I. Bremer; Klaus R. Schneider

A remark on solving large systems of equations in function spaces

Aplikace matematiky, Vol. 35 (1990), No. 6, 494–498

Persistent URL: `http://dml.cz/dmlcz/104432`

A REMARK ON SOLVING LARGE SYSTEMS OF EQUATIONS IN FUNCTION SPACES

I. BREMER, K. R. SCHNEIDER

(Received June 10, 1989)

Summary. In order to save CPU-time in solving large systems of equations in function spaces we decompose the large system in subsystems and solve the subsystems by an appropriate method. We give a sufficient condition for the convergence of the corresponding procedure and apply the approach to differential algebraic systems.

Keywords: large system, decomposition, block iterative algorithm, differential algebraic equations

AMS Classificanion: 65J15

1. INTRODUCTION

The simulation of highly integrated circuits in microelectronics requires the numerical solution of the Cauchy problem for very large systems of differential algebraic equations. The CPU-time needed for solving such problems by means of traditional solvers increases superlinearly ($O(N^{\beta})$, $1.1 < \beta < 1.5$, where N is the number of nodes of the given circuit [4]). Therefore, the search for methods reducing the computation time in the process of solving very large systems is an important recent task. One possibility to reach this aim consists in applying block iterative methods, usually called relaxation methods. The basic steps of such an approach are partitioning of the large system and independent solving of the subsystems. This method is well-known for systems of linear and nonlinear equations [5, 8]. Concerning the Cauchy problem for differential systems describing electrical circuits the so-called waveform relaxation method has been developed in [3].

In this note we first consider equations in metric spaces with Lipschitz operators and give a sufficient condition for convergence of relaxation methods. Finally, we apply this approach to the Cauchy problem for a differential algebraic system.

2. GENERAL PROBLEM. SUFFICIENT CONDITIONS FOR CONVERGENCE

Let (X, d) be a complete metric space. We consider in X the operator equation

(2.1) $$K(y) = 0$$

where K maps X into itself. We assume (2.1) to be equivalent to the fixed point problem

(2.2) $$y = \hat{T}(y) .$$

The operator $\hat{T}$ is not uniquely determined by the operator K. Generally, it depends also on the numerical procedure used to solve (2.1).

A fundamental step in applying block iteration methods to (2.2) is an appropriate partitioning, that is, we decompose y and T and get after some possible rearrangement and reassignment

(2.3) $$x_i = T_i(x) , \quad i = 1, \ldots, N ,$$

which is equivalent to (2.2). To be able to do this we assume

(A_1) There are n complete metric spaces $(X_1, d_1), \ldots, (X_n, d_n)$ such that (X, d) is the product space of these metric spaces.

Concerning the operators T_i we assume

(A_2) The operators T_i, $i = 1, \ldots, N$, map X into X_i. There are positive constants k_{ij}, $1 \leqq i, j \leqq N$, such that $\forall x_i, \tilde{x}_i \in X_i$

(2.4) $$d_i(T_i(x), T_i(\tilde{x})) \leqq \sum_{j=1}^{N} k_{ij} d_j(x_j, \tilde{x}_j) .$$

Let K be the matrix defined by $K = (k_{ij})$. We are interested in a condition on K ensuring the convergence of the iteration scheme

(2.5) $$x^k = T(x^{k-1}) , \quad k = 1, 2 \ldots$$

in some metric space $(X, \tilde{d})$ for any initial guess x^0.

Lemma 2.1. *Assume the hypotheses* (A_1), (A_2) *hold. Further we suppose that the spectral radius* $\varrho(K)$ *of* K *satisfies*

(2.6) $$\varrho(K) < 1 .$$

Then there is a metric $\tilde{d}$ *in* X *such that* (2.5) *converges in* $(X, \tilde{d})$ *for any initial guess* x^0.

Proof. The matrix K maps the cone $\boldsymbol{R}^n_+$ into itself. If K is not strictly positive, that is, there are elements in $\boldsymbol{R}^n_+$ which are mapped by K into the boundary of $\boldsymbol{R}^n_+$, then K contains zero elements. Under the assumption (2.6) we can replace these zero entries by a small positive number such that the perturbed matrix $\tilde{K}$ is strictly positive and satisfies $\varrho(\tilde{K}) = \varrho(\tilde{K}^{\mathsf{T}}) < 1.$[1]) According to the Frobenius-Perron theory [1, 2, 6] $\varrho(\tilde{K}^{\mathsf{T}})$ is an eigenvalue of $\tilde{K}^{\mathsf{T}}$ with an eigenvector $a = (a_1, \ldots, a_N)^{\mathsf{T}}$ in the interior of $\boldsymbol{R}^n_+$.

Therefore, we have

$$K^{\mathsf{T}} a < \tilde{K}^{\mathsf{T}} a = \varrho(\tilde{K}^{\mathsf{T}})\, a < a\,. \tag{2.8}$$

We use the vector a to define the metric $\tilde{d}$ by

$$\tilde{d} = a_1 d_1 + \ldots + a_N d_N\,, \tag{2.9}$$

Thus, from (2.4), (2.5), (2.7) and (2.8) we get

$$\begin{aligned}\tilde{d}(x^{k+1}, x^k) &= \tilde{d}(T(x^k), T(x^{k-1})) = \sum_{j=1}^{N} a_j d_j(T_j(x^k), T_j(x^{k-1})) \leqq \\ &\leqq \sum_{i=1}^{N} d_i(x^k, x^{k-1}) \sum_{j=1}^{N} \tilde{k}_{ji} a_j = \varrho(\tilde{K}^{\mathsf{T}}) \sum_{i=1}^{N} a_i d_i(x^k, x^{k-1}) = \\ &= \varrho(\tilde{K})\, \tilde{d}(x^k, x^{k-1}) \leqq \varrho(\tilde{K})^k\, \tilde{d}(x^1, x^0)\,.\end{aligned}$$

Since $\varrho(\tilde{K}) < 1$ the sequence (2.5) converges in $(X, \tilde{d})$ for any initial guess x^0, q.e.d.

3. APPLICATION TO DIFFERENTIAL ALGEBRAIC SYSTEMS

To give an application of Lemma 2.1 we consider the initial value problem for the differential algebraic system

$$\begin{aligned}&\frac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(x_1.\, x_2, t)\,, \\ &x_2 = f_2(x_1, x_2, t)\,, \quad x(0) = 0\,, \quad t \in (0, T)\end{aligned} \tag{3.1}$$

under the assumptions

(H_1) $f_1 \in C(\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}, \boldsymbol{R}^n)$, $f_2 \in C(\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}, \boldsymbol{R}^m)$.

(H_2) There are positive constants $l_{11}, l_{12}, l_{21}, l_{22}$ such that for all (x_1, x_2), $(\tilde{x}_1, \tilde{x}_2) \in \boldsymbol{R}^n \times \boldsymbol{R}^n$ and for all $t \in \boldsymbol{R}$

$$\begin{aligned}|f_1(x_1, x_2, t) - f_1(\tilde{x}_1, \tilde{x}_2, t)| &\leqq l_{11}|x_1 - \tilde{x}_1| + l_{12}|x_2 - \tilde{x}_2|\,, \\ |f_2(x_1, x_2, t) - f_2(\tilde{x}_1, \tilde{x}_2, t)| &\leqq l_{21}|x - \tilde{x}_1| + l_{22}|x_2 - \tilde{x}_2|\,.\end{aligned} \tag{3.2}$$

(H_3) (3.3) $l_{22} < 1$.

Let X_1 be the space of continuous functions x mapping $[0, T]$ into $\boldsymbol{R}^n$ and satisfying $x(0) = 0$. X_1 equipped with the norm

$$\|x\|_1 := \max_{[0,T]} \{\mathrm{e}^{-\alpha t}|x(t)|\} \tag{3.4}$$

[1] K^{T} denotes the transpose of $\boldsymbol{K}$.

is a Banach space where α is a (suitable chosen) positive number. Let X_2 be the space $C([0, T], \boldsymbol{R}^m)$ endowed with the norm $\|\cdot\|_2$ which coincides with the norm $\|\cdot\|_1$.

Let us introduce operators $T_1: X_1 \times X_2 \to X_1$, $T_2: X_1 \times X_2 \to X$ defined by

$$(3.5) \qquad \begin{aligned} T_1(x_1, x_2)(t) &:= \textstyle\int_0^t f_1(x_1(s), x_2(s), s)\,\mathrm{d}s\,, \\ T_1(x_1, x_2)(t) &:= f_2(x_1(t), x_2(t), t)\,. \end{aligned}$$

Then the system (3.1) is equivalent to the system

$$(3.6) \qquad \begin{aligned} x_1 &= T_1(x_1, x_2)\,, \\ x_2 &= T_2(x_1, x_1)\,. \end{aligned}$$

It is easy to verify that T_1 and T_2 satisfy the relations

$$(3.7) \qquad \begin{aligned} \|T_1(x_1, x_2) - T_1(\tilde{x}_1, \tilde{x}_2)\|_1 &\leqq \frac{l_{11}}{\alpha}\|x_1 - \tilde{x}_1\|_1 + \frac{l_{12}}{\alpha}\|x_2 - \tilde{x}_2\|_2\,, \\ \|T_2(x_1, x_2) - T_2(\tilde{x}_1, \tilde{x}_2)\|_2 &\leqq l_{21}\|x_1 - \tilde{x}_1\|_1 + l_{22}\|x_2 - \tilde{x}_2\|_2 \end{aligned}$$

for all $(x_1, x_2), (\tilde{x}_1, \tilde{x}_2) \in X_1 \times X_2$. In order to be able to apply Lemma 2.1 we have to verify the validity of the relation (2.6). Let K be the matrix defined by

$$K = \begin{pmatrix} \frac{l_{11}}{\alpha} & \frac{l_{12}}{\alpha} \\ l_{21} & l_{22} \end{pmatrix}.$$

It is easy to verify that under the condition (3.3) there is a positive number α such that for $\alpha > \alpha_0$ the spectral radius of K is less than one. Applying Lemma 2.1 we have the following result.

Theorem 3.1. *Assume the hypotheses* $(\mathrm{H}_1) - (\mathrm{H}_3)$ *hold. Then the initial value problem for the differential algebraic system* (4.1) *has a unique solution for any given* T, *which can be approximated by the iteration scheme*

$$\begin{aligned} x_1^{k+1}(t) &:= \textstyle\int_0^t f_1(x_1^k(s), x_2^k(s), s)\,\mathrm{d}s\,, \\ x_2^{k+1}(t) &:= f_2(x_1^k(t), x_2^k(t), t) \end{aligned}$$

where $(x_1^0(t), x_2^0(t))$ *is any initial guess.*

An other important application of Lemma 2.1 is concerned with the waveform relaxation method in circuit analysis (see [7]).

References

[1] *G. Frobenius:* Über Matrizen aus positiven Elementen. S.-B. Preuss. Akad. Wiss. Berlin 1908, 471–476, 1909, 514–518.

[2] *G. Frobenius:* Über Matrizen aus nichtnegativen Elementen. S.-B. Preuss. Akad. Wiss. Berlin 1912, 456–477.

[3] *E. Lelarasmee, A. E. Ruehli, A. L. Sangiovanni-Vincentelli:* The waveform relaxation method for the time domain analysis of large scale integrated circuits. IEEE Trans. CAD 1, (1982), 131—145.
[4] *A. R. Newton. A. L. Sangiovanni-Vincentelli*: Relaxation based electrical simulation. IEEE Trans ED 30 (1983), 1184—1207.
[5] *J. M. Ortega, W. C. Rheinboldt:* Iterative Solutions of Nonlinear Equations in Several Variables. New York: Academic Press, 1970.
[6] *O. Perron:* Zur Theorie der Matrizen. Math. Ann. 64 (1907), 248—263.
[7] *K. R. Schneider:* A remark on the waveform relaxation method. Int. J. Circuit Theory Appl. 18 (1990).
[8] *R. S. Varga:* Matrix iterative analysis, Prentic-Hall, Englewood Cliffs, N. J. 1962.

Souhrn

POZNÁMKA K ŘEŠENÍ VELKÝCH SOUSTAV ROVNIC V PROSTORECH FUNKCÍ

I. Bremer, K. R. Schneider

S cílem ušetřit čas základní jednotky při řešení velkých soustav rovnic v prostorech funkcí je daná soustava rozložena na menší, které se řeší vhodnou metodou. Je podána postačující podmínka konvergence příslušné procedury. Metoda je aplikována na diferenciální algebraické soustavy.

Authors' address: DM. *I. Bremer*, Dr. *K. R. Schneider*, Akademie der Wissenschaften der DDR, Karl-Weierstrass-Institut für Mathematik, Mohrenstrasse, 1086 Berlin.