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SOME FUNCTIONS OF EIGENVALUES OF NORMAL OPERATOR

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Summary. Kellogg's iterations in the eigenvalue problem are discussed with respect to the boundary spectrum of a linear normal operator.

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The Rayleigh-quotient iteration is often used for finding an approximation of the eigenvalue in the eigenvalue problem. The problem of approximative construction of the eigenvalues does not seem to be satisfactorily solved yet, particularly in the case of complex eigenvalues.

The purpose of this paper is to show how the knowledge of the Rayleigh-quotient iterations can be used for the construction of the complex eigenvalues. We will search for the eigenvalues of the equation

$$(1) \qquad Tx = \lambda x$$

in a complex Hilbert space X for a normal operator under certain conditions on its boundary spectrum. Some terms from the theory of spectral representation are used [1].

Let us denote by $(\cdot, \cdot)$ the scalar product in X, let the norm be defined as usual, $\|\cdot\| = (\cdot, \cdot)^{1/2}$. Let $[X]$ be the space of linear bounded operators on X, $\|T\|_X = \sup_{\|x\|_X = 1} \|Tx\|$. If there is no danger of misunderstanding the indices will be omitted. Similarly, the braces $\{\cdot\}$ denote sequences as well as sets. Let C be the open complex plane. We denote the spectrum of T by $\sigma(T)$, the spectral radius by $r(T)$ and the spectral radius circle of T by $\omega(T)$, i.e. $\omega(T) = \{\lambda \in C: |\lambda| = r(T)\}$. Given $T \in [X]$ we define its adjoint T^* for which $(Tx, y) = (x, T^*y)$ holds for every $x, y \in X$. We say that T is self-adjoint if $T = T^*$ and that T is normal if $TT^* = T^*T$. Iterations are constructed in the following way:

$$(2) \qquad x^{(n+1)} = Tx^{(n)},$$

$$\mu_n = \frac{(x^{(n+1)}, x^{(n+1)})}{(x^{(n)}, x^{(n)})}, \tag{3}$$

$$\nu_n = \frac{(x^{(n+1)}, x^{(n)})}{(x^{(n)}, x^{(n)})}, \tag{4}$$

where $x^{(0)} \in X$.

First, the convergence of the sequence of (3) is studied.

Theorem 1. *Let $T \in [X]$ be a normal operator and let $x^{(0)} \in X$ be such that $Tx^{(0)} \neq o$. Then the sequence $\{\mu_n\}$ generated by $x^{(0)}$ converges. Furthermore, if $x^{(0)}$ is such that $E_\lambda x^{(0)} \neq x^{(0)}$ for every $|\lambda| < r(T)$, then $\{\mu_n\}$ converges to the square of the spectral radius of T. In both cases the convergence is monotonous.*

The proof is quite similar to that in [3] and therefore is omitted. Nearly the same results were obtained by Kolomý for linear non-negative self-adjoint operators [4].

The following theorem explains how to understand the behaviour of the sequence $\{\nu_n\}$ from (4).

Theorem 2. *Let $T \in [X]$ be a normal operator and let $x^{(0)} \in X$ be such that $(E(\omega(T))\, x^{(0)}, x^{(0)}) \neq 0$, where E is the spectral measure of T (see [1]). Then the sequence $\{\nu_n\}$ converges.*

Proof. As $(x^{(n+1)}, x^{(n)}) = (T(T^*T)^n x^{(0)}, x^{(0)})$ due to (2), it is possible to represent this expression with the help of the Gelfand-Najmark theorem [1] in the form

$$(x^{(n+1)}, x^{(n)}) = \int_S \lambda |\lambda|^{2n} (E(\mathrm{d}\lambda)\, x^{(0)}, x^{(0)}) \tag{5}$$

where $S \subset C$ and $\sigma(T) \subset S$. Dividing both the numerator and the denominator of the ratio in (4) by $r^{2n}(T)$, we have

$$\nu_n = \frac{\int_S \lambda |\lambda/r(T)|^{2n} (E(\mathrm{d}\lambda)\, x^{(0)}, x^{(0)})}{\int_S |\lambda/r(T)|^{2n} (E(\mathrm{d}\lambda)\, x^{(0)}, x^{(0)})}. \tag{6}$$

As the both functions in (6) are bounded, the Lebesgue's dominated convergence theorem will be used [1]. After a short simple computation we obtain

$$\lim_{n\to\infty} \nu_n = \frac{\int_{\omega(T)} \lambda (E(\mathrm{d}\lambda)\, x^{(0)}, x^{(0)})}{(E(\omega(T))\, x^{(0)}, x^{(0)})}, \tag{7}$$

which complete the proof as $(E(\omega(T))\, x^{(0)}, x^{(0)}) \neq 0$ under the assumptions of the theorem.

This result is not positive at all because the eigenvalue of T need not be obtained as the result of this process. In which cases is the result positive at all?The answer is partially given by the assumptions of the former results obtained in [2], [3] and [4]. In these papers it is assumed that the operator T has at least the following property:

only one point of the spectrum of T lies on its spectral radius circle (or its „weight" is in some sense greater than the „weight" of some other points of the spectral radius circle). This description covers for example the case of self-adjoint non-negative operators.

Now we ask for some more information that we can obtain from the limit point of $\{v_n\}$ provided more assumptions are imposed on the spectrum of T.

Theorem 3. *Let $T \in [X]$ be a normal operator and let u_1, u_2 be its complex points of $\sigma(T)$ on $\omega(T)$ (not necessarily isolated). Further, let $x^{(0)} \in X$ be such that $(E(\omega(T))\, x^{(0)}, x^{(0)}) = (E(\{u_1, u_2\})\, x^{(0)}, x^{(0)}) \neq 0$. Then the limit point of $\{v_n\}$ lies on the line segment connecting u_1, u_2. In particular, if u_1 and u_2 are complex conjugate then the real part of $\{v_n\}$ converges to the real part of both u_1, u_2.*

Proof. The sequence $\{v_n\}$ converges owing to Theorem 2. The rest of the proof proceeds as follows. By the assumption $(E(\omega(T))\, x^{(0)}, x^{(0)}) = (E(\{u_1, u_2\})\, x^{(0)}, x^{(0)}) \neq 0$ we have $(E(\{u_1\})\, x^{(0)}, x^{(0)}) \neq 0$ or $(E(\{u_2\})\, x^{(0)}, x^{(0)}) \neq 0$. Let us assume that $(E(\{u_1\})\, x^{(0)}, x^{(0)}) \geqq (E(\{u_2\})\, x^{(0)}, x^{(0)})$ (without any loss of generality) and denote

$$k = \frac{(E(\{u_2\})\, x^{(0)}, x^{(0)})}{(E(\{u_1\})\, x^{(0)}, x^{(0)})}.$$

It is evident that $0 \leqq k \leqq 1$. For the sequence $\{v_n\}$ we obtain, similarly as in the proof of Theorem 2, that

$$\lim_{n\to\infty} v_n = \frac{\int_{\omega(T)} \lambda (E(d\lambda)\, x^{(0)}, x^{(0)})}{(E(\omega(T))\, x^{(0)}, x^{(0)})} = \frac{u_1 + k u_2}{1 + k} \tag{8}$$

holds owing to (7). Denoting $u_1 = \operatorname{Re} u + i \operatorname{Im} u$, $u_2 = \operatorname{Re} u - i \operatorname{Im} u$, we have from (8) that

$$\lim_{n\to\infty} v_n = \operatorname{Re} u + i \frac{k - 1}{k + 1} \operatorname{Im} u,$$

and the theorem is proved.

Remark. If u_1 is an eigenvalue (not necessarily an isolated point of $\sigma(T)$), then the convergence of the real part of $\{v_n\}$ does not depend on the type of u_2 due to its properties in $\sigma(T)$.

Comparing Theorem 1 and Theorem 3 we can seek for the values u_1 and u_2 provided u_1 and u_2 are complex conjugate.

Proposition 1. *Under the assumptions of Theorems 1 and 3 the values u_1 and u_2 satisfy*

$$u_1 = \bar{u}_2 = \lim_{n\to\infty} \sqrt{(\mu_n)} \exp\left(i \arccos \frac{\operatorname{Re} v_n}{\sqrt{\mu_n}}\right). \tag{9}$$

The proof of this assertion is obvious. For easier understanding of this fact we remark that if $u_1 = \bar{u}_2 = ue^{i\varphi}$ holds, then φ can be obtained from the expression

$$\cos \varphi = \lim_{n\to\infty} \frac{\operatorname{Re} v_n}{\sqrt{\mu_n}} . \tag{10}$$

This proposition gives us the tool for computation of complex conjugate points of $\sigma(T)$.

A similar method can by used also in the case, when u_1 and u_2 are not complex conjugate.

Propostiion 2. *Let $T \in [X]$ be a normal operator and let u_1 and u_2 be the only points of $\sigma(T)$ lying on $\omega(T)$. Further, let $x^{(0)} \in X$ and $\tilde{x}^{(0)} \in X$ be such that*

$$(E(\{u_1\})\, x^{(0)}, x^{(0)})\,(E(\{u_2\})\, \tilde{x}^{(0)}, \tilde{x}^{(0)}) \neq$$
$$\neq (E(\{u_1\})\, \tilde{x}^{(0)}, \tilde{x}^{(0)})\,(E(\{u_2\})\, x^{(0)}, x^{(0)})$$

and

$$(E(\omega(T))\, x^{(0)}, x^{(0)}) \neq 0 , \quad (E(\omega(T))\, \tilde{x}^{(0)}, \tilde{x}^{(0)}) \neq 0 .$$

Then the pair u_1, u_2 is the solution of the equation

$$(\operatorname{Re} v_x - \operatorname{Re} v_{\tilde{x}})(\operatorname{Im} u_1 - \operatorname{Im} u_2) = (\operatorname{Re} u_1 - \operatorname{Re} u_2)(\operatorname{Im} v_x - \operatorname{Im} v_{\tilde{x}})$$

under the condition

$$|u_1| = |u_2| = r(T) ,$$

where $v_x, v_{\tilde{x}}$ are the limit points of the sequences $\{v_n\}$ with the starting approximations $x^{(0)}, \tilde{x}^{(0)}$, respectively.

The proof requires only a short straightforward computation, se we omit it.

References

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Souhrn

NĚKTERÉ FUNKCE VLASTNÍCH ČÍSEL NORMÁLNHO OPERÁTORU

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Popisuje se chování Kelloggových iterací při řešení rovnice $Tx = \lambda x$ v Hilbertově prostoru pro normální operátor. Je studována situace, kdy na spektrální kružnici $\omega(T)$ operátoru T ($\omega(T) = \{\lambda \in C, \lambda = r(T)\}$) leží dva komplexní body spektra. K jejich určení se užijí posloupnosti (3) a (4) — jisté Rayleighovy podíly.

Резюме

НЕКОТОРЫЕ ФУНКЦИИ СОБСТВЕННЫХ ЗНАЧЕНИЙ НОРМАЛЬНОГО ОПЕРАТОРА

Tomáš Kojecký

Описывается отношение итераций Келлогга для решения уравнения $Tx = \lambda x$ с пространстве Банаха в случае; когда оператор T является нормальным. Рассматривается ситуациа; когда на спектральной окружности $\omega(T)$ оператора T находятся две комплехные точки его спектра. Для их нахождения берутся (3) и (4)-некоторые частные Ралея.

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