int_{s_0}^{s} d\xi\right) ds = \tfrac{1}{2}k_2[(s_0 - a)^2 +$$

$$+ (s_0 - b)^2] \leqq \tfrac{1}{2}k_2(b-a)^2 .$$

Combining these inequalities we have

$$|g'(s_0)| \leqq \frac{2}{b-a} \max(\varepsilon_1, \varepsilon_2) + \frac{k_2}{2}(b-a) .$$

Remark 1. Let Ω be a bounded domain with a lipschitzian boundary, $f(x, y)$ a function defined in Ω. Suppose that $f \in C(\bar{\Omega})$, $|D^i f| \leqq M_2$, $|i| = 2$. Then there exists a constant $M_1 > 0$ such that $|D^i f| \leqq M_1$ in Ω, $|i| = 1$.

Lemma 3. *Let T be a curved triangle, r the radius of the curved side. Suppose that T fulfils conditions 1–4. Let f be a continuous function in $\bar{T}$, $|D^i f| \leqq M_2$ in T, $|i| = 2$. Suppose that the mapping*

$$F \equiv \begin{cases} x = x_0 + v(r\cos u - x_0) \\ y = y_0 + v(r\sin u - y_0) \end{cases}$$

maps $\langle u_1, u_2\rangle \times \langle 0, 1\rangle$ onto $\bar{T}$.

Let us denote by d the greatest distance of vertices of T. Then

$$\left|\frac{\partial^2 f}{\partial u^2}\right| \leqq v(4M_2 r^2 + 2M_1 r) , \quad \left|\frac{\partial^2 f}{\partial v^2}\right| \leqq \frac{16}{3} M_2 d^2 ,$$

where M_1 is the constant from Remark 1.

Proof. Let us denote by p the diameter of T. Using Proposition 3 we have

$$\left|\frac{\partial x}{\partial u}\right| = |-vr\sin u| \leqq rv , \qquad \left|\frac{\partial y}{\partial u}\right| \leqq rv ,$$

$$\left|\frac{\partial x}{\partial v}\right| = |r\cos u - x_0| \leqq p \leqq \frac{2}{\sqrt{3}}\, d , \qquad \left|\frac{\partial y}{\partial v}\right| \leqq p \leqq \frac{2d}{\sqrt{3}} ,$$

$$\left|\frac{\partial^2 x}{\partial u^2}\right| = |-vr\cos u| \leqq rv , \qquad \left|\frac{\partial^2 y}{\partial u^2}\right| \leqq rv ,$$

$$\left|\frac{\partial^2 x}{\partial v^2}\right| = \left|\frac{\partial^2 y}{\partial v^2}\right| = 0 .$$

An easy computation gives the estimates in Lemma 3.

Lemma 4. *Let $g(u, v)$ be a continuous function in $\langle u_1, u_2\rangle \times \langle 0, 1\rangle$, suppose that there exist constants N_2, $Q_2 > 0$ such that*

$$\left|\frac{\partial^2 g}{\partial u^2}\right| \leqq vN_2 , \quad \left|\frac{\partial^2 g}{\partial v^2}\right| \leqq Q_2 \quad in \quad (u_1, u_2) \times (0, 1) .$$

Further we suppose that $g(u, 0) = 0$ for all

$$u \in \langle u_1, u_2\rangle , \quad g(u_1, 1) = g(u_2, 1) = 0 .$$

Then

(a) $|g(u, v)| \leqq \frac{1}{8}N_2(u_2 - u_1)^2 + \frac{1}{8}Q_2 , \qquad u \in \langle u_1, u_2\rangle , \quad v \in \langle 0, 1\rangle ;$

(b) $\left|\frac{\partial g}{\partial u}\right| \leqq \left[\frac{Q_2}{u_2 - u_1} + \frac{N_2}{2}(u_2 - u_1)\right] v , \quad u \in (u_1, u_2) , \quad v \in (0, 1) ;$

(c) $\left|\frac{\partial g}{\partial v}\right| \leqq \frac{N_2}{4}(u_2 - u_1)^2 + \frac{Q_2}{2} , \qquad u \in (u_1, u_2) , \quad v \in (0, 1) .$

Proof. (a) Let us choose $\varepsilon > 0$. There exists $\delta^- > 0$ such that $|g(u_1, 1 - \delta)| \leqq \varepsilon$, $|g(u_2, 1 - \delta)| \leqq \varepsilon$ *for all* $\delta > 0$, $\delta < \delta^-$. Let $[u_0, v_0] \in \langle u_1, u_2\rangle \times \langle 0, 1\rangle$ be arbitrary, $\delta < \delta^-$. The function $h(u) = g(u, 1 - \delta)$ fulfils $|h''(u)| \leqq vN_2$ and using Lemma 2 we get the inequality

$$|h(u)| \leqq \varepsilon + \tfrac{1}{8}vN_2(u_2 - u_1)^2 , \quad u \in \langle u_1, u_2\rangle .$$

In particular

$$|g(u_0, 1 - \delta)| \leqq \varepsilon + \tfrac{1}{8}vN_2(u_2 - u_1)^2 .$$

For $\delta \to 0$ we have

$$|g(u_0, 1)| \leqq \tfrac{1}{8}vN_2(u_2 - u_1)^2 .$$

For the function $k(v) = g(u_0, v)$ we use Lemma 2 obtaining

$$|k(v)| \leqq \tfrac{1}{8}vN_2(u_2 - u_1)^2 + \tfrac{1}{8}Q_2 , \quad v \in \langle 0, 1\rangle .$$

Finally,

$$|g(u_0, v_0)| \leqq \tfrac{1}{8}N_2(u_2 - u_1)^2 + \tfrac{1}{8}Q_2 .$$

(b) Let us choose $[u_0, v_0] \in (u_1, u_2) \times (0, 1)$, $\varepsilon > 0$. There exists $\delta^- > 0$ such that $|g(u_1 + \delta, 1)| < \varepsilon$, $|g(u_2 - \delta, 1)| < \varepsilon$ for arbitrary $\delta < \delta^-$. Let $\delta < \delta^-$. For the function $k(v) = g(u_1 + \delta, v)$, $v \in \langle 0, 1\rangle$ we have

$$|k(v)| \leqq \varepsilon v + \left|\frac{\partial^2 g}{\partial v^2}(u_1 + \delta, \xi)\frac{v(v - 1)}{2!}\right| ,$$

$\xi \in (0, 1)$ if we use a Lagrange interpolation polynomial of the first degree.

Therefore $|g(u_1 + \delta, v)| \leqq \varepsilon v + (Q_2/2)\, v$ and for $\delta \to 0$, $|g(u_1, v)| \leqq (Q_2/2)\, v$ for all $v \in \langle 0, 1\rangle$. In particular, $|g(u_1, v_0)| \leqq (Q_2/2)\, v_0$. Similarly we get $|g(u_2, v_0)| \leqq (Q_2/2)\, v_0$.

Using Lemma 2 for the function $h(u) = g(u, v_0)$, $u \in \langle u_1, u_2 \rangle$ we obtain an estimate

$$\left|\frac{\partial g}{\partial u}(u_0, v_0)\right| = |h'(u_0)| \leqq \frac{2}{u_2 - u_1} \cdot \frac{Q_2}{2} v_0 + \tfrac{1}{2} v_0 Q_2 (u_2 - u_1) .$$

(c) Again let $[u_0, v_0]$ be in $(u_1, u_2) \times (0, 1)$. From the proof of part (a) we know that $|g(u_0, 1)| \leqq (N_2/8)\, v(u_2 - u_1)^2$.
Applying Lemma 2 to the function $k(v) = g(u_0, v)$, $v \in \langle 0, 1 \rangle$ we have

$$\left|\frac{\partial g}{\partial v}(u_0, v_0)\right| \leqq 2 \frac{N_2}{8} v_0 (u_2 - u_1)^2 + \frac{1}{2} Q_2 \leqq \frac{N_2}{4} (u_2 - u_1)^2 + \frac{1}{2} Q_2 .$$

Theorem 1. *Let T be a curved triangle satisfying conditions 1–4, Sec. 3, let f be continuous in $\overline{T}$, $|D^i f| \leqq M_2$ in the interior of T, $|i| = 2$. Let φ be an interpolation function associated with the function f (see Definition 1). Then there exist constants $c_1, c_2 > 0$ independent of T such that*

(1) $\quad |f - \varphi| \leqq c_1 d^2 \quad$ *in* $\quad T$,

(2) $\quad |D^i(f - \varphi)| \leqq (c_2/\sin \alpha)\, d \quad$ *in the interior of* $\quad T$,

where α is the smallest angle of T and
d the greatest distance of vertices of T.

Proof. Let r be the radius of the curved side of T, $P_0[x_0, y_0]$, P_1, P_2 vertices of T. Let

$$F \equiv \begin{cases} x = x_0 + v(r \cos u - x_0) \\ y = y_0 + v(r \sin u - y_0) \end{cases}$$

maps $\langle u_1, u_2 \rangle \times \langle 0, 1 \rangle$ onto $\overline{T}$.
Let $M_1 > 0$ be a constant, $|D^i f| \leqq M_1$, $|i| = 1$ in T (see Remark 1). We have

$$\frac{\partial^2 \varphi_j}{\partial u^2} = \frac{\partial^2 \varphi_j}{\partial v^2} = 0 \quad \text{in} \quad (u_1, u_2) \times (0, 1), \quad j = 1, 2, 3$$

and using Lemma 3 for the function $g(u, v) = f(x(u, v), y(u, v)) - \varphi(u, v)$ we conclude

$$\left|\frac{\partial^2 g}{\partial u^2}\right| \leqq v(4M_2 r^2 + 2M_1 r), \quad \left|\frac{\partial^2 g}{\partial v^2}\right| \leqq \frac{16}{3} M_2 d^2 .$$

Further, $g(u, 0) = 0$ for all $u \in \langle u_1, u_2 \rangle$, $g(u_1, 1) = g(u_2, 1) = 0$. Therefore the function g satisfies conditions of Lemma 6 with constants

$$N_2 = 4M_2 r^2 + 2M_1 r, \quad Q_2 = \tfrac{16}{3} M_2 d^2$$

and this lemma implies

$$|g(u, v)| \leqq \tfrac{1}{8}(u_2 - u_1)^2 (4M_2 r^2 + 2M_1 r) + \tfrac{2}{3} M_2 d^2 .$$

Proposition 3 and the equality $l = r(u_2 - u_1)^2$ give

$$|g(u, v)| \leqq \frac{d^2}{12}\left[M_2(6H^2 + 1) + \frac{3M_1H^2}{r}\right], \quad \text{where} \quad H = 2\pi . \frac{\sqrt{3}}{9} .$$

Since $\max_{[x,y]\in T} |g(x, y)| = \max_{[u,v]\in\langle u_1,u_2\rangle\times\langle 0,1\rangle} |g(u, v)|$, (1) of Theorem 1 is proved with a constant

$$c_1 = \frac{1}{12}\left[M_2(6H^2 + 1) + \frac{3M_1H^2}{R}\right].$$

To prove (2) we use again Lemma 6 and Proposition 3 obtaining estimates for $D^i g$, $|i| = 1$ in $(u_1, u_2) \times (0, 1)$:

$$\left|\frac{\partial g}{\partial u}\right| \leqq v\left[\frac{Q_2}{u_2 - u_1} + \frac{N_2}{2}(u_2 - u_1)\right] \leqq$$

$$\leqq vd\left[2M_2\, r\left(H + \frac{8}{3 \sin \alpha}\right) + M_1 H\right],$$

$$\left|\frac{\partial g}{\partial v}\right| \leqq \frac{N_2}{4}(u_2 - u_1)^2 + \frac{Q_2}{2} \leqq d^2\left[M_2\left(H^2 + \frac{8}{3}\right) + \frac{M_1H^2}{2r}\right].$$

If J denotes the Jacobian of a mapping F, then

$$\frac{\partial g}{\partial x} = \frac{1}{J}\left(\frac{\partial g}{\partial u} . \frac{\partial y}{\partial v} - \frac{\partial g}{\partial v} . \frac{\partial y}{\partial u}\right),$$

$$\frac{\partial g}{\partial y} = \frac{1}{J}\left(- \frac{\partial g}{\partial u} . \frac{\partial x}{\partial v} + \frac{\partial g}{\partial v} . \frac{\partial x}{\partial u}\right).$$

Again we use Proposition 5 and the inequality

$$|r \sin u - y_0 \leqq \frac{2}{\sqrt{3}} d$$

and we conclude

$$\left|\frac{\partial g}{\partial x}\right| \leqq \frac{d}{\sin \alpha}\left[M_2\left(\frac{4}{\sqrt{3}} H + \frac{32}{3\sqrt{(3)} \sin \alpha} + H^2 + \frac{8}{3}\right) +\right.$$

$$\left. + M_1\left(\frac{2H}{R\sqrt{3}} + \frac{H^2}{2R}\right)\right] = c_2 \frac{d}{\sin \alpha}.$$

Analogously

$$\left|\frac{\partial g}{\partial y}\right| \leqq c_2 \frac{d}{\sin \alpha}.$$

Corollary 1. *Let Ω be a bounded domain in the plane with a lipschitzian boundary Γ consisting of a finite number of circular arcs and a finite number of line segments, let $\bar{\Omega} = \bigcup_i T_i$ be a triangulation of Ω satisfying conditions 1–4, Sec. 3. Suppose that f is a continuous function in $\bar{\Omega}$, $|D^i f| \leqq M_2$, $|i| = 2$ in Ω, φ is an interpolation function of f. Then there exist constants $k_1, k_2 > 0$ such that*

$$\|f - \varphi\|_{L_2(\Omega)} \leqq k_1 d^2 ,$$

$$\|D^i(f - \varphi)\|_{L_2(\Omega)} \leqq \frac{k_2}{\sin \alpha} d ,$$

where α is the smallest angle,

d the greatest distance of vertices in triangles T_i.

Proof. We use Theorem 1 and the well-known estimates for straight sided triangles (see [3]). We denote by c_1 the maximum of the constants c_1 from Theorem 1, similarly for c_2. Then

$$\|f - \varphi\|^2_{L_2(\Omega)} = \int_\Omega |f - \varphi|^2 \,\mathrm{d}x\,\mathrm{d}y =$$

$$= \sum_i \int_{T_i} |f - \varphi|^2 \,\mathrm{d}x\,\mathrm{d}y \leqq c_1^2 d^4 \sum_i \int_{T_i} \mathrm{d}x\,\mathrm{d}y = \mu(\Omega)\, c_1^2 d^4 ,$$

$$\|D^i(f - \varphi)\|_{L_2(\Omega)} = \int_\Omega |D^i(f - \varphi)|^2 \,\mathrm{d}x\,\mathrm{d}y \leqq \frac{c_2^2}{\sin^2 \alpha} \mu(\Omega)\, d^2 .$$

Theorem 2. *Let T be a curved triangle satisfying conditions 1–5, Sec. 3, let f be a continuous function in $\bar{T}$, $|D^i f| \leqq M_2$, $|i| = 2$ inside T, ψ the sine interpolation function of f. Then there exist constants $c_1, c_2 > 0$ independent of T such that*

(1) $$|f - \psi| \leqq c_1 d^2 \quad \text{in} \quad \bar{T};$$

(2) $$|D^i(f - \psi)| \leqq \frac{c_2}{\sin \alpha} d \quad \text{inside} \quad T,$$

α being the smallest angle of T,
d the greatest distance of vertices of T.

Proof. We denote by r the radius of the curved side of T, $P_0[x_0, y_0]$, P_1, P_2 vertices of T.

Let

$$F \equiv \begin{cases} x = x_0 + v(r \cos u - x_0) \\ y = y_0 + v(r \sin u - y_0) \end{cases}$$

be the mapping of $\langle u_1, u_2 \rangle \times \langle 0, 1 \rangle$ onto $\bar{T}$.

We have

$$\frac{\partial^2 \psi_0}{\partial u^2} = \frac{\partial^2 \psi_0}{\partial v^2} = 0 , \qquad \frac{\partial^2 \psi_1}{\partial v^2} = \frac{\partial^2 \psi_2}{\partial v^2} = 0 ,$$

$$\frac{\partial^2 \psi_1}{\partial u^2} = -\frac{v \sin (u - u_2)}{\sin (u_1 - u_2)} , \qquad \frac{\partial^2 \psi_2}{\partial u^2} = -\frac{v \sin (u - u_1)}{\sin (u_2 - u_1)}$$

and because $|u_2 - u_1| \leqq \pi/2$ we get the estimates

$$\left|\frac{\partial^2 \psi_1}{\partial u^2}\right| \leqq v\,, \quad \left|\frac{\partial^2 \psi_2}{\partial u^2}\right| \leqq v\,.$$

Now we use Lemma 5 for the function $g(u, v) = f(x(u, v), y(u, v)) - \psi(u, v)$ and we get the inequalities

$$\left|\frac{\partial^2 g}{\partial u^2}\right| \leqq v(4M_2 r^2 + 2M_1 r + 2B_1)\,,$$

$$\left|\frac{\partial^2 g}{\partial v^2}\right| \leqq \frac{16}{3} M_2 d^2\,,$$

where $B_1 = 2 \max_T |f(x, y)|$, M_1 is the constant from Remark 1. Further, $g(u, 0) = 0$ for $u \in \langle u_1, u_2 \rangle$, $g\langle u_1, 1\rangle = g(u_2, 1) = 0$, therefore the function g satisfies conditions of Lemma 6 with constants

$$N_2 = 4M_2 r^2 + 2M_1 r + 2B_1\,, \quad Q_2 = \tfrac{16}{3} M_2 d^2\,.$$

Using Lemma 6 we get (with $H = (2\sqrt{(3)}/9)\,\pi$):

$$|g(u, v)| \leqq \tfrac{1}{8}(4M_2 r^2 + 2M_1 r + 2B_1)(u_2 - u_1)^2 + \tfrac{2}{3} M_2 d^2 \leqq$$

$$\leqq \frac{d^2}{12}\left[M_2(6H^2 + 1) + \frac{2M_1}{r} H^2 + \frac{3B_1 H^2}{r^2}\right] \leqq \frac{d^2}{12} M_2(6H^2 + 1) +$$

$$+ \frac{3M_1 H^2}{R} + \frac{3B_1 H^2}{R^2}\Big]\,,$$

$$\left|\frac{\partial g}{\partial u}\right| \leqq v\left[\frac{16M_2 d^2 r}{3l} + (2M_2 r^2 + M_1 r + B_1)\frac{l}{r}\right] \leqq$$

$$\leqq vd\left[2M_2 r\left(\frac{8}{3 \sin \alpha} + H\right) + M_1 H + \frac{B_1 H}{r}\right]$$

$$\left|\frac{\partial g}{\partial v}\right| \leqq \left(M_2 r^2 + \frac{M_1}{2} r + \frac{B_1}{2}\right)\frac{l^2}{r^2} + \frac{8}{3} M_2 d^2 \leqq$$

$$\leqq d^2\left[\left(H^2 + \frac{8}{3}\right) M_2 + \frac{M_1 H^2}{2r} + \frac{B_1 H^2}{r^2}\right].$$

Now we use the inequalities $|r \sin u - y_0| \leqq (2/\sqrt{(3)})\, d$, $|r \cos u - x_0| \leqq (2/\sqrt{(3)})\, d$, Propositions 4, 5 and the equality $l = r(u_2 - u_1)^2$, obtaining

$$\left|\frac{\partial g}{\partial x}\right| \leqq \frac{c_2}{\sin \alpha}\, d\,, \quad \left|\frac{\partial g}{\partial y}\right| \leqq \frac{c_2}{\sin \alpha}\, d\,,$$

where

$$C_2 = 4\left[M_2\left(\frac{32}{3\sqrt{(3)}\sin\alpha} + \frac{4}{\sqrt{3}}H + H^2 + \frac{8}{3}\right) + \right.$$

$$\left. + M_1\left(\frac{2H}{\sqrt{(3R)}} + \frac{H^2}{2R}\right) + B_1\left(\frac{2H}{\sqrt{(3R)}} + \frac{H^2}{R^2}\right)\right].$$

Corollary 2. *Let Ω be a bounded domain in the plane with a lipschitzian boundary Γ consisting of a finite number of circular arcs and a finite number of line segments. Let $\bar{\Omega} = \bigcup_i T_i$ be a triangulation of Ω satisfying conditions 1–5, Sec. 3. Let f be a continuous function in $\bar{\Omega}$, $|D^i f| \leqq M_2$, $|i| = 2$ in Ω, let ψ be the sine-interpolation function of f. Then there exist constants $K_1, K_2 > 0$ such that*

$$\|f - \psi\|_{L_2(\Omega)} \leqq K_1 d^2\,,$$

$$\|D^i(f - \psi)\|_{L_2(\Omega)} \leqq \frac{K_2}{\sin\alpha}\, d\,,$$

where α is the smallest angle and

d the greatest distance of vertices in the triangles T_i.

Proof. Similar to the proof of Corollary 1.

Theorem 1 and 2 enable us to prove the convergence of the finite elements method for second order boundary value problems if we use interpolation functions from Sec. 4 in this method. The proof follows the standard procedure described for example in [3].

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Souhrn

SPECIÁLNÍ EXAKTNÍ ZAKŘIVENÉ KONEČNÉ PRVKY

JITKA KŘÍŽKOVÁ

V článku jsou zavedeny speciální exaktní zakřivené elementy, které jsou vhodné pro řešení kontaktních problémů druhého řádu v oblastech, jejichž hranice se skládají z konečného počtu kruhových oblouků a úseček. Pro tyto elementy jsou dokázány interpolační odhady.

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