Applications of Mathematics

Juraj Zeman

On existence of the weak solution for nonlinear diffusion equation

Applications of Mathematics, Vol. 36 (1991), No. 1, 9–20

Persistent URL: http://dml.cz/dmlcz/104440

36 (1991) APPLICATIONS OF MATHEMATICS No. 1, 9–20

ON EXISTENCE OF THE WEAK SOLUTION FOR NONLINEAR DIFFUSION EQUATION

JURAJ ZEMAN

(Received March 29, 1989)

Summary. The paper concerns the existence of bounded weak solutions of a nonlinear diffusion equation with nonhomogeneous mixed boundary conditions.

Keywords: Nonlinear diffusion, method of lines.
AMS Classification: 45K65.

1. INTRODUCTION

The main of this paper is to find a bounded weak solution of the initial-boundary value problem for the nonlinear diffusion equation of the form

$$\begin{aligned}
&\frac{\partial}{\partial t} b(u) - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) = f(x, u) \quad \text{in} \quad I \times D\,,\\
&u = u^D \quad \text{on} \quad I \times \Gamma_1\,,\\
&\sum_{i=1}^{N} a_i(x, u, \nabla u) \cos(\nu, x_i) = 0 \quad \text{on} \quad I \times \Gamma_2\,,\\
&u(x, 0) = u_0(x) \quad \text{in} \quad D\,,
\end{aligned}$$

where $b(z) = |z|^m \,.\, \operatorname{sgn}(z)$, $m > 0$, D is a bounded domain in R^N with smooth boundary ∂D, Γ_1, Γ_2 are open subdomains of ∂D such that $\Gamma_1 \cap \Gamma_2 = 0$, meas $\Gamma_1 +$ + meas $\Gamma_2 =$ meas ∂D, meas $\Gamma_1 > 0$, $a_i\colon D \times R \times R^N \to R$ $(i = 1, \ldots, N)$, $f\colon D \times R \to R$ satisfy Carathéodory's conditions.

The equation in (1.1) appears in various physical, chemical and biological models. For $0 < m < 1$ it is known as the slow diffusion equation, for $m = 1$ as the classical heat equation and for $m > 1$ as the fast diffusion equation.

A similar equation is solved in [1], but one of the main assumptions of that paper is ellipticity of the operator $A(=\sum a_i)$ while we assume only monotonicity. We also prove boundedness of the solution. In the case of linear operator A it is possible to get a smoother solution (see [6]). We solve Problem (1.1) using the method of lines which has been intensively studied in [2], and we apply it to the slow and fast diffusions simultaneously.

In the sequel we shall adopt the following notation: Let $I = (0, T)$, $T < \infty$, $Q = D \times I$, $V = \{v \in W_p^1,\ v = 0 \text{ in } \Gamma_1\}$, $B(z) = m/(m+1)\,|z|^{m+1}$,

$$a(u; v, w) = \sum_{i=1}^{N} \int_D a_i(x, u, \nabla v) \frac{\partial w}{\partial x_i}\, dx\,,$$

$$(f(v), w) = \int_D f(x, v)\, w\, dx\,,$$

$$\partial_t^h u(t) = \frac{u(t) - u(t-h)}{h}\,.$$

2. EXISTENCE OF THE WEAK SOLUTION

We will assume that the elliptic part in (1.1) is continuous in all variables and monotone in ∇u, i.e.

$$(2.1)\qquad \sum_{i=1}^{N} (a_i(x, \eta, \xi) - a_i(x, \eta, \zeta)) (\xi_i - \zeta_i) \geqq 0$$

for $x \in D$, $\eta \in R$, $\xi, \zeta \in R^N$, $a_i(x, \eta, 0) = 0$ for $i = 1, \ldots, N$, and satisfies

$$(2.2)\qquad \frac{\partial a_i(x, \eta, \xi)}{\partial \xi_j} = \frac{\partial a_j(x, \eta, \xi)}{\partial \xi_i}$$

is the sense of distribution,

$$(2.3)\qquad \sum_{i=1}^{N} a_i(x, \eta, \xi)\, \xi_i \geqq C_1 \sum_{i=1}^{N} |\xi_i|^p - C_2\,, \quad 1 < p < \infty\,.$$

The growth conditions are of the form

$$(2.4)_1\qquad \sum_{i=1}^{N} |a_i(x, \eta, \xi)| \leqq C_3 + B(\eta)^{1/q} + |\xi|^{p-1}\,(p^{-1} + q^{-1} = 1)\,,$$

$$(2.4)_2\qquad |f(x, \eta)| \leqq C_4(|d(x)| + |\eta|^\alpha)\,, \quad \alpha = \min\left(m, \frac{m+1}{q}\right), \quad d(x) \in L_\infty(D)\,.$$

Boundary and Initial Data satisfy

$$(2.5)\qquad u_0 \in L_\infty(D) \cap V,\quad u^D \in L_\infty(I, W_p^1(D)) \cap L_\infty(Q),\ \frac{d}{dt}\, u^D \in L_\infty(I \times D)\,,$$

$$u^D(t) \to u_0 \quad \text{for} \quad t \to 0 \quad \text{in} \quad L_\infty(D)\,.$$

Definition 2.6. *We call $u \in u^D + L_p(I, V)$ a weak solution of the initial boundary value problem* (1.1) *if the following two conditions are fulfilled:*

i) $b(u) \in L_\infty(Q)$, $(d/dt)\, b(u) \in L_q(I, V^*)$ *satisfy*

$$\int_I \left(\frac{d}{dt}\, b(u), v\right) dt = -\int_I \int_D \left(b(u) - b(u_0)\right) \frac{dv}{dt}\, dx\, dt$$

for every $v \in L_p(I, V) \cap L_\infty(Q)$, $dv/dt \in L_\infty(Q)$ *and* $v(T) = 0$;

ii) $f(u) \in L_q(I. V^*)$ *and the identity*

$$\int_I \left(\frac{\mathrm{d}}{\mathrm{d}t} b(u), v\right) \mathrm{d}t + \int_I a(u; u, v)\, \mathrm{d}t = \int_I (f(u), v)\, \mathrm{d}t$$

holds for every $v \in L_p(I, V)$.

The main result of this paper is

Theorem 2.7. *Suppose* (2.1), (2.2), (2.3), (2.4) *and* (2.5). *Then there exists a weak solution of Problem* (1.1) *in the sense of Definition* 2.6.

We now prove a series of assertions, which contain most of the essential elements for the proof of Theorem 2.7.

Suppose that an integer n is specified and set $h = T/n$. Applying a time discretization formula we use the approximation scheme

$$\frac{b(u_i) - b(u_{i-1})}{h} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u_{i-1}, \nabla u_i) = f(x, u_{i-1}), \tag{2.8}$$

$u_i = u_i^D + V$, where $u_i^D = (1/h) \int_{t_{i-1}}^{t_i} u^D(s)\, \mathrm{d}s$, $i = 1, \ldots, n$, $u_{i-1} = u_0$ for $i = 1$.

Definition 2.9. *We call* u_i, $i = 1, \ldots, n$ *a solution of Problem* (2.8) *in* V, *if* $u_i \in u_i^D + V$, $B(u_i) \in L_1(D)$, *the functional* $F_i(v) = \int_D b(u_i)\, v\, \mathrm{d}x$ *can be uniquely extended to* V *and the following identity holds for all* $v \in V$ $(i = 1, \ldots, n)$:

$$(\partial_i^h b(u_i), v) + a(u_{i-1}; u_i, v) = (f(u_{i-1}), v)\,. \tag{2.10}$$

Lemma 2.11. *There exists a unique solution* u_i *of Problem* (2.8) *in the sense of Definition* 2.9 *for any positive integers* $i, n \geqq n_0$. *Moreover, each* $u_i \in L_\infty(D)$.

Proof. By induction with respect to i.

Let us suppose the assertion is true for $j = 1, \ldots, i - 1$. Let us now prove it for $j = i$.

Existence. Let

$$\Phi(v) = \int_0^1 \mathrm{d}t \int_D \sum_{i=1}^{N} a_i(x, u_{i-1}, t\, \nabla v) \frac{\partial v}{\partial x_i}\, \mathrm{d}x + \int_0^1 \mathrm{d}t \int_D \frac{1}{h}\, b(tv)\, v\, \mathrm{d}x - \int_D \left(\frac{1}{h}\, b(u_{i-1}) - f(u_{i-1})\right) v\, \mathrm{d}x\,.$$

Due to (2.1), (2.2), (2.3) Φ is continuous, strictly convex and coercive over V and has a G-differential

$$D\Phi(u, v) = \left(\frac{b(u) - b(u_{i-1})}{h}, v\right) + a(u_{i-1}; u, v) - (f(u_{i-1}), v)\,.$$

The classical results concerning the minimization of Φ imply existence of a solution u_i^0 of Problem 2.8 for $j = i$ (with the homogeneous boundary condition (see [3])). Then the function $u_i = u_i^D + u_i^0$ is the solution of Problem 2.8.

Boundedness. Suppose the contrary (see [4]), that is, there exists $\{c_j\}_{j=1}^{\infty}$, $c_j \leqq c_{j+1}$, $c_j \to \infty$ for $j \to \infty$ and $K_j = \{x \in D, |u_i^0(x)| > c_j\}$ with meas $(K_j) > 0$. Putting

$$u_i^0|^{c_j} = u_i^0(x)|^{c_j} = \begin{cases} u_i^0(x) & x \in D - K_j , \\ c_j \operatorname{sgn}(u(x)) & x \in K_j \end{cases}$$

we easily obtain $\Phi(u_i^0|^{c_j}) < \Phi(u_i^0)$ for sufficiently large j, which contradicts the minimum property of u_i^0.

Let us construct sequences of step functions $\{u_n^D\}$, $\{u_n\}$ defined by

$$(2.12) \qquad u_n^D(t) = u_i^D \quad t_{i-1} \leqq t < t_i , \quad i = 1, \ldots, n ,$$
$$u_n(t) = u_i \quad t_{i-1} \leqq t < t_i , \quad i = 1, \ldots, n .$$

We can write (2.10) in the form

$$(2.13) \qquad (\partial_t^h b(u_n), v) + a(u_{nh}; u_n, v) = (f(u_{nh}), v)$$

where $n \geqq n_0$, $v \in V$, $u_{nh} = u_n(t - h)$ and $u_n(t) = u_0$ for $t \in (-h, 0)$. The main role in the proof of Theorem 2.7 is played by the uniform boundedness of the sequence $\{u_i\}$ in $L_\infty(D)$. First we have to prove a priori estimates.

Lemma 2.14. *The estimates*

i) $\int_I \|u_n\|_V^p \, dt \leqq C$,

ii) $\int_D B(u_n(t)) \, dx \leqq C$

hold for all $n \geqq n_0$.

Proof. Putting in (2.13) $v = u_n - u_n^D$ and integrating it over $(0, \tau)$ we have

$$(2.15) \qquad \int_0^\tau \int_D \partial_t^h b(u_n) (u_n - u_n^D) \, dx \, dt + \int_0^\tau a(u_{nh}; u_n, u_n - u_n^D) \, dt =$$
$$= \int_0^\tau \int_D f(u_{nh}) (u_n - u_n^D) \, dx \, dt .$$

First let us estimate the first term in (2.15). Using the inequality

$$(2.16) \qquad B(z) - B(z_0) \leqq (b(z) - b(z_0)) \, z ,$$

which follows from the fact that the function $f(x) = m/(m + 1) \, x^{m+1} - x^m + 1/(m + 1) \geqq 0$ for $x > 0$, we can write

$$(2.17) \qquad B(u_n) - B(u_{nh}) \leqq (b(u_n) - b(u_{nh})) \, u_n \quad \text{for a.e.} \quad t \in (h, T) .$$

We can integrate it over $(0, \tau) \times D$:

$$(2.18) \qquad 1/h \int_0^\tau \int_D (B(u_n) - B(u_{nh})) \, dx \, dt \leqq \int_0^\tau \int_D \partial_t^h b(u_n) \, u_n \, dx \, dt .$$

In virtue of the equality

$$\int_0^\tau \int_D \partial_t^h b(u_n) \, u_n \, dx \, dt = \int_0^\tau (\partial_t^h b(u_n), u_n - u_n^D) \, dt + \int_0^\tau \int_D \partial_t^h b(u_n) \, u_n^D \, dx \, dt$$

in (2.18) we have

$$1/h \int_0^\tau \int_D (B(u_n) - B(u_{nh})) \, dx \, dt \leqq \int_0^\tau (\partial_t^h b(u_n), u_n - u_n^D) \, dt + \int_0^\tau \int_D \partial_t^h b(u_n) \, u_n^D \, dx \, dt .$$

Using integration by parts in the above inequality, we obtain

$$(2.19)\qquad 1/h \int_{\tau-h}^{\tau} \int_D B(u_n)\,dx\,dt - \int_D B(u_0)\,dx \leqq \int_0^{\tau} (\partial_t^h b(u_n), u_n - u_n^D)\,dt -$$
$$- \int_0^{\tau-h} \int_D (b(u_n) - b(u_0))\, \partial_t^{-h} u_n^D\,dx\,dt + 1/h \int_{\tau-h}^{\tau} \int_D (b(u_n) - b(u_0))\, u_n^D\,dx\,dt\,,$$

which yields

$$(2.20)\qquad \int_0^{\tau} \int_D \partial_t^h b(u_n)(u_n - u_n^D)\,dx\,dt \geqq 1/h \int_{\tau-h}^{\tau} \int_D B(u_n)\,dx\,dt - \int_D B(u_0)\,dx +$$
$$+ \int_0^{\tau-h} \int_D (b(u_n) - b(u_0))\, \partial_t^{-h} u_n^D\,dx\,dt - 1/h \int_{\tau-h}^{\tau} \int_D (b(u_n) - b(u_0))\, u_n^D\,dx\,dt\,.$$

In virtue of (2.3), (2.4) and Young's inequality we estimate

$$(2.21)\qquad \int_0^{\tau} a(u_{nh}; u_n, u_n - u_n^D)\,dt \geqq C_1 \int_0^{\tau} \|u_n\|_V^p\,dt + C_2 \int_0^{\tau} \|u_n^D\|_V^p\,dt - C_3 \int_0^{\tau} \int_D B(u_{nh})\,dx\,dt\,,$$

$$(2.22)\qquad \left|\int_0^{\tau} \int_D f(u_{nh})(u_n - u_n^D)\,dx\,dt\right| \leqq \delta \int_0^{\tau} \|u_n\|_V^p\,dt + C_\delta \int_0^{\tau} \int_D B(u_{nh})\,dx\,dt + C \int_0^{\tau} \|u_n^D\|_V^p\,dt\,,$$

which together with (2.20) yields (if δ is sufficiently small)

$$(2.23)\qquad 1/h \int_{\tau-h}^{\tau} \int_D B(u_{nh})\,dx\,dt + \int_0^{\tau} \|u_n\|_V^p\,dt \leqq C_1 \int_0^{\tau} \int_D B(u_{nh})\,dx\,dt + C_2$$

in view of

$$\int_D B(u_0)\,dx \leqq C\,,$$
$$1/h \int_{\tau-h}^{\tau} \int_D |b(u_n)|\,dx\,dt \leqq \delta/h \int_{\tau-h}^{\tau} \int_D B(u_n)\,dx\,dt + C_\delta\,,$$
$$\int_0^{\tau-h} \int_D b(u_n)\, \partial_t^{-h} u_n^D\,dx\,dt \leqq C \int_0^{\tau-h} \int_D |b(u_n)|\,dx\,dt \leqq$$
$$\leqq C_1 \int_0^{\tau} \int_D B(u_n)\,dx\,dt + C_2\,.$$

Using Gronwall's lemma in the discrete form (letting $a_i = \int_D B(u_n(t))\,dx$ for $t_{i-1} \leqq \leqq t < t_i$) we obtain the required estimates.

Lemma 2.24. *For any $n \geqq n_0$ and $i = 1, \ldots, n$ the solution u_i satisfies*

$$\|u_i\|_{L_\infty(D)} \leqq C\,.$$

Proof. Putting $v = b^s(u_i) - b^s(u_i^D)$ (see [6]), where s is odd, $s > s_0(m)$ we obtain

$$(2.25)\qquad \int_D (b(u_i) - b(u_{i-1}))\, b^s(u_i)\,dx - \int_D (b(u_i) - b(u_{i-1}))\, b^s(u_i^D)\,dx +$$
$$+ h \int_D a(u_{i-1}, \nabla u_i)\, \nabla b^s(u_i)\,dx - h \int_D a(u_{i-1}, \nabla u_i)\, \nabla b^s(u_i^D)\,dx =$$
$$= \int_D f(u_{i-1})(b^s(u_i) - b^s(u_i^D))\,dx\,.$$

In virtue of (2.1) and $(2.4)_2$ we can estimate the third term on the left-hand side and the term on the right-side obtaining

$$(2.26)\qquad \int_D b^{s+1}(u_i) \leqq \int_D (1 + C_4 h)\, |b(u_{i-1})|\, |b^s(u_i)| + C_4 h \int_D |d|\, |b^s(u_i)| +$$
$$+ \int_D |b(u_i) - b(u_{i-1})|\, |b^s(u_i^D)| + h \int_D |a(u_{i-1}, \nabla u_i)|\, |\nabla b^s(u_i^D)| +$$
$$+ C_4 h \int_D |d|\, |b^s(u_i^D)| + C_4 h \int_D |b(u_{i-1})|\, |b^s(u_i^D)|\,.$$

Now we can successively estimate the third, the fourth, the fifth and the sixth terms on the right-side in (2.26):

$$\int_D |b(u_i) - b(u_{i-1})| \, |b^s(u_i^D)| \leqq |b^s(c_M)| \int_D (|b(u_i)| + |b(u_{i-1})|) \leqq$$
$$\leqq C|b^s(c_M)|$$

where $c_M = \sup |u_i^D|$, in view of (2.5), $(2.14)_{ii}$;

$$h \int_D |a(u_{i-1}, \nabla u_i)| \, |\nabla b^s(u_i^D)| = h \int_D |a(u_{i-1}, \nabla u_i) \, s b^{s-1}(u_i^D)| \, |\nabla u_i^D| \leqq$$
$$\leqq hs|b^{s-1}(c_M)| \left(\int_D B(u_{i-1}) + \int_D |\nabla u_i|^p + \int_D |\nabla u_i^D|^p \leqq Chs|b^{s-1}(c_M)|\right.$$

holds in view of (2.5) and Lemma 2.14;

$$C_4 h \int_D |d| \, |b^s(u_i^D)| \leqq C_4 h |b^s(c_M)| \int_D |d| \leqq Ch|b^s(c_M)| \, ;$$
$$C_4 h \int_D |b(u_{i-1})| \, |b^s(u_i^D)| \leqq C_4 h |b^s(c_M)| \int_D |b(u_{i-1})| \leqq Ch|b^s(c_M)| \, .$$

All these estimates together with (2.26) imply

$$\int_D b^{s+1}(u_i) \leqq \int_D (1 + C_4 h) \, |b(u_{i-1})| \, |b^s(u_i)| + C_4 h \int_D |d| \, |b^s(u_i)| +$$
$$+ sC|b^s(c_M)| \, .$$

Applying twice Young's inequality we obtain

$$\int_D b^{s+1}(u_i) \leqq (1 + \eta h)^{s+1} \int_D b^{s+1}(u_{i-1}) + \frac{(s + 1) \, C_4 h C(\varepsilon)}{1 - \varepsilon C_4 h} \int_D |d|^{s+1} +$$
$$+ \frac{sC}{1 - \varepsilon C_4 h} |b^s(c_M)| \, , \quad \text{where} \quad 0 < \varepsilon < \frac{1}{C_4 h} \quad \text{and} \quad \eta = \frac{C_4(\varepsilon + 1)}{1 - \varepsilon C_4 h} \, .$$

This inequality may be formally rewritten as $y_i \leqq a y_{i-1} + b$, from which we recurrently obtain

$$y_i \leqq a y_{i-1} + b \leqq a^i y_0 + b \frac{a^i - 1}{a - 1} \leqq a^i \left(y_0 + \frac{b}{a - 1} \right) .$$

So we have

$$\int_D b^{s+1}(u_i) \leqq (1 + \eta h)^{i(s+1)} \left(\int_D b^{s+1}(u_0) + C_1' \int_D |d|^{s+1} + \right. \tag{2.27}$$
$$\left. + C_2' |b^s(c_M)|\right)$$

where the constants C_i' $(i = 1, 2)$ are such that their $(s + 1)$-st root tends to 1 if $s \to \infty$.

Now taking the $(s + 1)$-st root of (2.27) and letting $s \to \infty$ we obtain

$$\|b(u_i)\|_{L_\infty(D)} \leqq (1 + \eta h)^i \left(\|b(u_0)\|_{L_\infty(D)} + \|d(x)\|_{L_\infty(D)} + |b(c_M)|\right) ,$$

where we can estimate $(1 + \eta h)^i \leqq \exp(\eta T)$. This completes the proof.

Lemma 2.28. *The sequence of functions* $\{u_n\}$ *defined by* (2.12) *is uniformly bounded in* $L_\infty(Q)$.

Proof. This lemma immediately follows from Lemma 2.24.

Lemma 2.29. *The estimate* $\int_0^{T-\tau}(b(u_n(t+\tau)) - b(u_n(t)), u_n(t+\tau) - u_n(t))\,dt \leqq$ $\leqq C\tau$ *holds for* $n \geqq n_0$ *and* $0 < \tau \leqq \tau_0 < T$.

Proof. Sum up (2.10) for $i = j+1, \ldots, j+k$ and then put $v = u_{j+k} - u_j - (u^D_{j+k} - u^D_j)$. We estimate

$$\begin{aligned}&\int_D (b(u_{j+k} - b(u_j))(u_{j+k} - u_j) \leqq \int_D |b(u_{j+k}) - b(u_j)|\,|u^D_{j+k} - u^D_j| + \\ &+ \sum_{i=j+1}^{j+k} \{|a(u_{i-1}; u_{j+k} - u_j - (u^D_{j+k} - u^D_j))| + \\ &+ \int_D |f(u_{i-1})|\,|u_{j+k} - u_j - (u^D_{j+k} - u^D_j)|\}\, h \leqq \\ &\leqq C\{(\int_D (B(u_{j+k}) + B(u_j)) + 1)\,\|u^D_{j+k} - u^D_j\|_{L_\infty(D)} + \\ &+ h\sum_{i=j}^{j+k} \int_D B(u_i) + h\sum_{i=j+1}^{j+k} \|u_i\|^p + kh(\|u_{j+k}\|^p + \|u_j\|^p + 1)\} \leqq \\ &\leqq C\{kh(1 + \|u_{j+k}\|^p + \|u_j\|^p) + h\sum_{i=j+1}^{j+k} \|u_i\|^p\} \leqq Ckh\end{aligned}$$

because of $(2.14)_{ii}$, (2.4) and (2.5).

Using our notation, we conclude that

$$\int_0^{T-kh} \int_D (b(u_n(t+kh) - b(u_n(t))(u_n(t+kh) - u_n(t))\,dx\,dt \leqq Ckh$$

and Lemma 2.29 is proved.

From Lemma 2.29 it follows that the sequence $\{b(u_n)\}$ is compact in $L_1(Q)$ (see [1], Lemma 1.8, 1.9); thus there exists a subsequence (in the sequel, we denote a subsequence of $\{u_n\}$ again by $\{u_n\}$) and a function u such that

(2.30) $\qquad b(u_n) \to b(u)$ in $L_1(Q)$,

$\qquad\qquad b(u_{nh}) \to b(u)$ in $L_1(Q)$.

From the fact that the operator $b(u) = |u|^m \operatorname{sgn}(u)$ is strictly monotone and from (2.30) it follows (see [7]) that

(2.31) $\qquad u_n \to u$ a.e. in Q

and $\qquad u_n \to u$ in $L_r(Q)$ for $r > 1$,

because u_n is bounded in $L_\infty(Q)$.

Lemma 2.32. *The sequence* $\{u_n\}$ *satisfies*

i) $\partial_t^h b(u_n) \rightharpoonup (d/dt)\, b(u)$ *in* $L_q(I, V^*)$,

ii) $f(u_{nh}) \rightharpoonup f(u)$ *in* $L_q(Q)$,

iii) $a_i(u_{nh}, \nabla u_n) \rightharpoonup a_i(u, \nabla u)$ *in* $L_q(Q)$, $i = 1, \ldots, N$.

Proof. i) In virtue of (2.14), (2.28) and (2.13) we have

$$(2.33)\qquad \sup_{\|v\|_{L_p(I,V)}\leqq 1,\, v\in L_\infty(D)} \left|\int_I \int_D \partial_t^h b(u_n)\, v\, \mathrm{d}x\, \mathrm{d}t\right| \leqq C\,.$$

So the sequence $\{\partial_t^h b(u_n)\}$ is uniformly bounded in $L_q(I, V^*)$. Then there exists a subsequence and $\chi \in L_q(I, V^*)$ such that

$$(2.34)\qquad \partial_t^h b(u_n) \rightharpoonup \chi \quad \text{in} \quad L_q(I, V^*)\,.$$

From the fact that $\|\partial_t^h b(u)\|_{L_q(I,V^*)} \leqq C$ it follows that there exsits a subsequence such that

$$(2.35)\qquad \partial_t^h b(u) \rightharpoonup \psi := \frac{\mathrm{d}}{\mathrm{d}t}\, b(u) \quad \text{in} \quad L_q(I, V^*)\,.$$

In virtue of (2.30) the identity

$$\int_I (\chi, v)\, \mathrm{d}t = -\int_I \int_D (b(u) - b(u_0)) \frac{\mathrm{d}v}{\mathrm{d}t}\, \mathrm{d}t$$

holds for $v \in L_p(I, V) \cap L_\infty(Q)$, $\mathrm{d}v/\mathrm{d}t \in L_\infty(Q)$, $v(T) = 0$.

Putting $v = \varphi_h = (1/h) \int_{t-h}^t \varphi\, \mathrm{d}s$, $\varphi \in L_p(I, V)$ and realizing that $\varphi_h \to \varphi$ in $L_p(I, V)$ we obtain

$$\int_I (\chi, \varphi) = \int_I \left(\frac{\mathrm{d}}{\mathrm{d}t}\, b(u), \varphi\right), \quad \text{which yeidls} \quad \chi = \frac{\mathrm{d}}{\mathrm{d}t}\, b(u)\,.$$

ii) In virtue of (2.31) we have

$$(2.36)\qquad f(u_{nh}) \to f(u) \quad \text{a.e. in } Q\,.$$

From $(2.4)_2$ and from Lemma 2.28 we obtain that

$$\|f(u_{nh})\|_{L_q(Q)}^q \leqq C_1 + C_2 \int_I \int_D B(u_{nh})\, \mathrm{d}x\, \mathrm{d}t \leqq C\,.$$

Hence there exists $\chi \in L_q(Q)$ such that

$$(2.37)\qquad f(u_{nh}) \rightharpoonup \chi \quad \text{in} \quad L_q(Q)\,.$$

(2.36), (2.37) together give the assertion ii) of Lemma 2.32.

iii) In virtue of $(2.4)_1$, Lemma 2.28 and (2.14) we obtain

$$\|a_i(u_{nh}, \nabla u_n)\|_{L_q(Q)} \leqq C\,, \quad i = 1, \ldots, N\,,$$

which implies that there exists $\chi_i \in L_q(Q)$ $(i = 1, \ldots, N)$ such that

$$a_i(u_{nh}, \nabla u_n) \rightharpoonup \chi_i \quad \text{in} \quad L_q(Q)\,, \quad i = 1, \ldots, N\,.$$

To show that $\chi_i = a_i(u, \nabla u)$ we use Minty's trick (see [3]), which is based on the relation

$$(2.38)\qquad \limsup \int_I a(u_{nh}; u_n, (u_n - u)) \leqq 0\,.$$

(We prove this inequality later on.)

From the monotonicity we have

$$\int_I \int_D (a_i(u_{nh}, \nabla u_n) - a_i(u_{nh}, w))\left(\frac{\partial}{\partial x_i} u_n - w\right) \geqq 0 ,$$

where $w \in [L_p(Q)]^N$,

and letting $n \to \infty$ we obtain

$$\int_I \int_D \sum_{i=1}^{N} (\chi_i - a_i(u, w))\left(\frac{\partial}{\partial x_i} u - w\right) \geqq 0 .$$

Putting $w = \nabla u + \varepsilon v$, $\varepsilon > 0$, $v \in L_\infty(Q)$, the above inequality (after $\varepsilon \to 0_+$) yields

$$\int_I \int_D \sum_{i=1}^{N} (\chi_i - a_i(u, \nabla u))\, v \,\mathrm{d}x\, \mathrm{d}t \geqq 0 . \tag{2.39}$$

Now putting $w = \nabla u - \varepsilon v$, $\varepsilon > 0$ we obtain

$$\int_I \int_D \sum_{i=1}^{N} (\chi_i - a_i(u, \nabla u))\, v \,\mathrm{d}x\, \mathrm{d}t \leqq 0 . \tag{2.40}$$

(2.39), (2.40) together yield

$$\int_I \int_D \sum_{i=1}^{N} (\chi_i - a_i(u, \nabla u))\, v \,\mathrm{d}x\, \mathrm{d}t = 0 ,$$

which holds for all $v \in L_\infty(Q)$. This implies that

$$\chi_i = a_i(u, \nabla u) \quad \text{a.e. in } Q , \quad i = 1, \ldots, N .$$

Now we prove (2.38).

Putting $v = u_n - u_n^D$ in (2.13) and integrating it over $(0, t)$ we obtain

$$\int_0^t a(u_{nh}; u_n, (u_n - u_n^D))\, \mathrm{d}t =$$
$$= \int_0^t \int_D f(u_{nh})\, (u_n - u_n^D)\, \mathrm{d}x\, \mathrm{d}t - \int_0^t (\partial_t^h b(u_n), (u_n - u_n^D)) .$$

Using (2.19) in the above equality we have

$$\int_0^t a(u_{nh}; u_n, u_n - u_n^D) \leqq \int_0^t f(u_{nh})\, (u_n - u_n^D) - 1/h \int_{t-h}^t B(u_n(t)) +$$
$$+ \int_D B(u_0) - \int_0^{t-h} \int_D (b(u_n(t)) - b(u_0))\, \partial_t^{-h} u_n^D +$$
$$+ 1/h \int_{t-h}^t \int_D (b(u_n(t)) - b(u_0))\, u_n^D .$$

Letting $n \to \infty$ and using (2.28), (2.30), (2.32) and Lemma 1.5 from [1] we obtain the estimate

$$\limsup \int_0^t a(u_{nh}; u_n, u_n - u_n^D) \leqq \int_0^t \int_D f(u)\, (u - u^D) - \tag{2.41}$$
$$- \int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}t}\, b(u), u - u^D\right),$$

which holds for a.e. $t \in (0, T)$.

Now let $\varphi \in L_p(I, V) \cap L_\infty(Q)$, $d\varphi/dt \in L_\infty(Q)$, $\varphi(s) = 0$ for $s \in (T - \delta, T)$, $\varphi(0) = 0$; then

$$(2.42)\qquad \int_I a(u_{nh}; u_n, \varphi) = \int_I \int_D f(u_{nh})\, \varphi - \int_I \int_D \partial_t^h b(u_n)\, \varphi =$$
$$= \int_I \int_D f(u_{nh})\, \varphi + \int_I \int_D b(u_n)\, \partial_t^{-h} \varphi \to \int_I \int_D f(u)\, \varphi +$$
$$+ \int_I \int_D b(u) \frac{d\varphi}{dt} = \int_I \int_D f(u)\, \varphi - \int_I \left(\frac{d}{dt} b(u), \varphi \right).$$

Since the set $\Phi = \{\varphi \in L_p(I, V) \cap L_\infty(Q),\ d\varphi/dt \in L_\infty(Q),\ \varphi(s) = 0,\ s \in (0, \delta), (T - \delta, T),\ \delta > 0\}$ is dense in $L_p(I, V)$ (see [8]) we can put $\varphi = u - u^D \in L_p(I, V)$ in (2.42) and obtain

$$(2.43)\qquad \int_I a(u_{nh}; u_n, (u - u^D) \to \int_I \int_D f(u)\,(u - u^D) - \int_I \left(\frac{d}{dt} b(u), u - u^D \right).$$

(2.41) and (2.42) together yield (2.38).

Proof of Theorem 2.7: Let us put $v \in L_p(I, V)$ in (2.13) and then integrate it over $(0, T)$. Taking the limit as $n \to \infty$ we obtain (in virtue of Lemma 2.32) that u is a weak solution of (1.1) in the sense of Definition 2.6.

3. GENERALIZATIONS

a) The growth of the coefficients a_i: Instead of the condition $(2.4)_1$ we can consider

$$(3.1)\qquad \sum_{i=1}^{N} |a_i(x, \eta, \xi)| \leqq \mu(|\eta|)\,(C + |\xi|^{p-1}),$$

where $\mu(z) \in C((0, \infty))$ is increasing

Theorem 3.2. *If* (2.1), (2.2), (2.3), (3.1), $(2.4)_2$ *and* (2.5) *are satisfied, then there exists a weak solution of Problem* (1.1).

Proof. We replace the coefficients a_i in (1.1) by

$$(3.3)\qquad a_i^R = a_i(x, \lambda_R \eta, \xi), \quad \text{where} \quad \lambda_R = \min\left(1, \frac{R}{|\eta|}\right), \quad R > 0.$$

In virtue of (3.1) we have

$$\sum_{i=1}^{N} |a_i^R(x, \eta, \xi)| = \sum_{i=1}^{N} |a_i(x, \lambda_R \eta, \xi)| \leqq \mu(|\lambda_R \eta|)\,(C + |\xi|^{p-1}) \leqq$$
$$\leqq \mu(R)\,(C + |\xi|^{p-1}),$$

where $\mu(R)$ is a constant, because of $|\lambda_R \eta| \leqq R$. This growth condition is a special case of $(2.4)_1$.

Now we consider the problem

$$(3.4) \qquad \frac{\partial}{\partial t} b(u) - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i^R(x, u, \nabla u) = f(x, u) ,$$

$$u = u^D \quad \text{on} \quad I \times \Gamma_1 ,$$

$$\sum_{i=1}^{N} a_i(x, u, \nabla u) \cos(\nu, x_i) = 0 \quad \text{on} \quad I \times \Gamma_2 ,$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad D .$$

In virtue of Theorem 2.7 there exists a solution of (3.4). Let us denote it by u_R. Lemma 2.38 yields

$$(3.5) \qquad \|u_R\|_{L_\infty(Q)} \leqq P , \quad \text{where} \quad P = C_1(\|u_0\|_{L_\infty(D)} + \|d(x)\|_{L_\infty(D)} + b(c_M)) \exp(C_2 T)$$

and P does not depend on R.

Putting $R > P$ in (3.3) and considering (3.5) we obtain $a_i^R \equiv a_i$ for $i = 1, \ldots, N$, because $\lambda_R = 1$. Now Problem (3.4) is identical with Problem (1.1) and the proof is complete.

b) Time dependent coefficients: All arguments remain the same, if we assume that the coefficients a_i and f depend on t. In this case we have to assume that $a_i(x, t, \eta, \xi)$ $(i = 1, \ldots, N)$ and $f(x, t, \eta)$ are smooth in t, η, ξ and we have to use approximations a_h, f_h in (2.8) which are piecewise constant in time, for example

$$a_h(x, t, u_{i-1}, \nabla u_i) = \frac{1}{h} \int_{t_{i-1}}^{t_i} a(x, s, u_{i-1}, \nabla u_i) \, ds ,$$

$$f_h(x, t, u_{i-1}) = \frac{1}{h} \int_{t_{i-1}}^{t_i} f(x, s, u_{i-1}) \, ds , \quad \text{for} \quad t_{i-1} \leqq s < t_i .$$

Acknowledgment. I would like to thank Dr. J. Kačur for valuable discussions concerning some topics of the material.

References

[1] *H. W. Alt, S. Luckhaus:* Quasilinear elliptic-parabolic differential equations. Math. Z. 183, 311—341 (1983).

[2] *J. Kačur:* Method of Rothe in evolution equations. Teubner-Texte zur Mathematik, 80, Leipzig, 1985.

[3] *S. Fučík, A. Kufner:* Nonlinear Differential Equations. Amsterdam–Oxford–New Nork, Elsevier 1980.

[4] *J. Kačur:* On boundedness of the weak solution for some class of quasilinear partial differential equations. Časopis pěst. mat. 98 (1973), 43—55.

[5] *J. Nečas:* Les méthodes directes en théorie des équations elliptiques, Academia, Prague, 1967.

[6] *J. Filo:* On solutions of a perturbed fast diffusion equation, Aplikace matematiky 32, 1987, 364—380.
[7] *J. L. Lions:* Quelques méthodes de résolution des problémes aux limites non linéaires, Russian translation, Moskva 1972.
[8] *H. Gajewski, K. Gröger, K. Zacharias:* Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen. Akademie-Verlag, Berlin, 1974.

Súhrn

O EXISTENCII SLABÉHO RIEŠENIA NELINEÁRNEJ DIFÚZNEJ ROVNICE

JURAJ ZEMAN

Práca je venovaná otázkam existencie ohraničeného slabého riešenia nelineárnej difúznej rovnice s nehomogénnymi zmiešanými okrajovými podmienkami.

Author's address: Dr. *Juraj Zeman,* CSc., FaF UK, Odbojárov 10, 832 32 Bratislava.