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APPROXIMATION OF A NONLINEAR THERMOELASTIC PROBLEM WITH A MOVING BOUNDARY VIA A FIXED-DOMAIN METHOD

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Summary. The thermoelastic stresses created in a solid phase domain in the course of solidification of a molten ingot are investigated. A nonlinear behaviour of the solid phase is admitted, too. This problem, obtained from a real situation by many simplifications, contains a moving boundary between the solid and the liquid phase domains. To make the usage of standard numerical packages possible, we propose here a fixed-domain approximation by means of including the liquid phase domain into the problem (in this way we get the fixed domain involving the whole ingot) and by replacing the liquid phase with a solid phase having, however, a small shear modulus. The weak L^2-convergence of thus approximated stresses in the solid phase domain is demonstrated. Besides, this convergence is shown to be strong on subsets whose closure belongs to the solid phase domain.

Keywords: nonlinear thermoelasticity, solidification, moving boundary.

AMS Classification: 35J70, 73C50.

1. MOTIVATION AND FORMULATION OF THE PROBLEM

The aim of the paper is to propose an effective approximation that enables us to evaluate thermoelastic stresses within solidification of a molten steel ingot in a sand form by using standard numerical packages. We admit only rather simple geometrical situations like that in Fig. 1. The ingot, occupying a fixed domain $\Omega^0 \subset \mathbb{R}^3$, is partially in the solid phase (i.e. the domain $\Omega = \Omega(t)$), the rest (i.e. $\Omega^0 \setminus \Omega(t)$) being in the liquid phase; $\Omega(t)$ depends on time t, hence the boundary $\partial\Omega_{\mathrm{m}} = \partial\Omega_{\mathrm{m}}(t)$ between the solid and the liquid phases is moving. The fixed boundary of the ingot $\partial\Omega^0$ is divided into the parts $\partial\Omega^0_{\mathrm{air}}$ and $\partial\Omega^0_{\mathrm{form}}$ corresponding to the open-air surface and the surface of the ingot/form contact, respectively. The parts corresponding to the solid phase are denoted by $\partial\Omega_{\mathrm{air}} = \partial\Omega_{\mathrm{air}}(t)$ and $\partial\Omega_{\mathrm{form}} = \partial\Omega_{\mathrm{form}}(t)$ (possibly, $\partial\Omega_{\mathrm{air}}(t)$ or $\partial\Omega^0_{\mathrm{form}} \setminus \partial\Omega_{\mathrm{form}}(t)$ may be empty).

As the real situation is extremely complicated, we are forced to make many simplifications. We neglect mainly any changes concerning a composition of the material, any influence of the stress on the temperature field, all viscous effects, and inertial forces. Particularly, for the stress, time t will play the role of a parameter only (and we will mostly omit it for brevity) and the heat-transfer equation is not coupled with the system of equations for stress. Therefore we may and will suppose that we are

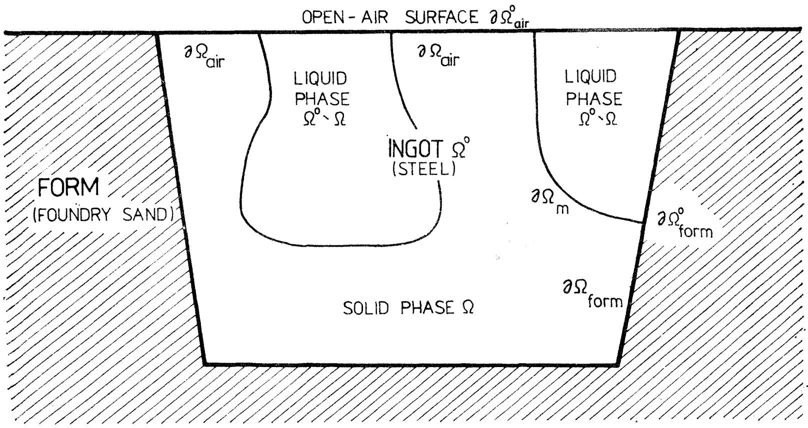

Fig. 1. An example of a possible geometrical situation in the course of solidification of a molten steel ingot Ω^0 in a sand form. The ingot is partially in a solid phase (= the domain Ω); the boundary $\partial\Omega_m$ between the solid and the liquid phases is moving.

given a temperature field $\theta = \theta(x, t)$ obtained by solving some nonlinear heat transfer problem, e.g. a Stefan problem which is the simplest model of the heat transfer in presence of phase transitions. Furthermore, denoting the solid/liquid transition temperature by θ_{SL}, we suppose:

$$\Omega(t) = \{x \in \Omega^0; \theta(x, t) < \theta_{SL}\} .$$

Besides, we employ small strains and a nonlinear Hook law in the solid phase; the nonlinear elasticity is also called a deformation theory of plasticity. The ingot/form contact is viewed as an elastic support, the friction being neglected. We are interested only in the stress in the solid phase, and thus we come to the following boundary value problem on the domain Ω (recall that the parameter t is omitted) for an unknown vector field of displacements $v = (v_1, v_2, v_3)$:

$$(1.1) \qquad \partial\tau_{ij}(v, \theta)/\partial x_j + f_i = 0 \quad \text{on} \quad \Omega,\ i = 1, 2, 3 ,$$

(i.e. the Lamé system) with the boundary conditions

$$(1.2) \qquad \tau_{ij}(v, \theta)\, n_j + pn_i = 0 \quad \text{on} \quad \partial\Omega_m , \quad i = 1, 2, 3 ,$$

$$(1.3) \qquad \tau_{ij}(v, \theta)\, n_j = 0 \quad \text{on} \quad \partial\Omega_{air} , \quad i = 1, 2, 3 ,$$

$$(1.4) \qquad \tau_{ij}(v, \theta)\, n_j + an_i v_n = 0 \quad \text{on} \quad \partial\Omega_{form} , \quad i = 1, 2, 3 ,$$

where we have used the summation convention and the following notation:

$\theta = \theta(x)$ temperature (which is supposed to be given),

$f = (f_1, f_2, f_3)$ the gravitational force loading,

$p = p(x)$ the pressure (i.e. the hydrostatic pressure in the liquid phase at the point $x \in \partial\Omega_{\mathrm{m}}$),

$n = (n_1, n_2, n_3)$ the vector of a unit outward normal to the boundary $\partial\Omega$,

$v_n = v_i n_i$ the normal displacement,

$a = a(x)$ the coefficient characterizing the elastic support of the ingot in the form,

$\tau_{ij}(v, \theta)$ the stress tensor components subjected to the nonlinear Hooke law:

$$\tau_{ij}(v, \theta) = \left(k(\theta) - \tfrac{2}{3}\mu(\Gamma(v), \theta)\right) \operatorname{div} v \, \delta_{ij} + 2\mu(\Gamma(v), \theta) \, e_{ij}(v) - 3 \, k(\theta) \, \alpha(\theta) \, \delta_{ij} , \tag{1.5}$$

with $e_{ij}(v) = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$ the small strain tensor components and $\Gamma(v) = (e_{ij}^*(v) \, e_{ij}^*(v))^{1/2}$ the intensity of shear strains, where $e_{ij}^*(v) = e_{ij}(v) - \frac{1}{3}\delta_{ij} \operatorname{div} v$, δ_{ij} is the Kronecker symbol, k the bulk modulus (depending on temperature), μ the shear modulus (depending on temperature and also on $\Gamma(v)$, which makes the problem nonlinear), and α: $\mathbb{R} \to \mathbb{R}$; typically $\alpha(\theta) = \alpha_0(\theta)(\theta - \theta_0)$ where α_0 is a temperature-dependent thermal dilatability related to a reference temperature θ_0.

Applying Green's formula once to (1.1) and employing the boundary conditions (1.2)–(1.4), we obtain in a usual way the weak formulation of the problem ($H^k(\Omega)$ will denote the Sobolev space of all functions from $L^2(\Omega)$ with k-th distributional partial derivatives from $L^2(\Omega)$):

Definition 1. *A function* $v \in [H^1(\Omega)]^3$ *is called a weak solution of the problem* (1.1)–(1.5) *if, for every* $z \in [H^1(\Omega)]^3$,

$$\int_\Omega \tau_{ij}(v, \theta) \, e_{ij}(z) \, \mathrm{d}x + \int_{\partial\Omega_{\mathrm{form}}} a v_n z_n \, \mathrm{d}S = \int_\Omega f_i z_i \, \mathrm{d}x - \int_{\partial\Omega_{\mathrm{m}}} p z_n \, \mathrm{d}S . \tag{1.6}$$

2. APPROXIMATION OF THE PROBLEM VIA A FIXED-DOMAIN METHOD

The moving boundary $\partial\Omega_{\mathrm{m}}$ is very unpleasant from the viewpoint of making a computer program. We propose here an approximation of the original problem by another problem that uses the fixed domain Ω^0. The idea is very simple indeed: we replace the liquid phase by a solid phase with a very low shear modulus (being equal to some small $\varepsilon > 0$). Thus we have got only a solid phase on the whole domain Ω^0. Of course, the temperature-dependent properties of the solid phase (i.e. k, μ, and α) must be now defined also for the temperature greater than the solid/liquid transition temperature θ_{SL}, at which some of them may have jumps. In other words, for $\theta > \theta_{SL}$ the Poisson ratio $\nu = (3k - 2\mu)/(6k + 2\mu)$ is very near to 1/2, while the Young modulus $E = 3k\mu/(3k + \mu)$ approaches zero. Besides, suppose f and a to be defined on Ω^0 and $\partial\Omega^0_{form}$, respectively.

Hence, we consider the following fixed-domain problem ($\varepsilon > 0$):

(2.1) $$\partial\tau^{\varepsilon}_{ij}(v, \theta)/\partial x_j + f_i = 0 \quad \text{on} \quad \Omega^0, \qquad i = 1, 2, 3,$$

(2.2) $$\tau^{\varepsilon}_{ij}(v, \theta)\, n_j = 0 \quad \text{on} \quad \partial\Omega^0_{air}, \qquad i = 1, 2, 3,$$

(2.3) $$\tau^{\varepsilon}_{ij}(v, \theta)\, n_j + a n_i v_n = 0 \quad \text{on} \quad \partial\Omega^0_{form}, \qquad i = 1, 2, 3,$$

with

(2.4) $$\tau^{\varepsilon}_{ij}(v, \theta) = (k^{\varepsilon}(\theta) - \tfrac{2}{3}\mu^{\varepsilon}(\Gamma(v), \theta)) \operatorname{div} v\, \delta_{ij} + 2\mu^{\varepsilon}(\Gamma(v), \theta)\, e_{ij}(v) - 3\, k^{\varepsilon}(\theta)\, \bar{\alpha}(\theta)\, \delta_{ij},$$

where

(2.5) $$k^{\varepsilon}(\theta) = \begin{cases} k(\theta) & \text{for} \quad \theta < \theta_{SL}, \\ \lambda + \tfrac{2}{3}\varepsilon & \text{for} \quad \theta \geqq \theta_{SL}, \end{cases}$$

(2.6) $$\mu^{\varepsilon}(\Gamma, \theta) = \begin{cases} \mu(\Gamma, \theta) & \text{for} \quad \theta < \theta_{SL}, \\ \varepsilon & \text{for} \quad \theta \geqq \theta_{SL}, \end{cases}$$

(2.7) $$\bar{\alpha}(\theta) = \begin{cases} \alpha(\theta) & \text{for} \quad \theta < \theta_{SL}, \\ 0 & \text{for} \quad \theta \geqq \theta_{SL}. \end{cases}$$

In the usual manner we define the weak solution of the problem (2.1)–(2.7).

Definition 2. *A function* $v^{\varepsilon} \in [H^1(\Omega^0)]^3$ *will be called a weak solution of the problem* (2.1)–(2.7) *if the following integral identity holds for every* $z \in [H^1(\Omega^0)]^3$:

(2.8) $$\int_{\Omega^0} \tau^{\varepsilon}_{ij}(v^{\varepsilon}, \theta)\, e_{ij}(z)\, dx + \int_{\partial\Omega^0_{form}} a v^{\varepsilon}_n z_n\, dS = \int_{\Omega^0} f_i z_i\, dx.$$

3. ASSUMPTIONS AND APRIORI ESTIMATES

Let us collect the assumptions imposed on the data we will need in what follows:

(3.1) Ω^0, $\Omega \subset \Omega^0$ and $\Omega^0 \setminus \bar{\Omega}$ are three Lipschitz domains, Ω^0 is connected, and, for very connected component Q of $\Omega^0 \setminus \bar{\Omega}$, $\bar{Q} \cap \partial\Omega^0_{air}$ has a positive two-dimensional Lebesgue measure (the bar denotes the closure),

(3.2) $a \in L^{\infty}(\partial\Omega^0_{form})$, $a(x) \geqq a_{min} > 0$ for a.a. $x \in \partial\Omega^0_{form}$,

(3.3) $$\forall z \in [H^1(\Omega)]^3 : e_{ij}(z) = 0 \text{ a.e. in } \Omega \ \&\ \int_{\partial\Omega_{form}} a\, z_n^2\, dS = 0 \Rightarrow \Rightarrow z = 0 \text{ a.e. in } \Omega,$$

(3.4) $$\exists \phi \in C^1(\bar{\Omega}^0) : f = -\nabla\phi \text{ and } \phi(\partial\Omega^0_{air}) = \phi_0 = \text{const.},$$

(3.5) $k: [-\infty, \theta_{SL}] \to \mathbb{R}$ is Lipschitzian, $\forall \theta: k(\theta) \leqq k_{max}$,

$\mu: \mathbb{R}^+ \times [-\infty, \theta_{SL}] \to \mathbb{R}$ fulfils:

$\exists L_\mu \in \mathbb{R},\ \lambda > 0,\ \mu_{min} > 0,\ \mu'_{min} > 0 \quad \forall \theta, \theta', \Gamma, \Gamma'$:

$|\mu(\Gamma, \theta) - \mu(\Gamma', \theta)| \leqq L_\mu |\Gamma - \Gamma'|$,

$|\mu(\Gamma, \theta) - \mu(\Gamma, \theta')| \leqq L_\mu (1 + \Gamma) |\theta - \theta'|$,

$\mu_{min} \leqq \mu(\Gamma, \theta) \leqq \frac{3}{2}(k(\theta) - \lambda)$,

$\mu(\Gamma, \theta) + 2\Gamma \frac{\partial}{\partial \Gamma} \mu(\Gamma, \theta) \geqq \mu'_{min}$,

(3.6) $\alpha: [-\infty, \theta_{SL}] \to \mathbb{R}$ is Lipschitzian,

(3.7) $\theta \in L^2(\Omega),\quad \forall S$ open, $\bar{S} \subset \Omega: \theta \in W^{1,\infty}(S)$.

Let us remark that the conditions (3.1) in particular does not admit liquid phase components disconnected with the open-air surface (which could not be properly described by our small strain model). The condition (3.3) means that the solid phase domain Ω is well fixed in the form, i.e. it cannot be moved without deformation. It is obvious that (3.3) together with (3.1) enables us to exploit the well-known Korn inequality both on Ω and Ω^0. The condition (3.4) means that the loading force f has a potential ϕ which is constant on the open-air surface of the body $\partial\Omega^0_{air}$. In most applications, ϕ will be just the gravitational potential of the earth multiplied by the specific mass of the material of the ingot.

Of course, the pressure p that appears as data in the moving-boundary condition (1.2) is related with the loading force potential by

(3.8) $p = \phi_0 - \phi$.

It is known (see [1], the proof of Thm. 8.2.1 after a slight modification) that under the conditions (3.1)–(3.7) there exists just one solution v by Def. 1 and, for every $\varepsilon > 0$, just one solution v^ε by Def. 2. To prove our convergence results, we need some apriori estimates. Let us emphasize that the problems (2.1)–(2.7) are not uniformly coercive with respect to $\varepsilon > 0$, and for the following estimates it is essential that the loading force is irrotational in the liquid phase domain (as in applications f is constant in time while Ω is not, we suppose f to be irrotational even over the whole domain Ω^0).

Lemma. *Let $\varepsilon > 0$, let (3.1)–(3.7) be fulfilled, and let v^ε be the unique solution by Def. 2. Then there is a constant C independent of ε such that for all $i, j = 1, 2, 3$ and all $\varepsilon > 0$:*

(3.9) $\|v_i^\varepsilon\|_{H^1(\Omega^0)} \leqq C/\sqrt{\varepsilon}$,

(3.10) $\|v_i^\varepsilon\|_{H^1(\Omega)} \leqq C$,

(3.11) $\|\tau_{ij}^\varepsilon(v^\varepsilon, \theta)\|_{L^2(\Omega^0)} \leqq C$,

(3.12) $\|v_n^\varepsilon\|_{L^2(\partial\Omega^0_{form})} \leqq C$.

Proof. We can put $z = v^\varepsilon$ in (2.8). By using

$$\int_{\Omega^0} f_i z_i \, dx = -\int_{\Omega^0} \frac{\partial \phi}{\partial x_i} z_i \, dx = \int_{\Omega^0} \phi \operatorname{div} z \, dx - \int_{\partial\Omega^0} \phi z_n \, dS$$

we obtain

$$\int_{\Omega^0}((k^\varepsilon(\theta) - \tfrac{2}{3}\mu^\varepsilon(\Gamma(v^\varepsilon), \theta))(\operatorname{div} v^\varepsilon)^2 + 2\mu^\varepsilon(\Gamma(v^\varepsilon), \theta)\, e_{ij}(v^\varepsilon)\, e_{ij}(v^\varepsilon))\, dx +$$
$$+ \int_{\partial\Omega^0_{\text{form}}} a \,.\, (v^\varepsilon_n)^2 \, dS = \int_{\Omega^0} (3k^\varepsilon(\theta)\, \bar{\alpha}(\theta) + \phi) \operatorname{div} v^\varepsilon \, dx - \int_{\partial\Omega^0} \phi v^\varepsilon_n \, dS \,.$$

Employing the assumptions (3.2) a (3.5), we get the estimate

$$\lambda \int_{\Omega^0} (\operatorname{div} v^\varepsilon)^2 \, dx + 2\mu_{\min} \int_\Omega e_{ij}(v^\varepsilon)\, e_{ij}(v^\varepsilon) \, dx +$$
$$+ 2\varepsilon \int_{\Omega^0\setminus\Omega} e_{ij}(v^\varepsilon)\, e_{ij}(v^\varepsilon) \, dx + a_{\min} \int_{\partial\Omega^0_{\text{form}}} (v^\varepsilon_n)^2 \, dS \leqq$$
$$\leqq \frac{\lambda}{2} \int_{\Omega^0} (\operatorname{div} v^\varepsilon)^2 \, dx + \frac{1}{2\lambda} \int_{\Omega^0} (3k^\varepsilon(\theta)\, \bar{\alpha}(\theta) + \phi)^2 \, dx +$$
$$+ \tfrac{1}{2}a_{\min} \int_{\partial\Omega^0_{\text{form}}} (v^\varepsilon_n)^2 \, dS + \frac{1}{2a_{\min}} \int_{\partial\Omega^0_{\text{form}}} \phi^2 \, dS + \int_{\partial\Omega^0_{\text{air}}} \phi v^\varepsilon_n \, dS =$$
$$= I_1 + I_2 + \ldots + I_5 \,.$$

We may and will suppose $\phi_0 = 0$. Then $I_5 = 0$. The terms I_1 and I_3 can be absorbed in the left-hand side, which offers the estimate

$$\frac{\lambda}{2} \int_{\Omega^0} (\operatorname{div} v^\varepsilon)^2 \, dx + 2\mu_{\min} \int_\Omega e_{ij}(v^\varepsilon)\, e_{ij}(v^\varepsilon) \, dx + \tag{3.13}$$
$$+ 2\varepsilon \int_{\Omega^0\setminus\Omega} e_{ij}(v^\varepsilon)\, e_{ij}(v^\varepsilon) \, dx + \tfrac{1}{2}a_{\min} \int_{\partial\Omega^0_{\text{form}}} (v^\varepsilon_n)^2 \, dS \leqq$$
$$\leqq \frac{1}{2\lambda} \int_{\Omega^0} (3k^\varepsilon(\theta)\, \bar{\alpha}(\theta) + \phi)^2 \, dx + \frac{1}{2a_{\min}} \int_{\partial\Omega^0_{\text{form}}} \phi^2 \, dS \leqq C_0 \,.$$

Note that, by (3.6), $\bar{\alpha}$ has a linear growth, and by (3.7) $\bar{\alpha}(\theta) \in L^2(\Omega)$. As k^ε is bounded independently of ε, the constant C_0 can be taken also independently of ε. In particular, we have proved:

$$\|v^\varepsilon_n\|_{L^2(\partial\Omega^0_{\text{form}})} \leqq \sqrt{(2C_0/a_{\min})} \,,$$

which is nothing else than (3.12).

In view of (3.1) and (3.3) we can apply the Korn inequality on Ω^0, which yields the estimate $\varepsilon\|v^\varepsilon_i\|^2_{H^1(\Omega^0)} \leqq C$ (C will be a generic constant independent of ε). Thus (3.9) has been proved.

From (3.13) we get also the estimate

$$\int_\Omega (\lambda/2\, (\operatorname{div} v^\varepsilon)^2 \, dx + 2\mu_{\min}\, e_{ij}(v^\varepsilon)\, e_{ij}(v^\varepsilon))\, dx + \tfrac{1}{2}a_{\min} \int_{\partial\Omega_{\text{form}}} (v^\varepsilon_n)^2 \, dS \leqq C_0$$

As Ω is Lipschitzian, we can employ on it the Korn inequality (using again (3.3)), which yields $\|v^\varepsilon_i\|_{H^1(\Omega)} \leqq C$, i.e. (3.10).

Now we only have to estimate the stress tensor components:

$$\|\tau^{\varepsilon}_{ij}(v^{\varepsilon}, \theta)\|_{L^2(\Omega^0)} \leqq \|(k^{\varepsilon}(\theta) - \tfrac{2}{3}\mu^{\varepsilon}(\Gamma(v^{\varepsilon}), \theta)) \operatorname{div} v^{\varepsilon}\|_{L^2(\Omega^0)} +$$
$$+ 2\|\mu^{\varepsilon}(\Gamma(v^{\varepsilon}), \theta))\, e_{ij}(v^{\varepsilon})\|_{L^2(\Omega^0)} + 3\|k^{\varepsilon}(\theta)\, \bar{\alpha}(\theta)\|_{L^2(\Omega^0)} \leqq$$
$$\leqq k_{\max}\|\operatorname{div} v^{\varepsilon}\|_{L^2(\Omega^0)} + 3k_{\max}\|e_{ij}(v^{\varepsilon})\|_{L^2(\Omega)} +$$
$$+ 2\varepsilon\|e_{ij}(v^{\varepsilon})\|_{L^2(\Omega^0\setminus\Omega)} + 3k_{\max}\|\bar{\alpha}(\theta)\|_{L^2(\Omega^0)} \leqq$$
$$\leqq k_{\max}(2C_0/\lambda)^{1/2} + 3k_{\max}(C_0/(2\mu_{\min}))^{1/2} + (2\varepsilon C_0)^{1/2} + 3k_{\max}\|\bar{\alpha}(\theta)\|_{L^2(\Omega^0)}\,.$$

The estimate (3.11) is thus proved. □

4. CONVERGENCE RESULTS

Now we are going to prove the convergence of displacements obtained by solving the problem (2.1)–(2.7). restricted to the solid phase domain Ω. It should be emphasized that, to pass through the nonlinearity describing the plastic behaviour of the solid phase, we must exploit some regularity results on sets having a (uniform) neighbourhood contained in Ω. We employ also the following interesting property of Nemytskii operators in L^p-spaces: any Nemytskii operator, which does not generally map weakly convergent sequences onto weakly convergent ones, does map them so provided possible oscillations (which deteriorate the strong convergence to a weak one) are "cumulated" on a set of zero measure – here on the boundary of Ω.

Theorem 1. *Let (3.1)–(3.8) be valid, let $\{v^{\varepsilon}\}_{\varepsilon>0}$ be a sequence of the weak solutions by Def. 2, and $\varepsilon \to 0$. Then*

(4.1) $\quad v^{\varepsilon}_i \to v_i$ *weakly in* $H^1(\Omega)$,

(4.2) $\quad \tau_{ij}(v^{\varepsilon}, \theta) \to \tau_{ij}(v, \theta)$ *weakly in* $L^2(\Omega)$,

where $v = (v_1, v_2, v_3)$ is the unique weak solution by Def. 1. Moreover,

(4.3) $\quad v^{\varepsilon}_i \to v_i$ *weakly in* $H^2(S)$ *and*

(4.4) $\quad \tau_{ij}(v^{\varepsilon}, \theta) \to \tau_{ij}(v, \theta)$ *strongly in* $L^2(S)$

for every open set S such that $\bar{S} \subset \Omega$.

Proof. In view of the apriori estimates (3.10)–(3.12) we can choose a subsequence $\{v^{\varepsilon}\}_{\varepsilon>0}$ which satisfies (4.1) and such that

(4.5) $\quad \tau^{\varepsilon}_{ij}(v^{\varepsilon}, \theta) \to \sigma_{ij}$ weakly in $L^2(\Omega^0)$ and

(4.6) $\quad v^{\varepsilon}_n \to w$ weakly in $L^2(\partial\Omega^0_{\text{form}})$.

Passing to the limit in (2.8), we evidently get

(4.7) $$\int_{\Omega^0} \sigma_{ij}\, e_{ij}(z)\,\mathrm{d}x + \int_{\partial\Omega^0} awz_n\,\mathrm{d}S = \int_{\Omega^0} f_i z_i\,\mathrm{d}x\,.$$

It is clear that

(4.8) $$w = v_n \quad \text{a.e. in} \quad \partial\Omega_{\text{form}}\,,$$

which follows from (4.1), the continuity of the trace operator $v \mapsto v_n$ from $[H^1(\Omega)]^3$ to $L^2(\partial\Omega_{\text{form}})$, and from (4.6).

Furthermore, we will prove

(4.9) $$\sigma_{ij} = \tau_{ij}(v, \theta) \quad \text{a.e. in} \quad \Omega\,,$$

from which we shall obtain in particular (4.2) due to (4.5). Take an open set S, $\bar{S} \subset \Omega$. We want to employ the H^2-regularity of the displacements in the domain S. By [2; Sec. 5.2] we need the regularity of the loading f, namely $f \in L^2(\Omega)$, which is surely valid because of (3.4). Furthermore, we need $k(\theta)\,\alpha(\theta)$ belonging to $H^1(S')$ for some open $S' \supset \bar{S}$, which is fulfilled for $\bar{S}' \subset \Omega$ as a consequence of (3.5), (3.6) and (3.7). Also we need a regularity of the coefficients, namely for a.a. $x \in S'$ and all $\Gamma \geqq 0$:

$$k(\theta(x)) + \left|\frac{\mathrm{d}k}{\mathrm{d}\theta}\,.\,\nabla\theta(x)\right| + \mu(\Gamma, \theta(x))$$

$$+ \left|\frac{\partial\mu}{\partial\theta}(\Gamma, \theta(x))\,.\,\nabla\theta(x)\right| \leqq C(1 + \Gamma)\,,$$

$$\left|\frac{\partial\mu}{\partial\Gamma}(\Gamma, \theta(x))\right| \leqq C\,,$$

which is guaranteed by the assumptions (3.5) and (3.7). Using also (3.10), by [2; Sec. 5.2] we get the H^2-regularity on S:

$$\|v_i^\varepsilon\|_{H^2(S)} \leqq C_S\,,$$

where C_S is a constant depending on S but not on ε. Thus we can choose a subsequence from $\{v^\varepsilon\}$ converging weakly in $[H^2(S)]^3$. In view of (4.1) its limit is again v (restricted on S), and therefore even the whole (already chosen) sequence $\{v^\varepsilon\}$ converges to v weakly in $[H^2(S)]^3$; i.e. we have proved (4.3).

Now we will prove (4.4), from which we immediately obtain (4.9) because, by (4.5), we then have $\sigma_{ij} = \tau_{ij}(v, \theta)$ a.e. on S and by the sets S the whole solid phase domain Ω can be covered.

To show (4.4) we use the compact imbedding of $[H^2(S)]^3$ into $[H^1(S)]^3$. Then, by (4.3), $e_{ij}(v^\varepsilon)$ converge strongly to $e_{ij}(v)$ in $L^2(S)$. The function from $\mathbb{R}^9$ to $\mathbb{R}$ defined by

$$e_{ij} \mapsto \mu(\sqrt{((e_{ij} - \delta_{ij}e_{kk}/3)\,(e_{ij} - \delta_{ij}e_{kk}/3))}, \theta)$$

is clearly bounded (independently of θ) and continuous, and therefore the corresponding Nemytskii operator is continuous from $[L^2(S)]^9$ to $L^2(S)$ (or even to $L^r(S)$ with every $p < +\infty$). We thus see that $\mu(\Gamma(v^\varepsilon), \theta) \to \mu(\Gamma(v), \theta)$ strongly in $L^2(S)$, and therefore also in the measure on S, i.e.

$$\text{(4.10)} \qquad \forall \delta > 0\colon \lim_{\varepsilon \to 0} \operatorname{meas}\left(S \setminus S(\varepsilon, \delta)\right) = 0\,,$$

where

$$S(\varepsilon, \delta) = \{x \in S;\ |\mu(\Gamma(v^\varepsilon), \theta)(x) - \mu(\Gamma(v), \theta)(x)| \leqq \delta\}\,.$$

Now we can estimate:

$$\begin{aligned}
&\|\mu(\Gamma(v^\varepsilon), \theta)\, e_{ij}(v^\varepsilon) - \mu(\Gamma(v), \theta)\, e_{ij}(v)\|^2_{L^2(S)} \leqq \\
&\leqq 2\|\mu(\Gamma(v^\varepsilon), \theta)\, e_{ij}(v^\varepsilon - v)\|^2_{L^2(S)} + \\
&+ 2\|(\mu(\Gamma(v^\varepsilon), \theta) - \mu(\Gamma(v), \theta))\, e_{ij}(v)\|^2_{L^2(S(\varepsilon,\delta))} + \\
&+ 2\|(\mu(\Gamma(v^\varepsilon), \theta) - \mu(\Gamma(v), \theta))\, e_{ij}(v)\|^2_{L^2(S \setminus S(\varepsilon,\delta))} \leqq \\
&\leqq \tfrac{9}{2} k^2_{\max} \|e_{ij}(v^\varepsilon - v)\|^2_{L^2(\Omega_0)} + 2\delta^2 \|e_{ij}(v)\|^2_{L^2(S)} + \tfrac{9}{2} k^2_{\max} \|e_{ij}(v)\|^2_{L^2(S \setminus S(\varepsilon,\delta))}\,.
\end{aligned}$$

By (4.3) with the compactness of the imbedding $[H^2(S)]^3 \subset [H^1(S)]^3$, by (4.10), by the absolute continuity of the Lebesgue integral, and by (3.10), we obtain the estimate

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \|\mu(\Gamma(v^\varepsilon), \theta)\, e_{ij}(v^\varepsilon) - \mu(\Gamma(v), \theta)\, e_{ij}(v)\|_{L^2(S)} \leqq \\
&\leqq 2^{1/2}\, \delta \|e_{ij}(v)\|_{L^2(S)} \leqq 2^{1/2}\, \delta C\,.
\end{aligned}$$

As $\delta > 0$ is arbitrary, we eventually get

$$\lim_{\varepsilon \to 0} \|\mu(\Gamma(v^\varepsilon), \theta)\, e_{ij}(v^\varepsilon) - \mu(\Gamma(v), \theta)\, e_{ij}(v)\|_{L^2(S)} = 0\,.$$

Hence (4.4) is proved because the convergence in the other terms in the stress tensor components, which are either linear with respect to the displacement or even constant, is obvious.

It remains to demonstrate that v is the weak solution by Def. 1. It is clear that $\tau^\varepsilon_{ij}(v^\varepsilon, \theta) = \lambda \operatorname{div} v^\varepsilon \delta_{ij} + 2\varepsilon\, e_{ij}(v^\varepsilon)$ in the liquid phase domain $\Omega^0 \setminus \Omega$. By (3.9) we have $\|2\varepsilon\, e_{ij}(v^\varepsilon)\|_{L^2(\Omega^0 \setminus \Omega)} = \mathcal{O}(\sqrt{\varepsilon})$ for $\varepsilon \to 0$. Since the convergence from (4.5) holds also in $L^2(\Omega^0 \setminus \Omega)$, we can see that $-\lambda \operatorname{div} v^\varepsilon \to p$ weakly in $L^2(\Omega^0 \setminus \Omega)$, where $-p$ is the diagonal component of the limit stress tensor which has the form

$$\text{(4.11)} \qquad \sigma_{ij} = -p\delta_{ij} \quad \text{in} \quad \Omega^0 \setminus \Omega\,.$$

Now we use this fact for (4.7). Besides, let us take a test function $z \in [C^1(\bar{\Omega}^0)]^3$ for (4.7) having its support in $\Omega^0 \setminus \bar{\Omega}$. Then

$$\int_{\Omega^0 \setminus \Omega} p \operatorname{div} z \, dx = \int_{\Omega^0 \setminus \Omega} \nabla\phi\, z \, dx\,,$$

hence by using Green's formula we have

$$\int_{\Omega^0 \setminus \Omega} \nabla(p + \phi)\, z \, dx = 0\,.$$

As z is arbitrary, we can see that $\nabla p = -\nabla\phi$ a.e. on $\Omega^0 \setminus \Omega$, and thus also $p = \phi_c - \phi$ on $\Omega^0 \setminus \Omega$, where ϕ_c is a piecewise constant function on each connected component of $\Omega^0 \setminus \Omega$.

Now we take $z \in [C^1(\Omega^0)]^3$ with the support in $(\Omega^0 \setminus \bar{\Omega}) \cup \partial\Omega^0_{\text{air}}$. From (4.7) and (4.11) with $p = \phi_c - \phi$ we get (by using Green's formula again)

$$\int_{\partial\Omega^0_{\text{air}}} \phi_c z \, dS = \int_{\partial\Omega^0_{\text{air}}} \phi z \, dS = \int_{\partial\Omega^0_{\text{air}}} \phi_0 z \, dS \,,$$

where ϕ_0 is the constant from (3.4). As z is arbitrary we have, thanks to (3.1), $\phi_c = \phi_0$ over the whole liquid phase domain $\Omega^0 \setminus \Omega$.

Finally, we take a test function $z \in [H^1(\Omega)]^3$ for (1.6). We may and will suppose $z \in [L^\infty(\Omega)]^3$ because $L^\infty(\Omega) \cap H^1(\Omega)$ is dense in $H^1(\Omega)$. Furthermore, we can take, for $\eta > 0$, some extensions $z^\eta \in [H^1(\Omega^0)]^3$ of z such that the sequence $\{z^\eta\}_{\eta>0}$ is bounded in $[L^\infty(\Omega^0)]^3$ and $z^\eta = 0$ on $\{x \in \partial\Omega^0_{\text{form}};\ \text{dist}(x, \Omega) \geqq \eta\}$. Let us put z^η into (4.7). Denote $\{x \in \partial\Omega^0_{\text{form}} \setminus \partial\Omega_{\text{form}};\ \text{dist}(x, \Omega) < \eta\}$ by M^η. Using again (4.11), we easily obtain

$$\int_\Omega \sigma_{ij}\, e_{ij}(z)\, dx + \int_{\partial\Omega_{\text{form}}} awz_n \, dS - \int_{\Omega^0\setminus\Omega} p \operatorname{div} z^\eta \, dx =$$
$$= \int_\Omega f_i z_i \, dx - \int_{\Omega^0\setminus\Omega} \nabla\phi\, z^\eta \, dx - \int_{M^\eta} awz^\eta_n \, dS \,.$$

Employing Green's formula on $\Omega^0 \setminus \Omega$ and the fact that $p = \phi_c - \phi = \phi_c - \phi_0 = 0$ on $\partial^0_{\text{air}} \setminus \partial\Omega_{\text{air}}$, we get

$$\int_{\Omega^0\setminus\Omega} p \operatorname{div} z^\eta \, dx - \int_{\Omega^0\setminus\Omega} \nabla\phi\, z^\eta \, dx = -\int_{\partial\Omega_m} pz_n \, dS + \int_{M^\eta} pz^\eta_n \, dS$$

(recall that the normal to $\partial\Omega_m$ has been oriented from Ω to $\Omega^0 \setminus \Omega$). This yields

$$\int_\Omega \sigma_{ij}\, e_{ij}(z)\, dx + \int_{\partial\Omega_{\text{form}}} awz_n \, dS = \int_\Omega f_i z_i \, dx - \int_{\partial\Omega_m} pz_n \, dS + \tag{4.12}$$
$$+ \int_{M^\eta} (aw + p)\, z^\eta_n \, dS \,.$$

Now we pass to the limit with $\eta \to 0$ in the last term. As $M_{\eta_1} \supset M_{\eta_2}$ for $\eta_1 \geqq \eta_2 > 0$ and $\bigcap_{\eta>0} M_\eta = \emptyset$, we have got $\lim_{\eta\to 0} \text{meas}\, M_\eta = 0$ because of the continuity of the Lebesgue measure. Since $\{z^\eta\}_{\eta>0}$ is bounded in $[L^\infty(\Omega^0)]^3$, the traces of z^η on $\partial\Omega^0_{\text{form}}$ are bounded in $[L^\infty(\partial\Omega^0_{\text{form}})]^3$, too. Besides, $(aw + p) \in H^{1/2}(\partial\Omega^0_{\text{form}}) \subset L^1(\partial\Omega^0_{\text{form}})$. All these facts yield $\lim_{\eta\to 0} \int_{M_\eta} (aw + p)\, z^\eta_n \, dS = 0$. From (4.12) combined with (4.8) and (4.9), we then obtain (1.6).

Thus we have shown that v from (4,1) is the weak solution by Def. 1. The uniqueness of this solution then implies the convergence of the whole sequence $\{v^\varepsilon\}$. □

5. MISCELLANEOUS REMARKS

Remark 1. (*A non-Lipschitzian moving boundary.*) To simplify the preceding considerations we have assumed the moving boundary to be Lipschitzian, which however may be sometimes not too realistic. Let us briefly outline what should be changed in the case of non-Lipschitz moving boundaries. It is essential that the Korn inequality could not be employed on Ω, but only on internal Lipschitz approximations of Ω. Hence the displacements would not belong to $[H^1(\Omega)]^3$, but only to a set $[\mathscr{H}^1(\Omega)]^3$ with $\mathscr{H}(\Omega) = \{z: \Omega \to \mathbb{R}; \forall \eta > 0 \ z \in H^1(\Omega_\eta)\}$, where $\Omega_\eta = \{x \in \Omega; \operatorname{dist}(x, \partial\Omega_m) > \eta\}$. As we could not speak about traces of $z \in \mathscr{H}^1(\Omega)$ on $\partial\Omega_m$, we would have to define the weak solution of the original problem as a couple $(v, p) \in \mathscr{H}^1(\Omega) \times L^2(\Omega^0 \setminus \Omega)$ such that $\tau_{ij}(v, \theta)$ defined by (1.5) belongs to $L^2(\Omega)$ and the integral identity

$$\int_\Omega \sigma_{ij}(v, \theta)\, e_{ij}(z)\, dx - \int_{\Omega^0 \setminus \Omega} p \operatorname{div} z\, dx + \int_{\partial\Omega_{\text{form}}} a v_n z_n\, dS = \int_{\Omega^0} f_i z_i\, dx$$

holds for every $z \in [H^1(\Omega)]^3$ such that $z_n = 0$ on $\partial\Omega^0_{\text{form}} \setminus \partial\Omega_{\text{form}}$ in the sense of traces. All the apriori estimates as well as the convergence results would have to be modified appropriately. Also the proof of convergence would have to be modified, particularly by using a construction of the displacement $v \in [\mathscr{H}^1(\Omega)]^3$ by successive extension from Ω_η to $\Omega_{\eta/2}$ and to $\Omega_{\eta/3}$ etc., and afterwards by choosing a subsequence by a diagonalization procedure. Unfortunately, in this case no uniqueness result are known, hence we would obtain the convergence only in terms of subsequences.

Remark 2. (*Numerical treatment of the fixed-domain problems* (2.1)–(2.6)). We will briefly mention how the weak solutions from Def. 2 can be obtained numerically. It is clear that (2.1)–(2.6) represents a problem of nonlinear thermoelasticity with temperature-dependent coefficients on the fixed domain Ω^0. Let us first employ a linearization by the so-called secant modulus method, i.e. for $\varepsilon > 0$ fixed and some $v^{\varepsilon 0} \in [H^1(\Omega^0)]^3$ we define the approximations $v^{\varepsilon k} \in [H^1(\Omega)]^3$ recursively for $k = 1, 2, \ldots$ by requiring the following integral identity to be valid for all $z \in [H^1(\Omega^0)]^3$:

$$\int_{\Omega^0} T^\varepsilon_{ij}(v^{\varepsilon k}; v^{\varepsilon, k-1}, \theta)\, e_{ij}(z)\, dx + \int_{\Omega^0_{\text{form}}} a v^{\varepsilon k}_n z_n\, dS = \int_{\Omega^0} f_i z_i\, dx\,,$$

where

$$T^\varepsilon_{ij}(v; \bar{v}, \theta) = \left(k^\varepsilon(\theta) - \tfrac{2}{3}\mu^\varepsilon(\Gamma(\bar{v}), \theta)\right) \operatorname{div} v\, \delta_{ij} + 2\mu^\varepsilon(\Gamma(\bar{v}), \theta)\, e_{ij}(v) - 3k^\varepsilon(\theta)\, \bar{\alpha}(\theta)\, \delta_{ij}\,,$$

with k^ε, μ^ε and $\bar{\alpha}$ from (2.4)–(2.6). Clearly we have replaced the nonlinear problem on Ω^0 by a sequence of linear problems on Ω^0. It is known (see [1; Thm. 11.5.1]) that the above introduced conditions together with $(\partial\mu/\partial\Gamma)(\Gamma, \theta) \leqq 0$ for all $\theta < \theta_{\text{SL}}$ guarantee the convergence $v^{\varepsilon k} \to v^\varepsilon$ for $k \to \infty$ strongly in $[H^1(\Omega^0)]^3$. Finally, we can employ a finite element approximation by using piecewise linear elements and numerical integration in the boundary terms and in the terms containing temperature θ (we can simply replace θ by a piecewise constant approximation θ^h such that $\theta^h \to \theta$ strongly in $L^2(\Omega^0)$ for $h \to 0$). The approximate solutions $v^{\varepsilon kh}$ thus obtained can be actually computed by means of any standard package for the Lamé system. The convergence $v^{\varepsilon kh} \to v^{\varepsilon k}$ in $[H^1(\Omega)]^3$ for the mesh parameter h tending to zero can be proved under some usual additional assumptions in a quite standard manner (using also induction according to k) but the fact that we cannot use continuity of the Nemytskii operators in question because the functions μ^ε and possibly also

k^ε and $\bar{\alpha}$ have jumps at the phase transition temperature θ_{SL}. Nevertheless, this continuity is valid at any $\theta \in L^2(\Omega^0)$ such that meas $\{x \in \Omega^0; \theta(x) = \theta_{SL}\} = 0$, which means that we must suppose, in addition, that there is no "mushy region" (i.e. a region of positive measure where the solid and the liquid phases co-exist).

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Souhrn

APROXIMACE NELINEÁRNÍ TEPELNĚ-ELASTICKÉ ÚLOHY S POHYBLIVOU HRANICÍ POMOCÍ METODY PEVNÉ OBLASTI

JINDŘICH NEČAS, TOMÁŠ ROUBÍČEK

Jsou zkoumána tepelně elastická napětí vznikající v oblasti tuhé fáze při tuhnutí roztaveného ingotu. Připouští se též nelineární chování tuhé fáze. Taková úloha, jež vznikne po mnoha zjednodušeních reálné situace, obsahuje pohyblivou hranici mezi tuhou a kapalnou fází. Aby bylo možno použít běžného programového vybavení, navrhuje se zde aproximace metodou pevné oblasti zahrnutím oblasti kapalné fáze do úlohy (pevná oblast pak znamená celý ingot) a změnou kapalné fáze na tuhou, avšak s malým modulem pružnosti ve smyku. Je dokázána slabá L^2-konvergence takto aproximovaných napětí v oblasti tuhé fáze. Navíc se ukazuje, že tato konvergence je silná na množinách, jejichž uzávěr leží v oblasti tuhé fáze.

Резюме

ПРИБЛИЖЕНИЕ НЕЛИНЕЙНОЙ ТЕРМОУПРУГОЙ ПРОБЛЕМЫ С ПОДВИЖНОЙ ГРАНИЦЕЙ МЕТОДОМ ПОСТОЯННОЙ ОБЛАСТИ

JINDŘICH NEČAS, TOMÁŠ ROUBÍČEK

Изучаются термо-упругие напряжения возникающие в области твердой фазы во время кристаллизации расплавленного слитка. Нелинейное поведение твердой фазы также допускается. Эта проблема, полученая многими упрощениями реальной ситуации, имеет подвижную границу между областями твердой и жидкой фазы. Для возможности использования стандартных вычислительных пакетов предлагается приближение методом постоянной области так, что жидкая фаза заменяется твердой, но обладающей малым модулем поперечной упругости. Доказывается L^2-сходимость приближенных напряжений кроме того, эта сходимость сильна на подмножествах, имеющих замыкание в области твердой фазы.

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