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Valter Šeda; Ján Pekár

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ON SOME PROPERTIES OF THE SOLUTION OF THE DIFFERENTIAL EQUATION $u'' + \frac{2u'}{r} = u - u^3$

VALTER ŠEDA, JÁN PEKÁR

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Summary. In the paper it is shown that each solution $u(r, \alpha)$ of the initial value problem (2), (3) has a finite limit for $r \to \infty$, and an asymptotic formula for the nontrivial solution $u(r, \alpha)$ tending to 0 is given. Further, the existence of such a solution is established by examining the number of zeros of two different solutions $u(r, \bar{\alpha})$, $u(r, \hat{\alpha})$.

Keywords: Spherically symmetric solution, trajectory of the solution, ω — limit point of the trajectory, asymptotic formula, antitone and contractive operator, zero of the solution.

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1. INTRODUCTION

If u is a spherically symmetric solution of the equation

(1) $$\Delta u = u - u^3$$

in R^3 which can be obtained from the Klein-Gordon equation in nuclear physics, then $u = u(r)$ satisfies the initial value problem (for short, IVP)

(2) $$u'' + \frac{2u'}{r} = u - u^3 , \quad 0 \leqq r < \infty$$

(3) $$u(0) = \alpha , \quad u'(0) = 0$$

where r denotes the distance from the origin, $\alpha \in R$ and the condition $u'(0) = 0$ arises due to the regularity of the solution to (1) at 0.

By the transformation

(4) $$y = ru$$

the IVP (2), (3) changes into the problem

(5) $$y'' = y - \frac{y^3}{r^2},$$

(6) $$y(0) = 0\,, \quad y'(0) = \alpha\,.$$

Under a solution of (2) in $[0, h)$ $(0 < h \leqq \infty)$ each function $u \in C^1([0, h)) \cap \cap C^2((0, h))$ which satisfies (2) in $(0, h)$ will be understood. Similarly the solution of (5) can be defined. Since for the solution y of (5), (6) we have $y(r) = y'(\varrho)\, r$ with $\varrho = \varrho(r)$, $0 < \varrho < r$, by substituting y into (5) we obtain that $\lim_{r \to 0+} y''(r) = 0$ and hence each solution y of the IVP (5), (6) in $[0, h)$ belongs to the class $C^2([0, h))$ and both IVP-s (2), (3) and (5), (6) are equivalent to each other.

Local existence and uniqueness theorems have been proved by G. Sansone in [5], pp. 7–13. They are given here as

Lemma 1. *For the IVP*

(7) $$y'' = y - \frac{y^k}{r^{k-1}}, \quad y(0) = 0\,, \quad y'(0) = \alpha$$

where $k > 1$ the following statements hold:

(i) *Uniqueness statement: For each $\alpha \in R$ there exists at most one solution of* (7) *in any interval $[0, h]$, $h > 0$.*

(ii) *Existence statement: If $0 < \alpha < 1$, or $\alpha > 1$, then there exists a solution y of* (7) *in an interval $[0, h]$.*

As to the remaining cases, the functions $y \equiv 0$ and $y(r) \equiv r$ are solutions of (7) for $\alpha = 0$ and $\alpha = 1$, respectively, while for $\alpha < 0$ the existence of a solution to (7) follows from the following statement:

If k is odd (and this is the case of (5)), and if y is a solution of (7) in $[0, h)$, then so is $-y$.

By this lemma as well as by the above mentioned equivalence of the IVP-s (2), (3) and (5), (6) we conclude that the problem (2), (3) has a unique solution for each $\alpha \in R$ in an interval $[0, h)$, $0 < h \leqq \infty$.

2. GLOBAL PROPERTIES OF SOLUTIONS OF EQUATION (2)

Consider a function $E \in C(R^2, R)$ which is defined by the relation

(8) $$E(x, y) = \tfrac{1}{2}y^2 + \tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\,, \quad (x, y) \in R^2\,.$$

It is easy to show the following properties of this function:

(a) $E(-1, 0) = E(1, 0) = -\frac{1}{4} < E(x, y)$ for each $(x, y) \in R^2$, $(x, y) \neq (-1, 0)$, $(x, y) \neq (1, 0)$.

(b) $E(R^2) = [-\frac{1}{4}, \infty)$.

(c) For each $c \in [-\frac{1}{4}, \infty)$ the set

(9) $$E(x, y) = c$$

is symmetric with respect to the x-axis, to the y-axis and with respect to the origin.

(d) If $-\frac{1}{4} < c < 0$, then the set (9) is a pair of closed curves the parts of which over the x-axis are determined by the function

$$y(x) = \sqrt{(2c - \tfrac{1}{2}x^4 + x^2)}\,, \quad 0 < x_1 < x < x_2\,, \quad -x_2 < x < -x_1$$

where $x_1 < x_2$ are positive roots of the equation

$$\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2 = c\,.$$

(e) If $c = 0$, then the set (9) is a closed curve passing through the origin the part of which in the first quadrant is described by the function

$$y(x) = x\sqrt{(1 - \tfrac{1}{2}x^2)}\,, \quad 0 \leqq x \leqq \sqrt{2}\,.$$

(f) If $c > 0$, then the set (9) is a closed curve the part of which in the first quadrant is described by the function

$$y(x) = \sqrt{(2c - \tfrac{1}{2}x^4 + x^2)}\,, \quad 0 \leqq x \leqq x_1\,,$$

where x_1 is a positive root of the equation

$$\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2 = c\,.$$

Using the properties of the function E we will prove

Theorem 1. *If u is a solution of the equation (2) on an interval (a, b) where $0 \leqq a < b < \infty$, then this solution can be extended in a unique way to the interval (a, ∞). Each solution of (2) in (a, ∞) is bounded together with its first and second derivative in a neighbourhood of ∞.*

Proof. The uniqueness of the extension follows from the fact that the function $f(r, u, v) = -2v/r + u - u^3$ is continuous and locally Lipschitz continuous in the variables u, v on the set $(0, \infty) \times R^2$.

Let u be a solution of (2) in (a, b). Multiplying the identity (2) in (a, b) by the function u' we get the identity

$$u''(r)\,u'(r) + u^3(r)\,u'(r) - u(r)\,u'(r) = -\frac{2u'^2(r)}{r}$$

and hence

(10) $$\frac{\mathrm{d}}{\mathrm{d}r} E(u(r), u'(r)) = -\frac{2\,u'^2(r)}{r}\,, \quad a < r < b\,.$$

Thus the composite function $E(u(\cdot), u'(\cdot))$ is nonincreasing and hence for $a < r_0 \leqq \leqq r < b$ we have $E(u(r), u'(r)) \leqq E(u(r_0), u'(r_0))$. As the set of all $(x, y) \in R^2$ for which $E(x, y) \leqq E(u(r_0), u'(r_0))$ is compact by the above analysis of the function E, both functions u, u' are bounded in $[r_0, b)$. This implies that the solution u can be extended to the interval (a, ∞). The extension will also be bounded together with its first derivative on each interval $[r_0, \infty)$ where $a < r_0$. On the basis of (2) the second derivative of this solution will be bounded in the same interval, too.

Corollary 1. *The problem* (2), (3) *has a unique solution in* $[0, \infty)$ *for each* $\alpha \in R$. *Any other IVP for the equation* (2) *at* r_0, $0 < r_0 < \infty$, *has a unique solution extending to an interval* (a, ∞) *where* $0 \leqq a < \infty$.

J. Chauvette and F. Stenger in [3], pp. 229–230, assert that numerical experiments indicate that each solution of (2) approaches either 1 or -1 or 0 as $r \to \infty$ whereby only countably many solutions tend to 0. This statement is certainly true for constant solutions $u_1(r) \equiv 0$, $u_2(r) \equiv 1$ and $u_3(r) \equiv -1$ of (2) in $[0, \infty)$.

Let u be a nonconstant solution of (2) in (a, ∞) where $0 \leqq a < \infty$. In the proof of Theorem 1 it was shown that the composite function $E(u(\cdot), u'(\cdot))$ is nonincreasing (in fact it is decreasing since the zeros of u' are isolated) and hence there exists

(11) $$\lim_{r \to \infty} E(u(r), u'(r)) = c_1 .$$

The property (b) of the function E implies $c_1 \in [-\frac{1}{4}, \infty)$.

Lemma 2. *Let u be a solution of the equation* (2) *in the interval* (a, ∞). *If* $c_1 = -\frac{1}{4}$, *then either* $\lim_{r \to \infty} (u(r), u'(r)) = (1, 0)$ *or* $\lim_{r \to \infty} (u(r), u'(r)) = (-1, 0)$.

Proof. If $c_1 = -\frac{1}{4}$, then for each c_2, $-\frac{1}{4} < c_2 < 0$ there exists an r_0, $a < r_0 < \infty$, such that for all $r > r_0$ the trajectory $(u(r), u'(r))$ of the solution u lies in the interior of exactly one of two closed curves $E(x, y) = c_2$. Suppose that it lies in the interior of the right curve φ_+. Then for an arbitrary neighbourhood U of the point $(1, 0)$ there exists a c_2^*, $-\frac{1}{4} < c_2^* < 0$ such that the right curve ψ_+ $E(x, y) = c_2^*$ together with its interior lies in U, and for c_2^* there exists an r_0^* such that for all $r > r_0^*$ the trajectory $(u(r), u'(r))$ of u lies in the interior of ψ_+ and hence in U. This means that

$$\lim_{r \to \infty} (u(r), u'(r)) = (1, 0) .$$

If the trajectory $(u(r), u'(r))$ of u for $r > r_0$ lies in the interior of the left curve φ_- of the system $E(x, y) = c_2$, then we get

$$\lim_{r \to \infty} (u(r), u'(r)) = (-1, 0)$$

and the lemma is proved.

Let $c_1 > -\frac{1}{4}$ and let r_0 be such that $a < r_0 < \infty$. Then the trajectory $(u(r), u'(r))$, $r_0 \leqq r < \infty$, of the solution u starts on the curve

$$E(x, y) = E(u(r_0), u'(r_0)) \tag{12}$$

(or on one of the curves if $E(u(r_0), u'(r_0)) < 0$) and the whole trajectory lies in the interior of that curve up to its initial point, intersects all curves (9) for $c_1 < c < E(u(r_0), u'(r_0))$ (whereby for $c < 0$ we consider only the curves (9) which all lie either in the right half-plane or in the left half-plane), in the upper half-plane it moves from the left to the right while in the lower half-plane it moves from the right to the left, and it lies in the exterior of the curve

$$E(x, y) = c_1 . \tag{13}$$

In the sequel we will need the following definitions.

Definition 1. *We shall say that the trajectory $(u(r), u'(r))$ of the solution u winds up infinitely many times around the curve* (13) (*if* $-\frac{1}{4} < c_1 < 0$, *then around one of the two curves* (13) *which we denote by* φ) *if for each half-line p starting at the origin for* $c_1 \geqq 0$ *while in the case* $-\frac{1}{4} < c_1 < 0$ *at exactly that point from the two-point set* $\{(1, 0), (-1, 0)\}$ *which lies in the interior of* φ, *there exists an increasing sequence* $\{r_n\}_{n=1}^{\infty}$, $r_0 < r_1 < \ldots < r_n < \ldots$ *tending to* ∞ *as* $n \to \infty$ *and the corresponding points of the trajectory* $(u(r_n), u'(r_n))$ *belong to p.*

In the opposite case, i.e. if there exists a half-line p_1 starting either from the point $(0, 0)$, or from the point $(1, 0)$, or from the point $(-1, 0)$ according to whether $c_1 \geqq 0$, or $-\frac{1}{4} < c_1 < 0$ and the curve φ lies in the right half-plane, or $-\frac{1}{4} < c_1 < 0$ and the curve φ lies in the left half-plane, and there exists a point $r_1 > r_0$ such that for all $r \geqq r_1$ we have $(u(r), u'(r)) \notin p_1$, then we say that *the trajectory* $(u(r), u'(r))$ *of the solution u winds up finitely many times around the curve* (13) (*around the curve* φ *if* $-\frac{1}{4} < c_1 < 0$).

In accordance with the definition of ω-limit points and ω-limit sets of a trajectory for the autonomous system we introduce the following definition.

Definition 2. *A point* $(b, c) \in R^2$ *is called an* ω*-limit point of the trajectory* $(u(r), u'(r))$ *of a solution u,* $r_0 \leqq r < \infty$, *if there exists an increasing sequence* $\{r_n\}_{n=1}^{\infty}$, $r_0 < r_1 < \ldots < r_n < \ldots$ *such that* $r_n \to \infty$ *as* $n \to \infty$ *and* $\lim_{n \to \infty} (u(r_n), u'(r_n)) = (b, c)$. *The set* Ω *of all* ω*-limit points of the trajectory of the solution u is called the* ω*-limit set of the trajectory of this solution.*

As the trajectory $(u(r), u'(r))$, $r_0 \leqq r < \infty$, of the solution u lies in a compact set the boundary of which is the curve (12) or one of the curves (12), the ω-limit set Ω of this trajectory is non-void. With respect to (11) and to the continuity of E, Ω is a subset of the curve (13) or one of the curves (13) if $-\frac{1}{4} < c_1 < 0$ which as above we denote by φ.

Directly from Definition 1 it follows that if the trajectory of the solution u winds up infinitely many times around the curve (13) (or around φ if $-\frac{1}{4} < c_1 < 0$), then the ω-limit set Ω of this trajectory coincides with the curve (13) or with the curve φ. In fact, for each point (b, c) of the curve (13) there exists a half-line p which starts at the point determined by Definition 1 and passes through (b, c), and an increasing sequence $\{r_n\}_{n=1}^{\infty}$ such that $r_0 < r_1 < \ldots < r_n < \ldots, \lim_{n\to\infty} r_n = \infty$ and $(u(r_n), u'(r_n)) \in$ $\in p$. The points $(u(r_n), u'(r_n))$ lie in a compact subset bounded by (12), hence there exists a subsequence $(u(r_{n_k}), u'(r_{n_k}))$ which converges to a point $(k, l) \in p$. At the same time (11) implies that $E(k, l) = c_1$, hence (k, l) belongs to the curve (13), thus $(k, l) = (b, c)$ and $(b, c) \in \Omega$.

Lemma 3. *If $-\frac{1}{4} < c_1$ and the trajectory of a solution u winds up finitely many times around the curve* (13) (*or the curve φ if $-\frac{1}{4} < c_1 < 0$*) *then there exists*

$$\lim_{r\to\infty} (u(r), u'(r)) = (0, 0) .$$

Proof. Let the half-line p_1 and the number $r_1 > r_0$ be such as in Definition 1, that is for all $r \geqq r_1$ let $(u(r), u'(r)) \notin p_1$. Clearly the limit set Ω_1 of the trajectory of the solution u in $[r_1, \infty)$ is the same as the limit set Ω of the trajectory of u in $[r_0, \infty)$.

We shall prove that Ω_1 is a one-point set. Since the trajectory of the solution u lies in a compact set, this means that there exists

$$\lim_{r\to\infty} (u(r), u'(r)) = (u_0, u'_0)$$

and at the same time (u_0, u'_0) belongs to the curve (13). Then $u'_0 = 0$, since otherwise u_0 would not be finite. Simultaneously, $E(u_0, u'_0) = c_1 > -\frac{1}{4}$, hence

$$\frac{1}{4}u_0^4 - \frac{1}{2}u_0^2 = c_1 \tag{14}$$

and further, on the basis of (2), there exists $\lim_{r\to\infty} u''(r) = u_0 - u_0^3$. This limit should be 0, otherwise u'_0 would not be finite. Therefore u_0 is equal either to 0, or to 1, or to -1. After substituting into (14) by virtue of $c_1 > -\frac{1}{4}$ we get that $u_0 = 0$.

Let P_1 be the intersection point of the half-line p_1 with the curve (13) (or with that curve from the pair given by (13) in the neighbourhood of which the trajectory of the solution u lies for all sufficiently great r). If $c_1 = 0$, we consider the intersection point $P_1 \neq 0$ if such a point exists. If Ω_1 has at least two different points P_2, P_3 which lie on the curve (13), then observing the rule that the movement along this curve in the upper half-plane goes from the left to the right and in the lower plane in the opposite direction we conclude that the movement from P_2 to P_3 or from P_3 to P_2 passes through P_1. Consider the case that the movement from P_2 to P_3 passes through P_1. Then the trajectory of the solution u for all sufficiently great r lies in the neighbourhood of the curve (13) and the movement from the neighbourhood of P_2 into the neighbourhood of P_3 intersects the half-line p_1. This contradicts the assumption of the lemma and hence the proof of the lemma is complete.

Now we prove

Theorem 2. *Let* u *be a solution of the equation* (2) *in the interval* (a, ∞), $0 \leqq a < \infty$. *Then*

$$\lim_{r \to \infty} u'(r) = 0$$

and either

(a) $$\lim_{r \to \infty} u(r) = 0$$

or

(b) $$\lim_{r \to \infty} u(r) = 1$$

or

(c) $$\lim_{r \to \infty} u(r) = -1 .$$

Proof. By Lemma 2, if $c_1 = -\frac{1}{4}$, then the case (b) or the case (c) occurs. Lemma 3 implies that for $c_1 > -\frac{1}{4}$ the case (a) occurs provided the trajectory of the solution u winds up finitely many times around the curve (13). Hence it is sufficient to show that this trajectory cannot wind up infinitely many times around the curve (13). Suppose the opposite and consider the case $c_1 \neq 0$, $c_1 > -\frac{1}{4}$.

Integrating (10) in the interval $[r_0, \infty)$ where $a < r_0 < \infty$ we get that

$$\int_{r_0}^{\infty} \frac{2\, u'^2(t)}{t} \, \mathrm{d}t < \infty . \tag{15}$$

Let $\varepsilon > 0$ be sufficiently small (so that the whole curve (13) does not lie in the strip $-\varepsilon \leqq y \leqq \varepsilon$). Let us introduce the system of intervals $\{(a_k, b_k)\}_{k=1}^{\infty}$ in $[r_0, \infty)$ in which

$$u'^2(r) > \varepsilon^2 . \tag{16}$$

Note that there exist $u_0 > 0$, $M > 0$ such that on each interval (a_k, b_k) we have

$$|u'(\xi_k)| = |u'(r)|_{\max} \geqq u_0 ,$$

and simultaneously in $[r_0, \infty)$ we have $|u''(r)| \leqq M$.

By these inequalities we get

$$u_0 - \varepsilon \leqq |u'(\xi_k) - u'(a_k)| \leqq M(\xi_k - a_k)$$

and

$$u_0 - \varepsilon \leqq |u'(b_k) - u'(\xi_k)| \leqq M(b_k - \xi_k) .$$

Hence

$$b_k - a_k \geqq 2 \frac{u_0 - \varepsilon}{M} = K , \quad k = 1, 2, \ldots . \tag{17}$$

On the other hand, for $r \in (b_k, a_{k+1})$ we have

$$|u''(r)| \geqq |u(r) - u^3(r)| - \left|\frac{2\,u'(r)}{r}\right| \geqq \varepsilon_1 > 0 \tag{18}$$

and thus

$$|u'(a_{k+1}) - u'(b_k)| = 2\varepsilon = |u''(\eta)|\,(a_{k+1} - b_k) \geqq \varepsilon_1(a_{k+1} - b_k)\,,$$

which implies that

$$a_{k+1} - b_k \leqq \frac{2\varepsilon}{\varepsilon_1} = L, \quad k = 1, 2, \ldots. \tag{19}$$

Let $k_0 > 0$ be such that

$$K \geqq k_0 L. \tag{20}$$

From (15), (16) it follows that

$$\infty > \int_{r_0}^{\infty} \frac{2\,u'^2(t)}{t}\,\mathrm{d}t \geqq 2\sum_{k=1}^{\infty} \int_{a_k}^{b_k} \frac{u'^2(t)}{t}\,\mathrm{d}t \geqq 2\sum_{k=1}^{\infty} \int_{a_k}^{b_k} \frac{\varepsilon^2}{t}\,\mathrm{d}t\,. \tag{21}$$

On the other hand, (17), (19), (20) yield

$$\sum_{k=1}^{\infty} \int_{a_k}^{b_k} \frac{\varepsilon^2}{t}\,\mathrm{d}t \geqq k_0 \sum_{k=1}^{\infty} \int_{b_k}^{a_{k+1}} \frac{\varepsilon^2}{t}\,\mathrm{d}t\,, \tag{22}$$

because

$$\int_{a_k}^{b_k} \frac{\varepsilon^2}{t}\,\mathrm{d}t \geqq k_0 \int_{b_k}^{a_{k+1}} \frac{\varepsilon^2}{t}\,\mathrm{d}t$$

is equivalent to the inequality

$$\frac{1}{\xi}(b_k - a_k) \geqq k_0(a_{k+1} - b_k)\frac{1}{\eta}, \quad k = 1, 2, \ldots$$

where $a_k < \xi < b_k < \eta < a_{k+1}$.

Then (22) implies that

$$\infty = \int_{a_1}^{\infty} \frac{\varepsilon^2}{t}\,\mathrm{d}t = \sum_{k=1}^{\infty} \left(\int_{a_k}^{b_k} \frac{\varepsilon^2}{t}\,\mathrm{d}t + \int_{b_k}^{a_{k+1}} \frac{\varepsilon^2}{t}\,\mathrm{d}t \right) \leqq \sum_{k=1}^{\infty} \left(1 + \frac{1}{k_0}\right) \int_{a_k}^{b_k} \frac{\varepsilon^2}{t}\,\mathrm{d}t\,.$$

Therefore $\sum_{k=1}^{\infty} \int_{a_k}^{b_k} (\varepsilon^2/t)\,\mathrm{d}t = \infty$, which contradicts (21).

It remains to investigate the case $c_1 = 0$. Then we can use (17), but (18) and (19) can be applied only to that part of the trajectory which does not lie in a neighbourhood of 0.

Again integrating (10) in (a, ∞) we get

$$E(u(r), u'(r)) = \int_r^\infty \frac{2\,u'^2(t)}{t}\,\mathrm{d}t$$

and hence

(23) $$u'^2(r) = u^2(r) - \frac{u^4(r)}{2} + \int_r^\infty \frac{4\,u'^2(t)}{t}\,\mathrm{d}t\,, \quad a < r < \infty\,.$$

If we put

$$f(r) = u^2(r) - \frac{u^4(r)}{2}\,,$$

$$v(r) = u'^2(r)\,, \quad a < r < \infty\,,$$

then we can write (23) in the form

$$v(r) = f(r) + \int_r^\infty \frac{4\,v(t)}{t}\,\mathrm{d}t$$

from where we get

$$v'(r) + \frac{4\,v(r)}{r} = f'(r)$$

and therefore

(24) $$v(r) = f(r) + \frac{1}{r^4}\left[c_1 - \textstyle\int_{r_0}^r 4t^3 f(t)\,\mathrm{d}t\right], \quad a < r < \infty\,,$$

where $r_0 \in (a, \infty)$ is a chosen point and $c_1 = [v(r_0) - f(r_0)]\,r_0^4$.

Comparing (24) with (23) we get that the function

$$g(r) = \frac{1}{r^4}\left[c_1 - \textstyle\int_{r_0}^r 4t^3 f(t)\,\mathrm{d}t\right], \quad a < r < \infty$$

is decreasing and $\lim_{r\to\infty} g(r) = 0$.

By means of this function we define a pair of functions

(25) $$y(r) = c_1 - \textstyle\int_{r_0}^r 4t^3 f(t)\,\mathrm{d}t\,,$$

$$x(r) = r^4\,, \quad a < r < \infty\,.$$

This pair defines a new function y_1 of the variable x by the relation

$$y_1(x) = y[r(x)]$$

where $r(x)$ is the inverse function of the function $x(r)$ given by (25). Hence y_1 is given parametrically by the pair (25). Then

$$y_1'(x) = \frac{y'(r)}{x'(r)} = -f(r)\,, \quad a < r < \infty$$

where $x = x(r)$. For any two points $x_2 > x_1 > a^4$ we get that

(26) $$y_1(x_2) - y_1(x_1) = \int_{x_1}^{x_2} y_1'(x)\,dx = -\int_{r_1}^{r_2} 4r^3 f(r)\,dr\,.$$

Since $g(r) = y_1(x)/x$ is decreasing to 0 for $x \to \infty$, we have that

(27) $$y_1(x) > 0 \quad \text{for all} \quad x > a^4\,.$$

By the properties of the function f it follows that in the interval (a, ∞) there exists a sequence of intervals (a_k, b_k), $k = 1, 2, \ldots$, in which $f(r) < 0$ while $f(r) \geqq 0$ in (b_k, a_{k+1}).

Let $\varepsilon > 0$ be arbitrary. Then there exists an $r_0 > a$ such that for all (a_k, b_k) in (r_0, ∞) we have

(28) $$|u'(r)| < \varepsilon \quad \text{and} \quad |f(r)| < \varepsilon\,.$$

Further, there exists a $c_k \in (a_k, b_k)$ such that

(29) $$b_k - a_k = \left|\frac{u'(b_k) - u'(a_k)}{u''(c_k)}\right| \leqq \frac{2\varepsilon}{\sqrt{2} - 2\varepsilon}$$

since

$$u''(c_k) = -\frac{2\,u'(c_k)}{c_k} + u(c_k) - u^3(c_k)$$

and

$$|u''(c_k)| \geqq |u(c_k) - u^3(c_k)| - 2\left|\frac{u'(c_k)}{c_k}\right| \geqq \sqrt{2} - 2\varepsilon\,.$$

In the interval (b_k, a_{k+1}) there exist two subintervals $[c_k, d_k]$, $[f_k, g_k]$, $b_k < c_k < d_k < f_k < g_k < a_{k+1}$, with the property that

$$|u(d_k) - u(c_k)| \geqq \sqrt{2} - 2\varepsilon \quad \text{and} \quad |u(g_k) - u(f_k)| \geqq \sqrt{2} - 2\varepsilon\,.$$

As $|u'(r)| \leqq 1/\sqrt{2} + \varepsilon$ for $r > r_0$, we have

(30) $$d_k - c_k = \frac{|u(d_k) - u(c_k)|}{|u'(\xi_k)|} = \frac{2 - 2\varepsilon}{\dfrac{1}{\sqrt{2}} + \varepsilon} \geqq 1$$

and similarly

(31) $$g_k - f_k \geqq 1\,.$$

At the same time

(32) $$f(r) \geqq \varepsilon \quad \text{in} \quad [c_k, d_k] \quad \text{as well as in} \quad [f_k, g_k]\,.$$

In view of (28), (29) and (30), (31), (32) we have

$$-\int_{a_k}^{b_k} 4r^3 f(r)\,dr - \int_{c_k}^{d_k} 4r^3 f(r)\,dr \leqq 0$$

and

$$-\int_{f_k}^{g_k} 4r^3 f(r)\,\mathrm{d}r \leqq -\varepsilon \int_{f_k}^{g_k} 4r^3\,\mathrm{d}r < -\varepsilon\,.$$

Hence

$$-\int_{a_k}^{b_k} 4r^3 f(r)\,\mathrm{d}r - \int_{b_k}^{a_{k+1}} 4r^3 f(r)\,\mathrm{d}r < -\varepsilon$$

and by (26) we conclude

$$\sum_{k=k_0}^{\infty} \left[-\int_{a_k}^{b_k} 4r^3 f(r)\,\mathrm{d}r - \int_{b_k}^{a_{k+1}} 4r^3 f(r)\,\mathrm{d}r\right] =$$

$$= \lim_{x\to\infty} y_1(x) - y_1(x_1) = -\infty\,,$$

which contradicts (27). This contradiction completes the proof of Theorem 2.

3. ASYMPTOTIC FORMULAE FOR THE SOLUTIONS OF (2) TENDING TO 0

First we derive the estimates for the solutions from below. To this aim we need the following lemma.

Lemma 4. *Let* $0 \leqq \delta < 1$. *Then the general solution* z *of the differential equation*

$$z'' + \frac{2z'}{r} - (1-\delta)^2 z = 0 \tag{33}$$

is the function

$$z(r) = c_1 \frac{1}{r} \mathrm{e}^{-(1-\delta)r} + c_2 \frac{1}{r} \mathrm{e}^{(1-\delta)r}\,, \quad r > 0$$

where c_1, c_2 *are arbitrary real constants, and hence its Cauchy function satisfies*

$$K_\delta(r,t) = \frac{1}{1-\delta}\,\frac{t}{r}\,\frac{\mathrm{e}^{(1-\delta)(r-t)} - \mathrm{e}^{-(1-\delta)(r-t)}}{2} > 0$$

for $0 < t < r$.

The proof can be done by direct calculation.

Suppose that u is a nontrivial solution of (2) on (a,∞) such that $\lim_{r\to\infty} u(r) = 0$. Then there is an $r_0 > a$ (without loss of generality we may assume that $r_0 > 1$) such that

$$|u(r)| < 1 \quad \text{for} \quad r_0 \leqq r < \infty\,. \tag{34}$$

It is easy to see from the equation (2) that $u'(r) = 0$ cannot occur in $[r_0, \infty)$ and thus we have two posibilities:

either

$$u(r) > 0\,, \quad u'(r) < 0\,, \quad u''(r) > 0 \quad \text{in} \quad [r_0, \infty) \tag{35}$$

or

$$u(r) < 0\,, \quad u'(r) > 0\,, \quad u''(r) < 0 \quad \text{in} \quad [r_0, \infty)\,.$$

Suppose that $u(r) > 0$ in $[r_0, \infty)$. The other case can be dealt with by considering the function $-u$ which is also a solution of (2).

By Lemma 4 u satisfies the equation

$$u(r) = A\frac{1}{r}\,e^{-r} + B\frac{1}{r}\,e^{r} - \int_{r_0}^{r} K_0(r, t)\,u^3(t)\,dt \tag{36}$$

where

$$u(r_0) = A\frac{1}{r_0}\,e^{-r_0} + B\frac{1}{r_0}\,e^{r_0} = c_0 > 0\,, \tag{37}$$

$$u'(r_0) = Ae^{-r_0}\left(\frac{-1}{r_0^2} - \frac{1}{r_0}\right) + Be^{r_0}\left(\frac{1}{r_0} - \frac{1}{r_0^2}\right) = c_1 < 0\,.$$

After simple calculations we get that

$$A = \tfrac{1}{2}e^{r_0}[c_0(r_0 - 1) - c_1 r_0]\,,$$

$$B = \tfrac{1}{2}e^{-r_0}[c_0(r_0 + 1) + c_1 r_0]\,.$$

Since $r_0 > 1$, $c_0 > 0$, $c_1 < 0$, we have that $A > 0$. The inequality $K_0(r, t) > 0$ and hence

$$0 < u(r) \leqq A\frac{1}{r}\,e^{-r} + B\frac{1}{r}\,e^{r}\,, \quad r \geqq r_0$$

yield that $B \geqq 0$. On the other hand, this implies that $c_0(r_0 + 1) + c_1 r_0 \geqq 0$ which, on the basis of (37), is equivalent to the inequality

$$\frac{u'(r_0)}{u(r_0)} + \frac{r_0 + 1}{r_0} \geqq 0\,.$$

But we can vary r_0 and thus we obtain the inequality

$$\frac{u'(r)}{u(r)} \geqq -1 - \frac{1}{r} \quad \text{for all} \quad r \geqq r_0\,. \tag{38}$$

By integrating (38) we prove the following lemma.

Lemma 5. *If u is a nontrivial solution of the equation (2) in (a, ∞) with the property* $\lim_{r\to\infty} u(r) = 0$, *then there is an* $r_0 > \max(1, a)$ *such that for all* $r \geqq r_0$

$$\operatorname{sgn} u(r) \neq \operatorname{sgn} u'(r) \neq \operatorname{sgn} u''(r)$$

and

$$u(r) \geqq u(r_0)\,r_0 e^{r_0}\frac{1}{r}\,e^{-r} \quad \textit{if} \quad u(r) > 0 \quad \textit{in} \quad [r_0, \infty)$$

or

$$u(r) \leqq u(r_0)\, r_0 e^{r_0} \frac{1}{r} e^{-r} \quad \textit{if} \quad u(r) < 0 \quad \textit{in} \quad [r_0, \infty)\,.$$

Let $0 < \delta < 1$. As the equation (2) can be written in the form

$$u'' + \frac{2u'}{r} - (1-\delta)^2 u = u - (1-\delta)^2 u - u^3\,,$$

u satisfies the equation

$$u(r) = A_1 \frac{1}{r} e^{-(1-\delta)r} + B_1 \frac{1}{r} e^{(1-\delta)r} +$$

$$+ \int_{r_1}^{r} K_\delta(r, t) \left[(1 - (1-\delta)^2)\, u(t) - u^3(t)\right] dt\,.$$

Let $|u(r)| < 1$ in $[r_0, \infty)$, $u(r) > 0$, $u'(r) < 0$ in the same interval. Then there exists an $r_1 > r_0$ such that

$$\left[1 - (1-\delta)^2\right] u(r) - u^3(r) > 0 \quad \text{for all} \quad r \geqq r_1\,.$$

This implies that

$$u(r) > A_1 \frac{1}{r} e^{-(1-\delta)r} + B_1 \frac{1}{r} e^{(1-\delta)r}\,, \quad r \geqq r_1\,. \tag{39}$$

Further,

$$u(r_1) = A_1 \frac{1}{r_1} e^{-(1-\delta)r_1} + B_1 \frac{1}{r_1} e^{(1-\delta)r_1} = d_0 > 0\,, \tag{40}$$

$$u'(r_1) = A_1 e^{-(1-\delta)r_1} \left(-\frac{1}{r_1^2} - \frac{1}{r_1}(1-\delta)\right) +$$

$$+ B_1 e^{(1-\delta)r_1} \left(-\frac{1}{r_1^2} + \frac{1}{r_1}(1-\delta)\right) = d_1 < 0\,.$$

After some calculations we get the relation

$$A_1 = \frac{1}{2(1-\delta)} \left[d_0(r_1(1-\delta) - 1) - d_1 r_1\right] e^{(1-\delta)r_1} > 0\,,$$

because $d_0 > 0$, $d_1 < 0$ and r_1 can be taken sufficiently great.

Further,

$$B_1 = \frac{1}{2(1-\delta)} \left[d_0(r_1(1-\delta) + 1) + d_1 r_1\right] e^{-(1-\delta)r_1}\,. \tag{41}$$

By (39) we obtain that

$$1 > u(r) > A_1 \frac{1}{r} e^{-(1-\delta)r} + B_1 \frac{1}{r} e^{(1-\delta)r}\,, \quad r \geqq r_1\,.$$

hence $B_1 \leqq 0$. On the basis of (40), (41), the last inequality is equivalent to the inequality

$$\frac{u'(r_1)}{u(r_1)} \leqq -\left(1 - \delta + \frac{1}{r_1}\right).$$

As we can vary r_1, we come to the inequality

$$\frac{u'(r)}{u(r)} \leqq -\left((1 - \delta) + \frac{1}{r}\right), \quad r \geqq r_1$$

and integrating we conclude that

$$u(r) \leqq u(r_1)\, r_1 e^{(1-\delta)r_1} \frac{1}{r} e^{-(1-\delta)r}, \quad r \geqq r_1 .$$

Hence the following lemma is true.

Lemma 6. *If u is a nontrivial solution of the equation (2) in (a, ∞) with the property $\lim_{r\to\infty} u(r) = 0$, then there exists an $r_1 \geqq r_0 > \max(1, a)$ where r_0 was determined in Lemma 5, such that*

(42) $$u(r) \leqq u(r_1)\, r_1 e^{(1-\delta)r_1} \frac{1}{r} e^{-(1-\delta)r}, \quad r \geqq r_1 \quad \text{if} \quad u(r) > 0 \quad \text{in} \quad [r_0, \infty),$$

and

$$u(r) \geqq u(r_1)\, r_1 e^{(1-\delta)r_1} \frac{1}{r} e^{-(1-\delta)r}, \quad r \geqq r_1 \quad \text{if} \quad u(r) < 0 \quad \text{in} \quad [r_0, \infty).$$

Remark. Lemma 5 and Lemma 6 describe the asymptotic behaviour of the solution u of (2) which tends to 0 as $r \to \infty$. The estimate (42) can be improved. To this aim we write (36) in the form

$$u(r) = \frac{1}{r} e^{-r} \left(A + \tfrac{1}{2} \int_{r_0}^{r} t e^{t} u^3(t)\, dt\right) + \frac{1}{r} e^{r}\left(B - \tfrac{1}{2} \int_{r_0}^{r} t e^{-t} u^3(t)\, dt\right).$$

Further,

$$B - \tfrac{1}{2} \int_{r_0}^{r} t e^{-t} u^3(t)\, dt = B - \tfrac{1}{2} \int_{r_0}^{\infty} t e^{-t} u^3(t)\, dt + {}$$
$$+ \tfrac{1}{2} \int_{r}^{\infty} t e^{-t} u^3(t)\, dt = B - a + \tfrac{1}{2} \int_{r}^{\infty} t e^{-t} u^3(t)\, dt .$$

As

$$\lim_{r\to\infty} u(r) = \lim_{r\to\infty} \frac{1}{2re^{r}} \int_{r_0}^{r} t e^{t} u^3(t)\, dt = \lim_{r\to\infty} \frac{1}{2re^{-r}} \int_{r}^{\infty} t e^{t} u^3(t)\, dt = 0 ,$$

the equality $B - a = 0$ must be valid and hence the modified form of (36) is

(43) $$u(r) = \frac{1}{r} e^{-r}\left(A + \tfrac{1}{2} \int_{r_0}^{r} t e^{t} u^3(t)\, dt\right) + \frac{1}{r} e^{r} \tfrac{1}{2} \int_{r}^{\infty} t e^{-t} u^3(t)\, dt .$$

Further, on the basis of Lemma 6 we have that

$$\int_{r_1}^{r} te^t u^3(t)\,dt \leqq c^3 \int_{r_1}^{r} \frac{1}{t^2} e^{t-3(1-\delta)t}\,dt = c^3 \int_{r_1}^{r} \frac{1}{t^2} e^{-2t+3\delta t}\,dt\,, \quad r \geqq r_1$$

and the last integral is convergent as $r \to \infty$ if $\delta \leqq \frac{2}{3}$. Then

$$\tfrac{1}{2}\int_{r_1}^{r} te^t u^3(t)\,dt = \tfrac{1}{2}\int_{r_1}^{\infty} te^t u^3(t)\,dt - \tfrac{1}{2}\int_{r}^{\infty} te^t u^3(t)\,dt\,,$$

and the following lemma is true.

Lemma 7. *If u is a nontrivial solution of the equation (2) in (a, ∞) with the property $\lim_{r\to\infty} u(r) = 0$, then there exists a constant A_2 such that u satisfies the integral equation*

$$u(r) = A_2 \frac{1}{r} e^{-r} + \int_r^\infty K_0(r, t)\, u^3(t)\,dt\,. \tag{44}$$

To investigate the existence of a unique solution of (44) we begin with the following lemma.

Lemma 8. *Let $0 < \delta < \frac{2}{3}$, $K \geqq A_2 > 0$, $r_2 \geqq K^3/(2A_2)$ and let*

$$E = \left\{ u \in C([r_2, \infty)) : 0 \leqq u(r) \leqq \frac{K}{r} e^{-(1-\delta)r},\ r_2 \leqq r < \infty \right\}.$$

Then the operator T defined in E by the relation

$$Tu(r) = A_2 \frac{1}{r} e^{-r} + \int_r^\infty K_0(r, t)\, u^3(t)\,dt\,, \quad r_2 \leqq r < \infty\,, \quad u \in E \tag{45}$$

maps E into E and is antitone.

Proof. As $\delta < \frac{2}{3}$, for each $u \in E$ the integral $\int_{r_0}^{r} u^3(t)\, e^t t\,dt$ converges as $r \to \infty$, hence T is well defined and the function Tu is continuous. As $K_0(r, t) < 0$ for $t > r$, T is antitone in E, that is if $u_1, u_2 \in E$ and

$$u_1(r) \leqq u_2(r) \quad \text{for all} \quad r \geqq r_2\,, \quad \text{then} \quad Tu_1(r) \geqq Tu_2(r)$$
$$\text{in} \quad [r_2, \infty)\,.$$

Further,

$$T(0) = A_2 \frac{1}{r} e^{-r} \leqq \frac{K}{r} e^{-r+\delta r}\,, \quad r_2 \leqq r < \infty\,, \tag{46}$$

and

$$T\left(\frac{K}{r} e^{-(1-\delta)r}\right) \geqq 0 \tag{47}$$

if and only if

$$\frac{K^3}{2r} e^{-r} \int_r^\infty \frac{1}{t^2} e^{-2t+3\delta t} dt - \frac{K^3}{2r} e^r \int_r^\infty \frac{1}{t^2} e^{-4t+3\delta t} dt \leqq A_2 \frac{1}{r} e^{-r},$$
$$r \geqq r_2 .$$

The left-hand side of the last inequality is smaller or equal to

$$\frac{K^3}{2r} e^{-r} \int_r^\infty \frac{1}{t^2} e^{-2t+3\delta t} dt \leqq \frac{K^3}{2r} e^{-r} \int_r^\infty \frac{1}{t^2} dt = \frac{K^3}{2r^2} e^{-r} .$$

Hence (47) will be satisfied if

$$\frac{K^3}{2r^2} e^{-r} \leqq A_2 \frac{1}{r} e^{-r}$$

and this is true for all $r \geqq r_2$. As T is an antitone mapping, (46) and (47) imply $T(E) \subset E$.

In $C([r_2, \infty))$ consider the sup-norm $\|\cdot\|$. Then the following lemma holds.

Lemma 9. *If* $0 < \delta < \frac{1}{2}$ *and* $r_2 \geqq K^3/2A_2$ *is sufficiently great, then the operator* T *given by* (45) *is contractive in* E.

Proof. Let $u, v \in E$. Then $|u^2(t) + u(t)\, v(t) + v^2(t)| \leqq (3K^2/t^2)\, e^{-2(1-\delta)t}$, $r_2 \leqq \leqq t < \infty$ and hence

$$|T(u)(r) - T(v)(r)| \leqq \int_r^\infty |K_0(r, t)| \frac{3K^2}{t^2} e^{-2(1-\delta)t} dt \|u - v\| ,$$
$$r_2 \leqq r < \infty .$$

As $\lim_{r \to \infty} \frac{3}{2}(K^2/r) \int_r^\infty (1/t)\, e^{-t-r+2\delta t} dt = 0$ in view of the inequality $\delta < \frac{1}{2}$, there exists an $r_2 \geqq K^3/2A_2$ such that for all $r \geqq r_2$ the last expression is smaller than 1 and hence T is contractive.

Finally we prove a theorem which improves the result by G. Sansone in Theorem 5, [5], p. 18.

Theorem 3. *For each nontrivial solution* u *of the differential equation* (2) *with the property* $\lim_{r \to \infty} u(r) = 0$ *there exist numbers* $r_2 > 1$, $0 < A_3 < A_2$ *such that*

$$A_3 \frac{1}{r} e^{-r} \leqq |u(r)| \leqq A_2 \frac{1}{r} e^{-r} , \quad r \geqq r_2 . \tag{48}$$

Proof. We know that there exists an interval $[r_0, \infty)$ such that either $u(r) > 0$ holds in this interval or $u(r) < 0$ in $[r_0, \infty)$. Consider only the case (35). Lemma 5 gives an estimate from below

$$u(r) \geqq A_3 \frac{1}{r} e^{-r}, \quad r_0 \leqq r < \infty$$

while Lemma 6 yields an estimate from above

$$u(r) \leqq K \frac{1}{r} e^{-(1-\delta)r}, \quad r_1 \leqq r < \infty .$$

Lemma 7 says that u is a fixed point of the operator T. As $u \in E$ and by Lemma 9 T is contractive in E when r_2 is sufficiently large, u is the unique fixed point of this operator and the method of successive approximations is applicable. Since T is antitone, the inequalities $0 \leqq u(r) \leqq (K/r) e^{-(1-\delta)r}$ yield the estimates $u = Tu \leqq T(0) = A_2(1/r) e^{-r}$ for all sufficiently large r. Hence (48) is true and the theorem is proved.

Lemma 8 and Lemma 9 imply the following theorem.

Theorem 4. *There exists an uncountable one-parametric system of solutions u of the equation* (2) *with the property*

$$\text{(49)} \qquad \lim_{r\to\infty} u(r) = 0 .$$

Proof. Let $A_2 \in R$ and consider the integral equation (44). If $A_2 > 0$, then in Lemma 8 we choose $K = A_2$, $\delta = \frac{1}{4}$ and $r_2 \geqq A_2^2/2$ sufficiently large. By this lemma the operator T defined by (45) maps E into itself. From the proof of Lemma 9 it follows that the operator T is contractive if r_2 has the property

$$\frac{3A_2^2}{2r_2} e^{-r_2} \int_{r_2}^{\infty} \frac{1}{t} e^{-t/2} \, dt < 1 .$$

Then the equation (44) has a unique solution in E and hence it satisfies the inequalities

$$\text{(50)} \qquad 0 \leqq u(r) \leqq \frac{A_2}{r} e^{-3/4r}, \quad r_2 \leqq r < \infty .$$

The same conclusion holds if $A_2 < 0$. In this case the proof proceeds similarly with the only exception that $E = \{u \in C([r_2, \infty)): 0 \geqq u(r) \geqq (A_2/r) e^{-3/4r}, r_2 \leqq r < \infty\}$. If $A_2 = 0$, then (44) has a trivial solution in $(0, \infty)$.

Let $A_2 > 0$. By (50) it follows that $0 \leqq (r/2) e^r u^3(r) \leqq (A_2^3/2r^2) e^{-5/4r}$ in $[r_2, \infty)$ and therefore $\int_{r_2}^{\infty} (t/2) e^t u^3(t) \, dt$ exists and the integral equation (44) can be written in the form

$$u(r) = \left(A_2 - \int_{r_2}^{\infty} \frac{t}{2} e^t u^3(t) \, dt\right) \frac{1}{r} e^{-r} +$$

$$+\int_{r_2}^{\infty}\frac{t}{2}\,\mathrm{e}^{-t}\,u^3(t)\,\mathrm{d}t\,\frac{1}{r}\,\mathrm{e}^{r}-\int_{r_2}^{r}K_0(r,t)\,u^3(t)\,\mathrm{d}t\,,$$

$$r_2 \leqq r < \infty\,.$$

By virtue of Lemma 4 u is also a solution of the differential equation (2) which satisfies (49).

4. ZEROS OF THE SOLUTIONS OF (2)

Denote by $u(\cdot,\alpha)$ the solution of the initial value problem (2), (3) in $[0,\infty)$. By Corollary 1 it exists and is uniquely determined by α. With help of Lemma 4 we get that it satisfies the integral equation

$$u(r)=\frac{\alpha}{2}\,\frac{\mathrm{e}^{r}-\mathrm{e}^{-r}}{r}-\int_{0}^{r}K_0(r,t)\,u^3(t)\,\mathrm{d}t\,,\quad 0<r<\infty\,. \tag{51}$$

Conversely, if $u \in C([0,\infty))$ is a solution of (51), then it satisfies the initial value problem (2), (3) in $[0,\infty)$.

Similarly, if we write (2) in the form

$$u''+\frac{2u'}{r}=u-u^3\,,$$

we get that $u'=u'(\cdot,\alpha)$ satisfies the integral equation

$$u'(r)=\frac{1}{r^2}\int_{0}^{r}t^2[u(t)-u^3(t)]\,\mathrm{d}t\,,\quad 0<r<\infty \tag{52}$$

and hence

$$\lim_{r\to 0+}\frac{2\,u'(r)}{r}=\lim_{r\to 0+}\tfrac{2}{3}(u(r,\alpha)-u^3(r,\alpha))=\tfrac{2}{3}(\alpha-\alpha^3)\,.$$

Putting this result into (2) we get $u(\cdot,\alpha)$ has the second derivative at 0 and

$$u''(0,\alpha)=\lim_{r\to 0+}u''(r,\alpha)=\tfrac{1}{3}(\alpha-\alpha^3)\,.$$

By means of these considerations we prove the following theorem.

Theorem 5. *The solution $u=u(r,\alpha)$ of the initial value problem (2), (3) continuously depends on α on each compact interval $[0,r_0]$, $r_0>0$. In other words, if $\alpha_n\to\alpha$ for $n\to\infty$, then the sequences $\{u(\cdot,\alpha_n)\}$ and $\{u'(\cdot,\alpha_n)\}$ converge locally uniformly to $u(\cdot,\alpha)$ and $u'(\cdot,\alpha)$, respectively, in the interval $[0,\infty)$.*

Proof. It suffices to show that the above mentioned convergence is uniform on each compact interval $[0, r_0]$. Let E be the function determined by (8). Since the sequence $E(u(0, \alpha_n), u'(0, \alpha_n)) = \frac{1}{4}\alpha_n^4 - \frac{1}{2}\alpha_n^2$ is bounded, and by (10) the functions $E(u(r, \alpha_n), u'(r, \alpha_n))$ are nonincreasing in the variable r, the curves $(u(r, \alpha_n), u'(r, \alpha_n))$, $0 \leqq r \leqq r_0$, lie in a compact set. Thus the sequences $\{u(r, \alpha_n)\}$, $\{u'(r, \alpha_n)\}$ are uniformly bounded in $[0, r_0]$ and from (52) it follows that the sequence $\{(2u'(r, \alpha_n)/r\}$ has the same property. By (2) we finally get that $\{u''(r, \alpha_n)\}$ is uniformly bounded in $[0, r_0]$, too. The Ascoli theorem then gives that there exists a subseuence $\{\alpha_{n_k}\}$ of the sequence $\{\alpha_n\}$ such that the sequences $\{u(r, \alpha_{n_k})\}$ and $\{u'(r, \alpha_{n_k})\}$ converge uniformly in $[0, r_0]$ to the functions v and v', respectively.

By (51) we have

$$u(r, \alpha_{n_k}) = \frac{\alpha_{n_k}}{2} \frac{e^r - e^{-r}}{r} - \int_0^r K_0(r, t)\, u^3(t, \alpha_{n_k})\, dt\,, \quad 0 < r \leqq r_0\,.$$

The limit process leads to the relation

$$v(r) = \frac{\alpha}{2} \frac{e^r - e^{-r}}{r} - \int_0^r K_0(r, t)\, v^3(t)\, dt\,, \quad 0 < r \leqq r_0\,.$$

By the uniqueness of the solution of the initial value problem (2), (3) we get that $v(r) = u(r, \alpha)$, $0 \leqq r \leqq r_0$. Since each subsequence of the sequence $\{u(r, \alpha_n)\}$ has a subsequence, say $\{u(r, \alpha_{n_m})\}$, such that the sequences $\{u(r, \alpha_{n_m})\}$ and $\{u'(r, \alpha_{n_m})\}$ converge uniformly on $[0, r_0]$ to the same functions $u(r, \alpha)$ and $u'(r, \alpha)$, respectively, the whole sequences $\{u(r, \alpha_n)\}$ and $\{u'(r, \alpha_n)\}$ converge uniformly on $[0, r_0]$ to the functions $u(r, \alpha)$ and $u'(r, \alpha)$, respectively. The proof of the theorem is complete.

By Theorem 2, there exists $\lim_{r\to\infty} u(r, \alpha) = l(\alpha) \in \{-1, 0, 1\}$. Denote

$$N_{-1} = \{\alpha \in R: \lim_{r\to\infty} u(r, \alpha) = -1\}\,,$$

$$N_1 = \{\alpha \in R: \lim_{r\to\infty} u(r, \alpha) = 1\}\,,$$

$$N_0 = \{\alpha \in R: \lim_{r\to\infty} u(r, \alpha) = 0\}\,.$$

Clearly $R = N_{-1} \cup N_1 \cup N_0$, $0 \in N_0$ and $N_i \cap N_j = \emptyset$ for $i \neq j$, $i, j \in \{-1, 0, 1\}$. If u is a solution of (2), $-u$ is also a solution of this equation. Hence $\alpha \in N_1$ $(\alpha \in N_{-1})$ implies that $-\alpha \in N_{-1}$ $(-\alpha \in N_1)$ and $\alpha \in N_0$ iff $-\alpha \in N_0$. Further properties of the sets N_{-1}, N_1, N_0:

1. *The sets N_{-1}, N_1 are open (in R).*

Proof. We only prove that N_{-1} is open. Similarly the openness of N_1 can be proved. Let $\alpha_0 \in N_{-1}$. Then there exists an $r_0 > 0$ such that the trajectory $(u(r, \alpha_0), u'(r, \alpha_0))$ of the solution $u(r, \alpha_0)$ lies in the interior of the left curve $E(x, y) = c_1$

for a c_1, $-\frac{1}{4} < c_1 < 0$, for all $r \geqq r_0$. By Theorem 5 this implies that for a sufficiently small $\varepsilon > 0$ there exists a $\delta = \delta(r_0, \varepsilon) > 0$ such that for all α, $|\alpha - \alpha_0| < \delta$ we have $|u(r, \alpha) - u(r, \alpha_0)| < \varepsilon$, $|u'(r, \alpha) - u'(r, \alpha_0)| < \varepsilon$ in $[0, r_0]$. Hence for these $\alpha - s$, the point $(u(r_0, \alpha), u'(r_0, \alpha))$ also lies in the interior of the left curve $E(x, y) = c_1$ and therefore $\lim_{r \to \infty} u(r, \alpha) = -1$ for α, $|\alpha - \alpha_0| < \delta$.

2. $(0, \sqrt{2}] \subset N_1$.

Proof. If $\alpha \in (0, \sqrt{2}]$, then the initial point $(u(0, \alpha), u'(0, \alpha))$ of the trajectory of the solution $u(r, \alpha)$ lies in the interior (or on the boundary if $\alpha = \sqrt{2}$) of the right part of the curve $E(x, y) = 0$ and, since $E(u(r, \alpha), u'(r, \alpha))$ is decreasing in r, the cases $\lim_{r \to \infty} u(r, \alpha) = 0$ and $\lim_{r \to \infty} u(r, \alpha) = -1$ cannot occur. By Theorem 2 the only possibility is that $\lim_{r \to \infty} u(r, \alpha) = 1$ and hence $\alpha \in N_1$.

The statement 2 implies

3. $[-\sqrt{2}, 0) \subset N_{-1}$.

Due to the uniqueness of the initial value problem for the equation (2), if $\lim_{r \to \infty} u(r, \alpha) \neq 0$, then $u(r, \alpha)$ has only finitely many zeros, or no zero. By Theorem 3 the same is true for nontrivial solutions $u(r, \alpha)$ with $\lim_{r \to \infty} u(r, \alpha) = 0$. Hence it makes sense to define a function

$$n: (-\infty, 0) \cup (0, \infty) \to R$$

by the relation

$$n(\alpha) \textit{ is the number of zeros of the solution } u(r, \alpha)\,.$$

The following statement is true.

4. *The function n is constant on each component of the set N_1 and on each component of the set N_{-1}.*

Proof. We prove the statement only for N_1. For N_{-1} it could be proved analogously. The open set N_1 consists of at most countable many components which are open intervals by the statement 1. Let (a, b) be such an interval. We assert: The function n is continuous in (a, b). Indeed, if $\alpha_0 \in (a, b)$, then $\lim_{r \to \infty} u(r, \alpha_0) = 1$ and there exists a point $r_0 > 0$ such that the point $(u(r_0, \alpha_0), u'(r_0, \alpha_0))$ of the trajectory of the solution $u(r, \alpha_0)$ lies in the interior of the right curve $E(x, y) = c_1$ for a c_1, $-\frac{1}{4} < c_1 < 0$. By Theorem 5 it follows that for all α sufficiently close to α_0 the graph of the solution $u(r, \alpha)$ lies in an ε-neighbourhood of the graph of the solution $u(r, \alpha_0)$ on the interval $[0, r_0]$, and the point $(u(r_0, \alpha), u'(r_0, \alpha))$ also lies in the interior of the right curve $E(x, y) = c_1$. This implies that neither $u(r, \alpha_0)$ nor $u(r, \alpha)$ has a zero in the interval (r_0, ∞). By the fact that the local maximum of the solutions $u(r, \alpha)$ lies in the set $(-1, 0) \cup (1, \infty)$ while the local minimum of these solutions belongs to the set $(-\infty, -1) \cup (0, 1)$, as well as by the fact that the graphs of the solutions $u(r, \alpha)$ are near to the graph of $u(r, \alpha_0)$ it follows that $u(r, \alpha)$ have the same number of zeros in $[0, r_0]$ as the function $u(r, \alpha_0)$. Hence $n(\alpha) = n(\alpha_0)$ for all α sufficiently close to α_0. This implies the continuity of the function n in (a, b) and, as this function attains only values from the set of nonnegative integers, we get that it is constant in (a, b).

From the proof of the statement 2 we get the following statement.

5. *The equality* $n(r) = 0$ *holds in* $(0, \sqrt{2}] \cup [-\sqrt{2}, 0)$.

Using again the continuity of $u(r, \alpha)$ with respect to α we come to the next statement:

6. *If* $\alpha_0 \neq 0$, *then there exists a* $\delta = \delta(\alpha_0) > 0$ *such that*

$$n(\alpha) \geqq n(\alpha_0) \quad \textit{for each} \quad \alpha \in (\alpha_0 - \delta, \alpha_0 + \delta).$$

In other words, the function n *is lower semicontinuous.*

7. *Let* $\alpha > 0$ $(\alpha < 0)$. *Then*

$$\alpha \in N_1 \textit{ implies that } n(\alpha) \textit{ is even } (n(\alpha) \textit{ is odd}).$$

$$\alpha \in N_{-1} \textit{ implies that } n(\alpha) \textit{ is odd } (n(\alpha) \textit{ is even}).$$

On the other hand,

$$n(\alpha) \textit{ is even } (n(\alpha) \textit{ is odd}) \textit{ gives that } \alpha \in N_1 \cup N_0.$$

$$n(\alpha) \textit{ is odd } (n(\alpha) \textit{ is even}) \textit{ gives that } \alpha \in N_{-1} \cup N_0.$$

Suppose that $\alpha \neq 0$. If $u(\cdot, \alpha)$ attains a local maximum (a local minimum) at $r_0 \geqq 0$, then by the equation (2) either $u(r_0, \alpha) \geqq 1$ or $-1 \leqq u(r_0, \alpha) < 0$ (either $u(r_0, \alpha) \leqq -1$ or $0 < u(r_0, \alpha) \leqq 1$). Since the function $E(u(\cdot, \alpha), u'(\cdot, \alpha))$ is nonincreasing, we conclude that if $E(u(r_0, \alpha), u'(r_0, \alpha)) \leqq 0$, then $u(\cdot, \alpha)$ has no zeros in $[r_0, \infty)$ and either $\lim_{r\to\infty} u(r, \alpha) = 1$ or $\lim_{r\to\infty} u(r, \alpha) = -1$. Thus the following statement is true:

8. *Suppose that* $\alpha \neq 0$. *If* $u(\cdot, \alpha)$ *attains a local maximum at* $r_0 \geqq 0$ *and*

$$1 \leqq u(r_0, \alpha) \leqq \sqrt{2} \quad (-1 \leqq u(r_0, \alpha) < 0),$$

then $u(\cdot, \alpha)$ *has no zeros in* $[r_0, \infty)$ *and* $\alpha \in N_1$ $(\alpha \in N_{-1})$.

Further, if $u(\cdot, \alpha)$ *attains a local minimum at* $r_0 \geqq 0$ *and*

$$-\sqrt{2} \leqq u(r_0, \alpha) \leqq -1 \quad (0 < u(r_0, \alpha) \leqq 1),$$

then $u(\cdot, \alpha)$ *has no zeros in* $[r_0, \infty)$ *and* $\alpha \in N_{-1}$ $(\alpha \in N_1)$.

Theorem II in [4], p. 479, implies the existence of a sequence $\{\alpha_k\}_{k=1}^{\infty}$ such that

$u(\cdot, \alpha_k)$ has exactly $k - 1$ zeros in $(0, \infty)$ and $\lim_{r \to \infty} u(r, \alpha_k) = 0$, hence $\alpha_k \in N_0$ and $n(\alpha_k) = k - 1$, $k = 1, 2, \ldots$. Theorem 4.1 in [1], p. 86, asserts that there exists at most one value α_1 with the above mentioned property. The numerical evaluation of some α_k can be based on the following theorem.

Theorem 6. *If* $0 < \bar{\alpha} < \hat{\alpha}$ *and* $n(\bar{\alpha}) \neq n(\hat{\alpha})$, *then there is* $\alpha_0 \in [\bar{\alpha}, \hat{\alpha}]$ *such that*

$$\lim_{r \to \infty} u(r, \alpha_0) = 0 .$$

Proof. If at least one of the numbers $\bar{\alpha}, \hat{\alpha}$ belongs to N_0, the theorem is true. If both $\bar{\alpha}, \hat{\alpha} \in N_1 \cup N_{-1}$, then in view of the statement 4, $\bar{\alpha}, \hat{\alpha}$ belong to different components of $N_1 \cup N_{-1}$ which are disjoint open intervals. Hence there exists an $\alpha_0 \in (\bar{\alpha}, \hat{\alpha})$ from the complement of $N_1 \cup N_{-1}$, i.e. $\alpha_0 \in N_0$.

Remark. The values of the function n can be calculated with the help of the statement 8.

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Súhrn

O NIEKTORÝCH VLASTNOSTIACH RIEŠENÍ DIFERENCIÁLNEJ ROVNICE

$$u'' + \frac{2u'}{r} = u - u^3$$

VALTER ŠEDA, JÁN PEKÁR

V práci sa dokazuje, že každé riešenie $u(r, \alpha)$ začiatočnej úlohy (2), (3) má konečnú limitu, ak $r \to \infty$ a určený je asymptotický vzorec pre netriviálne riešenie $u(r, \alpha)$ idúce k 0. Ďalej sa existencia takéhoto riešenia dokazuje pomocou skúmania počtu koreňov dvoch rôznych riešení $u(r, \bar{\alpha})$, $u(r, \alpha)$.

Authors' addresses: Prof. Dr. Valter Šeda, DrSc., Katedra matematickej analýzy MFF UK, Mlynská dolina, 842 15 Bratislava, *Dr. Ján Pekár, CSc.,* Katedra numerických a optimalizačných metód MFF UK, Mlynská dolina, 842 15 Bratislava.