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AN ITERATIVE METHOD OF ALTERNATING TYPE FOR SYSTEMS WITH SPECIAL BLOCK MATRICES

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Summary. An iterative procedure for systems with matrices originating from the domain decomposition technique is proposed. The procedure introduces one iteration parameter. The convergence and optimization of the method with respect to the parameter is investigated. The method is intended not as a preconditioner for the CG method but for the independent use.

Keywords: iterative methods, block matrix, domain decomposition.

AMS Classification: 65N30.

We propose an iterative procedure for solving systems of linear algebraic equations through solving smaller systems with matrices that are slightly modified submatrices of the given matrix. This procedure can be useful for solving algebraic systems originating from the discretization of boundary-value problems for second-order elliptic equations on composite domains. Many authors established, in a similar way, different preconditioners for the CG method, we refer only to [1], [2], [3]. The present method is intended for the independent use and is close to that of [4].

1. THE METHOD

Let a system of linear algebraic equations

$$Mw = d\,, \tag{1}$$

be given, where

$$M = \begin{bmatrix} A & D^{\mathsf{T}} & O \\ D & B & E^{\mathsf{T}} \\ O & E & C \end{bmatrix} \quad w = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad d = \begin{bmatrix} f \\ g \\ h \end{bmatrix}. \tag{2}$$

We suppose that the matrices A, B, C are square and symmetric, O are null matrices of the corresponding orders.

Let us recall that the Schur complement of a regular block A_{11} in

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is the matrix $A_{22} - A_{21}A_{11}^{-1}A_{12}$ and that the Schur complement is symmetric and positive definite assuming that A and A_{11} are symmetric and positive definite as well.

Let further B_1 and B_2 be symmetric matrices such that $B_1 + B_2 = B$ and that the matrices

$$\begin{bmatrix} A & D^T \\ D & B_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B_2 & E^T \\ E & C \end{bmatrix}$$

are positive definite. It is easily seen that then the matrix M is positive definite too.

Let us denote by S_U the Schur complement of A in

$$\begin{bmatrix} A & D^T \\ D & B_1 \end{bmatrix}$$

and by S_L the Schur complement of C in

$$\begin{bmatrix} B_2 & E^T \\ E & C \end{bmatrix}.$$

We denote further $p = DA^{-1}f + E^TC^{-1}h$.

Our procedure consists in the construction of a sequence of vectors y_n converging to the y-component of the true solution of (1). It is easily calculated that this exact value is

$$(3) \qquad y = (S_U + S_L)^{-1}(g - p).$$

Let c be a real parameter, $0 < c < 1$.

Let an approximation y_n to y be given. Let now y'_{n+1} be the y-component of the solution of the system

$$(4_1) \qquad Ax + D^Ty = f,$$

$$(4_2) \qquad Dx + B_1y = (1 - c)g + c(DA^{-1}f + S_Uy_n) - (1 - c)(E^TC^{-1}h + S_Ly_n),$$

and let y''_{n+1} be the y-component of the solution of

$$(5_1) \qquad B_2y + E^Tz = cg - c(DA^{-1}f + S_Uy_n) + (1 - c)(E^TC^{-1}h + S_Ly_n),$$

$$(5_2) \qquad Ey + Cz = h.$$

The new iteration y_{n+1} is defined by

$$(6) \qquad y_{n+1} = cy'_{n+1} + (1 - c)y''_{n+1}.$$

Remark. Let us substitute y_n for y into the system (4), calculate x from (4_1) and calculate the value of the left-hand side of (4_2) with this value of x. We have $x = A^{-1}(f - D^{\mathrm{T}}y_n)$ and

$$Dx + B_1 y_n = DA^{-1}f - DA^{-1}D^{\mathrm{T}}y_n + B_1 y_n = DA^{-1}f + S_U y_n .$$

An analogous manipulation with the system (5) gives $E^{\mathrm{T}}C^{-1}h + S_L y_n$.

This elucidates a little the origin of the right-hand side terms in (4_2) and (5_1). On the basis of this consideration we can say that one step of our iteration method consists in the following sequence of operations:

1. Substitute y_n into (4) and (5).
2. Calculate the unknowns x and z from (4_1) and (5_2), respectively.
3. With these values of x and z and y_n calculate the value of the left-hand sides of (4_2) and (5_1).
4. Use these values to construct the right-hand side in (4_2) and (5_1).
5. Solve the systems (4) and (5) for the unknowns x, y and y, z, respectively. Only the y-components will be used in the sequel.
6. Calculate the new iteration y_{n+1} from (6).

The solution of the systems (4) and (5) with respect to y'_{n+1} and y''_{n+1}, respectively, and the substitution into (6) gives

$$y_{n+1} = \{[c^2 + (1-c)^2]I - c(1-c)(T + T^{-1})\}\, y_n + c(1-c)(S_U^{-1} + S_L^{-1})(g - p) . \tag{7}$$

Here $T = S_U^{-1}S_L$.

By K we denote the iteration matrix

$$K = [c^2 + (1-c)^2]I - c(1-c)(T + T^{-1}) .$$

Before studying the convergence of the process, we will show that if the sequence defined by (7) is convergent it converges to the true value. Namely, the limit y_∞ of the sequence y_n satisfies

$$y_\infty = (I - K)^{-1} c(1-c)(S_U^{-1} + S_L^{-1})(g - p) .$$

However, we have

$$I - K = 2c(1-c)I + c(1-c)(T + T^{-1}) = c(1-c)(S_U^{-1} + S_L^{-1})(S_U + S_L) .$$

Therefore y_∞ is the value from (3).

In order to study the convergence, i.e. the spectral radius of K, we put $e = c - \frac{1}{2}$. Let λ be an arbitrary eigenvalue of T. Then λ is real positive. We define μ as $\mu = \lambda + 1/\lambda - 2$, therefore $\mu \geqq 0$. The corresponding eigenvalue of K is then (in dependence on e)

$$\varkappa(\mu, e) = 4e^2 - (\tfrac{1}{4} - e^2)\,\mu .$$

We can restrict ourselves to the interval $\langle 0, \frac{1}{2})$ for e, i.e. $c \in \langle \frac{1}{2}, 1)$. The shape of the function $|\varkappa(\mu, e)|$ in dependence on e^2 is in Fig. 1.

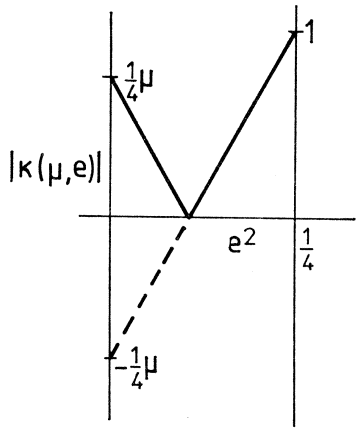

Fig. 1.

Now, let λ_i, $i = 1, \ldots, N$, be eigenvalues of T and let μ_i be the corresponding quantities. Let $\mu_{\max} = \max_i \mu_i$ and $\mu_{\min} = \min_i \mu_i$. Let $\sigma(e) = \max(|\varkappa(\mu_{\max}, e)|, |\varkappa(\mu_{\min}, e)|)$.

Theorem. *The spectral radius of K – for the chosen value of e – is equal to $\sigma(e)$.*

Proof. It is clear (cf. Fig. 1) that $\varkappa(\mu_i, e) > \varkappa(\mu_j, e)$ for $\mu_i < \mu_j$. Therefore $-\varkappa(\mu_i, e) > -\varkappa(\mu_j, e)$. We have

$$\sigma(e) \geqq -\varkappa(\mu_{\max}, e) > -\varkappa(\mu_i, e)\,,$$
$$\sigma(e) \geqq \varkappa(\mu_{\min}, e) > \varkappa(\mu_i, e) \quad \text{for} \quad i = 1, \ldots, N\,.$$

Therefore $\sigma(e) \geqq |\varkappa(\mu_i, e)|$ and it is equal either to $|\varkappa(\mu_{\max}, e)|$ or to $|\varkappa(\mu_{\min}, e)|$, q.e.d.

The process is convergent if $\sigma(e) < 1$. It is seen e.g. that if $\mu_{\max} < 4$, then $\sigma(e) < 1$ for all e. This case corresponds to the eigenvalues of T in the interval $(3 - 2\sqrt{2}, 3 + 2\sqrt{2})$.

The function $\sigma(e)$ has its minimum for $e^2 = s/(4(8 + s))$ where $s = \mu_{\max} + \mu_{\min}$. For this optimal value of e the spectral radius is $\sigma(e_{\text{opt}}) = r/(8 + r)$, where $r = \mu_{\max} - \mu_{\min}$. This quantity is always less that 1. By an appropriate choice of e (or alternatively c) we are always able to achieve the convergence of the process.

In order to find e_{opt} it is however necessary to know the eigenvalues of T and the corresponding μ's. Generally, this may be rather difficult. Therefore we try to illustrate the process by an example of a discretized boundary-value problem.

2. EXAMPLE

Let Ω be a domain composed of two squares according to Fig. 2. On the domain Ω the Dirichlet problem for the Laplace equation is to be solved.

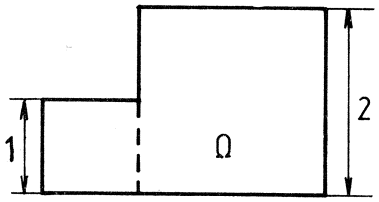

Fig. 2.

For the discretization we use the uniform square mesh with the step $h = 1/n$. We use the standard five point approximation. Let us denote by

$$A_1 = \begin{bmatrix} 4 & -1 & & \\ -1 & 4 & -1 & \\ & \ldots & & \\ & & -1 & 4 \end{bmatrix}$$

the square matrix of the order $n - 1$ and by A_2 an analogous matrix, but of the order $2n - 1$. Our discretized problem has the form

$$Ax = f,$$

where

(8)
$$A = \left[\begin{array}{cccc|c|cccc} A_1 & -I_1 & & & & & & & & \\ -I_1 & A_1 & -I_1 & & & & & & & \\ & \ldots & & & & & & & & \\ & -I_1 & A_1 & -I_1 & & & & & & \\ \hline & & -I_1 & A_1 & J & & & & & \\ \hline & & & J^{\mathrm{T}} & A_2 & -I_2 & & & & \\ & & & & -I_2 & A_2 & -I_2 & & & \\ & & & & & \ldots & & & & \\ & & & & & & -I_2 & A_2 & & \end{array}\right] \begin{array}{l} n-1 \text{ times} \\ \\ \\ \\ \\ \\ \\ \\ 2n-1 \text{ times} \end{array}$$

Here, I_1 and I_2 are identity matrices of the order $n - 1$ and $2n - 1$, respectively, $J = [-I_1, 0]$, where 0 is the null matrix of the order $(n - 1) \times n$. The vector of the unknowns x is a block vector with blocks corresponding to all the meshpoints on a vertical meshline, the vector f of the right-hand side originates from the values of the boundary condition and is blocked in the same way. The vertical and horizontal

dashed lines in (8) indicate the blocking corresponding to that of (2). The matrix B from (2) is now equal to A_1 and we put $B_1 = B_2 = A_1/2$. The steps of the above iterative procedure may be now described as follows:

1. Choose the values of the approximate solution on the interface between the two squares arbitrarily.
2. Solve the Dirichlet problems for the two squares separately.
3. Calculate the approximate values of the normal derivatives of these solutions on the interface from both sides. (This is due to our choice of the matrices B_1 and B_2.)
4. Put together the right-hand sides in the equations (4_2) and (5_1) and thus establish the right-hand side for the Neumann condition.
5. Solve the Neumann problems for the two squares separately.
6. Combine the values of both solutions on the interface and obtain the new Dirichlet condition on the interface.

For the numerical experiment, the exact solution of the problem was taken as the plane with values equal to zero on the left-hand side of the small square, to one on the interface, and to three on the right-hand side of the big square. The Dirichlet boundary conditions were taken from this exact solution. The null vector was taken as the initial approximation for the values on the interface, intentionally a bad approximation. For c the value 0·5 was taken.

The results for different values of n and a few first iterations are given in Table 1. The values shown are the maximum norms of the error on the interface. The standard semilogarithmic form of real numbers is used.

Tab. 1.

Iter. No	n					
	4	6	8	10	15	20
1	1·42 −3	3·19 −3	5·20 −3	7·08 −3	1·11 −2	1·43 −2
2	1·79 −6	9·07 −6	2·32 −5	4·22 −5	1·02 −4	1·71 −4
3				2·52 −7	9·39 −7	2·03 −6

Finally, it should be mentioned that the matrices D and E from the blocking of (2) are in this example of special form. It simplifies a little the computation. A further simplification is obtained when using the elimination method, because for the computation of the values and the values of the normal derivatives on the interface, only one or two block steps of the back substitution are necessary. The whole solution is computed only after the process converges.

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Souhrn

ITERAČNÍ METODA ALTERNUJÍCÍHO TYPU
PRO SOUSTAVY SE SPECIÁLNÍ BLOKOVOU MATICÍ

MILAN PRÁGER

Je navržena iterační metoda pro soustavy s maticí vznikající při metodě dekompozice oblasti. Metoda obsahuje jeden iterační parametr. Je vyšetřena konvergence metody a její optimalizace vzhledem k iteračnímu parametru. Metoda je určena k samostatnému použití, nikoli jako předpodmínění pro metodu sdružených gradientů.

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