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DOMAIN OPTIMIZATION IN 3D-AXISYMMETRIC ELLIPTIC PROBLEMS BY DUAL FINITE ELEMENT METHOD

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Summary. An axisymmetric second order elliptic problem with mixed boundary conditions is considered. The shape of the domain has to be found so as to minimize a cost functional, which is given in terms of the cogradient of the solution. A new dual finite element method is used for approximate solutions. The existence of an optimal domain is proven and a convergence analysis presented.

Keywords: Shape optimal design, finite elements, dual variational formulation.

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INTRODUCTION

The present paper is devoted to the analysis of an axisymmetric optimal design problem, where a part of the boundary of the meridional section plays the role of a design variable. As the state problem, a mixed boundary value problem for a second order elliptic operator is considered. The Dirichlet homogeneous boundary condition is prescribed on the variable part of the boundary. Since the cost functional is given in terms of the cogradient of the solution, we employ a dual finite element technique, the analysis of which has been presented recently in [3]. Thus the present paper extends the results of [1] and [4] to some three-dimensional axisymmetric problems.

In Section 1 the optimal design problem is given and the dual variational formulation of the state problem recalled. We introduce finite element approximations in Section 2 and prove their convergence to some optimal solution.

1. FORMULATION OF THE OPTIMIZATION PROBLEM

Let us consider the following model problem: $D(\alpha) \subset \mathbb{R}^2$ be the domain

$$D(\alpha) = \{(r, z) \mid 0 < r < \alpha(z),\ 0 < z < 1\},$$

where the function $\alpha(z)$ – the design variable – belongs to the set of admissible functions

$$U_{ad} = \{\alpha \in C^{(0),1}([0, 1]) , \quad \text{(i.e., Lipschitz function)} ,$$
$$\alpha_{\min} \leqq \alpha(z) \leqq \alpha_{\max}, \ |d\alpha/dz| \leqq C_1, \ \textstyle\int_0^1 \alpha^2(z)\, dz = C_2\} ,$$

with given positive constants $\alpha_{\min}, \alpha_{\max}, C_1, C_2$. Assume that U_{ad} is non-empty and denote the graph of the function α by $\Gamma(\alpha)$.

Here r and z denote the radial and axial coordinate, respectively. The following *Optimal Design Problem* will be studied:

$$(1.1) \qquad \alpha^0 = \arg\min_{\alpha \in U_{ad}} J(\alpha, y(\alpha)) ,$$

where

$$(1.2) \qquad J(\alpha, y) = \int_{D(\alpha)} F(\text{cograd}\ y)\, r\, dr\, dz$$

and $y(\alpha)$ is the solution of the following boundary value problem

$$(1.3) \qquad -\frac{1}{r}\frac{\partial}{\partial r}\left(r\, a_r \frac{\partial y}{\partial r}\right) - \frac{\partial}{\partial z}\left(a_z \frac{\partial y}{\partial z}\right) = f \quad \text{in} \quad D(\alpha) ,$$

$$y = 0 \quad \text{on} \quad \Gamma(\alpha) ,$$

$$a_z \frac{\partial y}{\partial z} = 0 \quad \text{on} \quad \partial D(\alpha) \dot{-} \Gamma(\alpha) \dot{-} \Gamma_0 ;$$

$$\text{cograd}\ y = \left(a_r \frac{\partial y}{\partial r} , \ a_z \frac{\partial y}{\partial z}\right)^{\mathrm{T}} , \quad a_r, a_z \in L^\infty(\hat{D}) , \quad f \in L_r^2(\hat{D}) ,$$

$$\hat{D} = \{(r, z) | \ 0 < r < \delta, \ 0 < z < 1\} , \quad \delta > \alpha_{\max} ,$$

$$\Gamma_0 = \{(r, z) | \ r = 0, \ z \in [0, 1]\} .$$

Assume that a positive constant a_0 exists such that

$$(1.4) \qquad a_r \geqq a_0 , \quad a_z \geqq a_0 \quad \text{a.e.} \quad \text{in } \hat{D} .$$

We denote by $L_r^m(\hat{D})$ the space of measurable functions u, for which

$$\|u\|_{0,r,\hat{D}}^m = \int_{\hat{D}} |u|^m\, r\, dr\, dz < +\infty , \quad m = 1, 2 .$$

The space of bounded measurable functions on D will be denoted by $L^\infty(D)$. Let $k \geqq 0$ and n be integers. We shall denote by $W_{r^n}^{k,2}(D)$ the weighted Sobolev space with the weight r^n and the norm

$$\|u\|_{k,r^n,D} = \Big(\sum_{|\beta| \leqq k} \int_D |D^\beta u|^2\, r^n\, dr\, dz\Big)^{1/2} ,$$

where $D^\beta u$ denotes any partial derivative of the order β. The same notation will be used also for vector-functions.

We shall use the weak formulation of the problem (1.3); the weak solution of (1.3) is the function $y \equiv y(\alpha) \in V(\alpha)$ such that

$$(1.5) \qquad \int_{D(\alpha)} \left[a_r \frac{\partial y}{\partial r} \frac{\partial v}{\partial r} + a_z \frac{\partial y}{\partial z} \frac{\partial v}{\partial z} \right] r \, \mathrm{d}r \, \mathrm{d}z = \int_{D(\alpha)} f v r \, \mathrm{d}r \, \mathrm{d}z$$

holds for all $v \in V(\alpha)$, where

$$V(\alpha) = \{ v \in W_r^{1,2}(D(\alpha)) \mid \gamma v = 0 \text{ on } \Gamma(\alpha) \} .$$

Note that there exists a continuous mapping

$$\gamma \colon W_r^{1,2}(D(\alpha)) \to L_r^2(\Gamma(\alpha))$$

such that $\gamma u = u|_{\Gamma(\alpha)}$ for any $u \in C(\mathscr{C}\ell\, D(\alpha))$. (The proof can be found e.g. in [2] – Sect. 1).

It is easy to find that there exists a positive constant C_3 such that

$$(1.6) \qquad \int_{D(\alpha)} |\operatorname{grad} u|^2 \, r \, \mathrm{d}r \, \mathrm{d}z \geqq C_3 \|u\|^2_{1,r,D(\alpha)}$$

holds for all $u \in V(\alpha)$ and $\alpha \in U_{\mathrm{ad}}$. Using (1.4) and (1.6) we derive that the problem is $V(\alpha)$-elliptic and therefore uniquely solvable for any $\alpha \in U_{\mathrm{ad}}$. Then cograd $y(\alpha) \in \in [L_r^2(D(\alpha))]^2$.

Assume that the mapping F, occuring in (1.2), is continuous from $[L_r^2(\hat{D})]^2$ into $L_r^1(\hat{D})$.

Example. Let us consider the function

$$F(\mathbf{q}) = [b_r(q_r - K_r)^2 + b_z(q_z - K_z)^2]^{1/2} - k_0 ,$$

where $\mathbf{q} = (q_r, q_z)^{\mathrm{T}}$, $K = (K_r, K_z)^{\mathrm{T}} \in [L_r^2(\hat{D})]^2$, $b_r, b_z \in L^\infty(\hat{D})$, $b_r \geqq b_0$ and $b_z \geqq b_0$ with some positive b_0 holds a.e. in $\hat{D}$, $k_0 \in L_r^1(\hat{D})$. Then $F(\mathbf{q})$ is continuous in $L_r^1(\hat{D})$. In fact, let

$$\lim_{n \to \infty} \|\mathbf{q}^n - \mathbf{q}\|_{0,r,\hat{D}} = 0 .$$

If we denote

$$F(\mathbf{q}) + k_0 = \|\mathbf{q} - \mathbf{K}\|_B ,$$

then $\|\cdot\|_B$ represents a norm in $\mathbb{R}^2$ almost everywhere in $\hat{D}$. We may write

$$\int_{\hat{D}} |F(\mathbf{q}) - F(\mathbf{q}^n)| \, r \, \mathrm{d}r \, \mathrm{d}z \leqq \int_{\hat{D}} \|\mathbf{q}^n - \mathbf{q}\|_B \, r \, \mathrm{d}r \, \mathrm{d}z \leqq$$

$$\leqq C_B \|\mathbf{q}^n - \mathbf{q}\|_{0,r,\hat{D}} \to 0 .$$

Setting e.g. $\mathbf{K} = 0$, $k_0 = 1$, $b_r = b_z = b^{-2}$, $b = \text{const} > 0$, then

$$F(\mathbf{q}) = (|\mathbf{q}(\alpha)| - b)/b$$

and b has the meaning of an „admissible" magnitude of the cogradient vector.

Since the cost functional is given in terms of the cogradient of the solution, we shall employ the dual variational formulation of the state problem (1.3). Let us recall the latter formulation, the derivation of which can be found e.g. in the paper [3] – Part I, Section 2.

Let us introduce the notation $\mathscr{H}(\alpha) = [L_r^2(D(\alpha))]^2$ and the following bilinear form in $\mathscr{H}(\alpha) \times \mathscr{H}(\alpha)$

$$(\boldsymbol{q}, \boldsymbol{p})_{\mathscr{H}(\alpha)} = \int_{D(\alpha)} (a_r^{-1} q_r p_r + a_z^{-1} q_z p_z)\, r \,\mathrm{d}r \,\mathrm{d}z\,,$$
$$\|\boldsymbol{q}\|_{\mathscr{H}(\alpha)} = (\boldsymbol{q}, \boldsymbol{q})_{\mathscr{H}(\alpha)}^{1/2}\,.$$

It is readily seen that the norms $\|\cdot\|_{\mathscr{H}(\alpha)}$ and $\|\cdot\|_{0,r,D(\alpha)}$ are equivalent by virtue of (1.4). Moreover, let us define

$$B(\alpha; \boldsymbol{q}, v) = \int_{D(\alpha)} \left(q_r \frac{\partial v}{\partial r} + q_z \frac{\partial v}{\partial z} \right) r \,\mathrm{d}r \,\mathrm{d}z\,,$$
$$L(v) = \int_{D(\alpha)} f v r \,\mathrm{d}r \,\mathrm{d}z\,,$$
$$Q_f(\alpha) = \{\boldsymbol{q} \in \mathscr{H}(\alpha) \mid B(\alpha; \boldsymbol{q}, v) = L(v)\ \forall v \in V(\alpha)\}\,.$$

Then the functional

$$\tfrac{1}{2} \|\boldsymbol{q}\|_{\mathscr{H}(\alpha)}^2 \tag{1.7}$$

attains its minimum over the set $Q_f(\alpha)$ at the point $\boldsymbol{q}(\alpha)$ if and only if $\boldsymbol{q}(\alpha) =$ $=$ cograd $y(\alpha)$.

We assume that

$$\int_0^r t f(t, z)\,\mathrm{d}t \in L_{r^{-1}}^2(\hat{D})\,. \tag{1.8}$$

(Note that (1.8) is fulfilled e.g. if $f = r^\beta f_0(z)$, $\beta > -2$, $f_0 \in L^2((0, 1))$). Then the following vector-function

$$\boldsymbol{q}^* = (-r^{-1} \int_0^r t f(t, z)\,\mathrm{d}t, 0)^{\mathrm{T}} \tag{1.9}$$

belongs to the set $Q_f(\alpha)$ for any $\alpha \in U_{\mathrm{ad}}^0$, where

$$U_{\mathrm{ad}}^0 = \{\alpha \in C^{(0),1}([0, 1]) \mid \alpha_{\min} \leqq \alpha(z) \leqq \alpha_{\max}\}\,.$$

Defining the subspace

$$Q(\alpha) = \{\boldsymbol{q} \in \mathscr{H}(\alpha) \mid B(\alpha; \boldsymbol{q}, v) = 0\ \forall v \in V(\alpha)\}\,, \tag{1.10}$$

we may write $Q_f(\alpha) = \boldsymbol{q}^* + Q(\alpha)$. Substituting $\boldsymbol{q} = \boldsymbol{q}^* + \boldsymbol{p}$, $\boldsymbol{p} \in Q(\alpha)$ into (1.7), we conclude that the functional

$$\|\boldsymbol{p}\|_{\mathscr{H}(\alpha)}^2 + (\boldsymbol{q}^*, \boldsymbol{p})_{\mathscr{H}(\alpha)}$$

attains its minimum over the subspace $Q(\alpha)$ if and only if $\boldsymbol{p}(\alpha) =$ cograd $y(\alpha) - \boldsymbol{q}^*$.

The sufficient and necessary condition for the minimizer $\boldsymbol{p}(\alpha) \in Q(\alpha)$ is

$$(1.11) \qquad (\boldsymbol{p}(\alpha), \mathbf{t})_{\mathscr{H}(\alpha)} = -(\boldsymbol{q}^*, \mathbf{t})_{\mathscr{H}(\alpha)} \quad \forall \mathbf{t} \in Q(\alpha).$$

The latter minimum problem has a unique solution $\boldsymbol{p}(\alpha)$ for any $\alpha \in U_{\text{ad}}$.

Now the optimization problem (1.1) can be replaced by the following equivalent Optimal Design Problem:

$$(1.12) \qquad \alpha^0 = \arg\min_{\alpha \in U_{\text{ad}}} J^*(\alpha, \boldsymbol{q}(\alpha))$$

where

$$J^*(\alpha, \boldsymbol{q}) = \int_{D(\alpha)} F(\boldsymbol{q})\, r \,\mathrm{d}r \,\mathrm{d}z,$$

$\boldsymbol{q}(\alpha) = \boldsymbol{q}^* + \boldsymbol{p}(\alpha)$, $\boldsymbol{q}^*$ is defined by the formula (1.9) and $\boldsymbol{p}(\alpha)$ is the solution of (1.11).

2. APPROXIMATIONS BY THE FINITE ELEMENT METHOD

First we introduce piecewise linear approximations of the set U_{ad}. Let N be a positive integer and $h = 1/N$. We denote the subintervals $[(j-1)h, jh]$ by e_j and define

$$U_{\text{ad}}^h = \{\alpha_h \in U_{\text{ad}} \mid \alpha_h|_{e_j} \in P_1(e_j)\ \forall j\},$$

where $P_1(e_j)$ is the set of linear polynomials defined on e_j. Let $D(\alpha_h) = D_h$ be the domain bounded by the graph $\Gamma_h = \Gamma(\alpha_h)$ of the function $\alpha_h \in U_{\text{ad}}^h$.

The dual state problem (1.11) can be solved approximately by means of the finite element method, proposed in the paper [3]. Let us recall some results of the latter paper and apply them to the model problem (1.11).

We have to introduce the space

$$X_1(D(\alpha)) = W_r^{1,2}(D(\alpha)) \cap L_{r^{-1}}^2(D(\alpha))$$

with the norm

$$\|\varphi\|_{X_1(D(\alpha))} = \left(\int_{D(\alpha)} (\varphi^2 r^{-2} + |\text{grad}\, \varphi|^2)\, r \,\mathrm{d}r \,\mathrm{d}z\right)^{1/2}$$

and the subspace

$$W(\alpha) = \{\varphi \in X_1(D(\alpha)) \mid \gamma\varphi = 0 \text{ on } \partial D(\alpha) \dot{-} \Gamma(\alpha) \dot{-} \Gamma_0\}.$$

The operator

$$\text{curl}\, \varphi = \left(\frac{\partial \varphi}{\partial z}, -\frac{\varphi}{r} - \frac{\partial \varphi}{\partial r}\right)^{\mathrm{T}}$$

is well-defined on $W(\alpha)$.

For any $\alpha \in U_{\text{ad}}^0$ the space $Q(\alpha)$ can be identified with

$$\text{curl}\, W(\alpha) = \{\boldsymbol{q} \in \mathscr{H}(\alpha) \mid \exists \varphi \in W(\alpha) \text{ such that } \boldsymbol{q} = \text{curl}\, \varphi\}.$$

Moreover curl: $W(\alpha) \to Q(\alpha)$ is a one-to-one mapping ([3] – Thm. 4.6) and

$$(2.1) \qquad \|\text{curl}\ \varphi\|_{0,r,D(\alpha)} \leqq 2^{1/2} \|\varphi\|_{X_1(D(\alpha))} \, .$$

The function $u \in X_1(D(\alpha))$ if and only if $u/r \in X_3(D(\alpha))$, where

$$X_3(D(\alpha)) = W_{r^3}^{1,2}(D(\alpha)) \cap L_r^2(D(\alpha)) \, ,$$

with the norm

$$\|v\|_{X_3(D(\alpha))} = (\int_{D(\alpha)} (v^2 + |\text{grad}\ v|^2\ r^2)\ r\ \mathrm{d}r\ \mathrm{d}z)^{1/2} \, .$$

For any $\alpha \in U_{\text{ad}}^0$ and any $u \in X_1(D(\alpha))$ we have the inequalities

$$(2.2) \qquad 3^{-1/2} \|u\|_{X_1(D(\alpha))} \leqq \|u/r\|_{X_3(D(\alpha))} \leqq 3^{1/2} \|u\|_{X_1(D(\alpha))} \, .$$

If we construct the approximations of $\boldsymbol{q} \in Q(\alpha)$, we may therefore write

$$(2.3) \qquad \boldsymbol{q} = \text{curl}\ \varphi = \text{curl}\,(r\psi) \, , \quad \psi \in X_3(D(\alpha))$$

and approximate the function ψ.

The polygonal domain D_h will be carved into triangles by the following way. We choose $\alpha_0 \in (0, \alpha_{\min})$ and introduce a uniform triangulation of the rectangle $\mathscr{R} = [0, \alpha_0] \times [0, 1]$, independent of α_h, if $h = 1/N$ is fixed. In the remaining part $D_h \dot{-} \mathscr{R}$ let the vertices of triangles divide the segments $[\alpha_0, \alpha_h(jh)]$, $j = 0, 1, \ldots, N$, into M equal segments, where $M = 1 + [(\alpha_{\max} - \alpha_0) N]$ and the square brackets denote the integer part of the number inside. In this way, we obtain a regular family $\{\mathscr{T}_h(\alpha_h)\}$, $h \to 0$, $\alpha_h \in U_{\text{ad}}^h$, of triangulations, with

$$(2.4) \qquad h_{\max} = \max_{K \in \mathscr{T}_h(\alpha_h)} (\text{diam}\ K) \leqq h/\sin \omega_0 \, ,$$

$$\omega_0 = \text{arctg}\,((\alpha_{\min} - \alpha_0)(\alpha_{\max} - \alpha_0)^{-1} (1 + C_1 + C_1^2)^{-1}) \, .$$

Here K denotes any (closed) triangle of $\mathscr{T}_h(\alpha_h)$.

Let us define finite element spaces Σ_h^k by the standard manner, i.e.,

$$\Sigma_h^k = \{u \in C(\mathscr{C}\ell\ D_h) |\ u|_K \in P_k(K)\ \forall K \in \mathscr{T}_h(\alpha_h)\} \, , \quad k = 1, 2 \, .$$

We introduce the local Lagrange interpolation Π_K^k of the degree k on any $K \in \mathscr{T}_h(\alpha_h)$ so that

$$\Pi_K^k \colon C(K) \to P_k(K) \quad \text{and} \quad \Pi_K^k\, u = u$$

at all the nodes of the triangle K. For $k = 1$ the nodes are only vertices, for $k = 2$ they are vertices and mid-points of sides. We define the global interpolation

$$\Sigma_h^k \colon C(\mathscr{C}\ell\ D_h) \to \Sigma_h^k$$

so that

$$\Pi_h^k\, u|_K = \Pi_K^k\,(u|_K) \quad \forall K \in \mathscr{T}_h(\alpha_h) \, .$$

In the paper [3] (Corollary 5.7 and Lemma 5.9) the following estimate has been derived for any regular family of triangulations $\mathscr{T}_h(D)$

$$\|u - \Pi_h^k u\|_{X_3(D)} \leqq C_k(u)\, h_{\max}^k\,, \quad k = 1, 2\,, \tag{2.5}$$

where the constant $C_k(u)$ depends on u but not on $h_{\max}$, D is a fixed polygonal domain.

We shall construct subspaces $S_h \subset Q(\alpha_h)$. Let us define the set

$$Y_h = \{u_h |\ u_h = r w_h,\ w_h \in \Sigma_h^k,\ w_h = 0 \text{ on } \partial D_h \dot{-} \Gamma_h \dot{-} \Gamma_0\}$$

and

$$S_h = \operatorname{curl} Y_h\,. \tag{2.6}$$

It is easy to verify $Y_h \subset W(\alpha_h)$ and then $S_h \subset Q(\alpha_h)$ follows. Note that

$$\boldsymbol{q}^h \in S_h \Rightarrow \boldsymbol{q}^h = \left(r \frac{\partial w_h}{\partial z}\,,\ -2w_h - r \frac{\partial w_h}{\partial r}\right)^{\mathrm{T}}, \quad w_h \in \Sigma_h^k\,;$$

consequently, the components q_r^h and q_z^h are piecewise polynomial.

Instead of the state problem (1.11) we can solve the *Approximate State Problem*: find $\boldsymbol{p}^h(\alpha_h) \in S_h$ such that

$$(\boldsymbol{p}^h(\alpha_h), \boldsymbol{t}^h)_{\mathscr{H}(\alpha_h)} = -(\boldsymbol{q}^*, \boldsymbol{t}^h)_{\mathscr{H}(\alpha_h)} \quad \forall \boldsymbol{t}^h \in S_h\,. \tag{2.7}$$

By virtue of (1.4) and the boundedness of a_r, a_z, there exist positive constants C and C^* such that

$$C\|\boldsymbol{p}\|_{0,r,D(\alpha)} \leqq \|\boldsymbol{p}\|_{\mathscr{H}(\alpha)} \leqq C^*\|\boldsymbol{p}\|_{0,r,D(\alpha)} \tag{2.8}$$

holds for all $\boldsymbol{p} \in \mathscr{H}(\alpha)$ and any $\alpha \in U_{\mathrm{ad}}^0$. The approximate problem (2.7) has a unique solution for any $h = 1/N$ and any $\alpha_h \in U_{\mathrm{ad}}^h$.

Next we prove the following

Proposition 1. *Let $\{\alpha_h\}$, $h \to 0$ be a sequence of $\alpha_h \in U_{\mathrm{ad}}^h$, converging to a function α in $C([0, 1])$. Then*

$$\boldsymbol{p}^{0h}(\alpha_h) \to \boldsymbol{p}^0(\alpha) \quad \text{in} \quad [L_r^2(\hat{D})]^2 \quad \text{for} \quad h \to 0\,, \tag{2.9}$$

where $\boldsymbol{p}^{0h}(\alpha_h)$ is the solution of (2.7), extended by zero to the domain $\hat{D} \dot{-} \bar{D}(\alpha_h)$ and $\boldsymbol{p}^0(\alpha)$ is the solution of (1.11), extended by zero to $\hat{D} \dot{-} \bar{D}(\alpha)$.

Proof. 1^0. We can find easily that $\alpha \in U_{\mathrm{ad}}$. It follows from (2.7) that

$$\|\boldsymbol{p}^h\|_{\mathscr{H}(\alpha_h)} \leqq \|\boldsymbol{q}^*\|_{\mathscr{H}(\alpha_h)} \quad \forall h\,.$$

Consequently, using (2.8) we obtain

$$C\|\boldsymbol{p}^h\|_{0,r,D_h} \leqq \|\boldsymbol{q}^*\|_{\mathscr{H}(\alpha_h)} \leqq \|\boldsymbol{q}^*\|_{\mathscr{H}(\delta)}\,. \tag{2.10}$$

Therefore a subsequence of $\{\boldsymbol{p}^{0h}\}$ exists (and we shall denote it by the same symbol) such that

$$(2.11) \qquad \boldsymbol{p}^{0h} \rightharpoonup \boldsymbol{p}^0 \quad \text{(weakly) in } [L^2_r(\hat{D})]^2 .$$

2^0. We can show that $\boldsymbol{p}^0|_{D(\alpha)} \in Q(\alpha)$. In fact, let us consider a function $w \in V(\alpha)$ and denote by $\tilde{w}$ its extension to $\hat{D} \dot{-} D(\alpha)$ by means of zero. There exists a sequence $\{w_\varkappa\}$, $\varkappa \to 0$, such that

$$w_\varkappa \in C^\infty(\mathscr{C}\ell\, \hat{D}), \quad w_\varkappa = 0 \quad \text{in} \quad \hat{D} \dot{-} D(\alpha),$$
$$(2.12) \qquad \operatorname{supp} w_\varkappa \cap \Gamma(\alpha) = \emptyset, \quad \|w_\varkappa - \tilde{w}\|_{1,r,\hat{D}} \to 0 .$$

(The proof of this assertion is analogous to that of Lemma 2 in [2]). There exists a $h_0(\varkappa)$ such that $w_\varkappa$ vanishes on $\Gamma(\alpha_h)$ for $h < h_0(\varkappa)$ so that

$$w_\varkappa|_{D_h} \in V(\alpha_h) \quad \forall h < h_0(\varkappa) .$$

Since $\boldsymbol{p}^h \in S_h \subset Q(\alpha_h)$, we have

$$B(\alpha_h; \boldsymbol{p}^h, w_\varkappa) = 0 \quad \forall h < h_0(\varkappa) .$$

Using the weak convergence (2.11), we obtain

$$0 = B(\delta; \boldsymbol{p}^{0h}, w_\varkappa) \to B(\delta; \boldsymbol{p}^0, w_\varkappa) \quad \text{for} \quad h \to 0 .$$

Passing to the limit with $\varkappa \to 0$ and using (2.12), we arrive at

$$0 = B(\delta; \boldsymbol{p}^0, \tilde{w}) = B(\alpha; \boldsymbol{p}^0, w) .$$

Consequently, $\boldsymbol{p}^0|_{D(\alpha)} \in Q(\alpha)$ follows.

3^0. Next we show that

$$(2.13) \qquad \boldsymbol{p}^0 = 0 \quad \text{a.e. in} \quad \hat{D} \dot{-} D(\alpha) .$$

In fact, let $\boldsymbol{p}^0 \neq 0$ on a set $E \subset \hat{D} \dot{-} D(\alpha)$, meas $E > 0$. Denote the characteristic function of the set E by χ_E. Using (2.11), we obtain for $h \to 0$

$$(\boldsymbol{p}^{0h}, \chi_E \boldsymbol{p}^0)_{0,r,\hat{D}} \to (\boldsymbol{p}^0, \chi_E \boldsymbol{p}^0)_{0,r,\hat{D}} = \|\boldsymbol{p}^0\|^2_{0,r,E} > 0 .$$

On the other hand, we may write

$$(\boldsymbol{p}^{0h}, \chi_E \boldsymbol{p}^0)_{0,r,\hat{D}} = (\boldsymbol{p}^h, \boldsymbol{p}^0)_{0,r,D_h \cap E} \leqq \|\boldsymbol{p}^h\|_{0,r,D_h} \|\boldsymbol{p}^0\|_{0,r,D_h \cap E} \to 0 ,$$

since (2.10) holds and meas $(D_h \cap E) \to 0$. Thus we come to a contradiction.

4^0. Let us show that $\boldsymbol{p}^0$ is a solution of the problem (1.11). Let us consider a $\mathbf{t} \in Q(\alpha)$. We know that a function $\varphi \in W(\alpha)$ exists such that $\mathbf{t} = \operatorname{curl} \varphi$. Let us extend φ "symmetrically with respect to $\Gamma(\alpha)$ in the radial direction" to get $\tilde{\varphi} \in W(\delta)$. By lemma 6.1 in [3] there exists a sequence $\{\varphi_n\}$, $n \to \infty$, such that

$$\varphi_n \in C^\infty(\mathscr{C}\ell\, \hat{D}), \quad \operatorname{supp} \varphi_n \cap (\partial\hat{D} \dot{-} \Gamma(\delta)) = \emptyset ,$$
$$(2.14) \qquad \|\tilde{\varphi} - \varphi_n\|_{X_1(\hat{D})} \to 0 .$$

We set

$$\psi_n = \varphi_n / r\,, \quad \mathbf{t}^n = \operatorname{curl}\left(r\psi_n\right), \quad \mathbf{t}^{hn} = \operatorname{curl}\left(r\Pi_h^k\psi_n\right),$$

where Π_h^k is the Lagrange linear or quadratic interpolation. Obviously, the triangulations $\mathscr{T}_h(\alpha_h)$ can be extended to cover the domain $\hat{D}$ in such a way, that the family of extended triangulations remains regular. We have

$$\psi_n \in C^\infty(\mathscr{C}\ell\,\hat{D})\,, \quad \operatorname{supp}\psi_n = \operatorname{supp}\varphi_n\,, \quad \mathbf{t}^{hn}|_{D_h} \in S_h \subset Q(\alpha_h)\,,$$

since $\Pi_h^k\psi_n$ vanishes on $\partial\hat{D} \dot{-} \Gamma(\delta)$. Consequently, $\mathbf{t}^{hn}$ can be inserted into (2.7). We may write

$$\left|(\boldsymbol{p}^{0h}, \mathbf{t}^{hn})_{\mathscr{H}(\delta)} - (\boldsymbol{p}^0, \mathbf{t}^n)_{\mathscr{H}(\delta)}\right| \leqq \left|(\boldsymbol{p}^{0h}, \mathbf{t}^{hn} - \mathbf{t}^n)_{\mathscr{H}(\delta)}\right| +$$
$$+ \left|(\boldsymbol{p}^{0h}, \mathbf{t}^n)_{\mathscr{H}(\delta)} - (\boldsymbol{p}^0, \mathbf{t}^n)_{\mathscr{H}(\delta)}\right| = I_1 + I_2\,,$$
$$I_1 \leqq \left\|\boldsymbol{p}^{0h}\right\|_{\mathscr{H}(\delta)} \left\|\mathbf{t}^{hn} - \mathbf{t}^n\right\|_{\mathscr{H}(\delta)} \to 0 \quad \text{for} \quad h \to 0\,,$$

using (2.10) and the following estimate

$$\text{(2.15)} \qquad \left\|\mathbf{t}^{hn} - \mathbf{t}^n\right\|_{\mathscr{H}(\delta)} \leqq C^*\left\|\operatorname{curl}\left(r(\Pi_h^k\psi_n - \psi_n)\right)\right\|_{0,r,\hat{D}} \leqq$$
$$\leqq C^*\,6^{1/2}\left\|\Pi_h^k\psi_n - \psi_n\right\|_{X_3(\hat{D})} \leqq C_k(\psi_n)\,h^k\,,$$

which follows from (2.8), (2.1), (2.2), (2.5). (Note that an analogue of (2.4) can be derived for the family of extended triangulations and employed here).

Making use of (2.11), we obtain that I_2 tends to zero. Altogether, we have

$$\text{(2.16)} \qquad (\boldsymbol{p}^h, \mathbf{t}^{hn})_{\mathscr{H}(\alpha_h)} = (\boldsymbol{p}^{0h}, \mathbf{t}^{hn})_{\mathscr{H}(\delta)} \to (\boldsymbol{p}^0, \mathbf{t}^n)_{\mathscr{H}(\delta)} = (\boldsymbol{p}^0, \mathbf{t}^n)_{\mathscr{H}(\alpha)}\,.$$

Next, we may write

$$\text{(2.17)} \qquad \left|(\boldsymbol{q}^*, \mathbf{t}^{hn})_{\mathscr{H}(\alpha_h)} - (\boldsymbol{q}^*, \mathbf{t}^n)_{\mathscr{H}(\alpha)}\right| \leqq$$
$$= \left|(\boldsymbol{q}^*, \mathbf{t}^{hn})_{\mathscr{H}(\alpha_h)} - (\boldsymbol{q}^*, \mathbf{t}^n)_{\mathscr{H}(\alpha_h)}\right| + \left|(\boldsymbol{q}^*, \mathbf{t}^n)_{\mathscr{H}(\alpha_h)} - (\boldsymbol{q}^*, \mathbf{t}^n)_{\mathscr{H}(\alpha)}\right| \leqq$$
$$= \left\|\boldsymbol{q}^*\right\|_{\mathscr{H}(\delta)}\left(\left\|\mathbf{t}^{hn} - \mathbf{t}^n\right\|_{\mathscr{H}(\delta)} + C^*\left\|\mathbf{t}^n\right\|_{0,r,\Delta(\alpha_h,\alpha)}\right) \to 0\,,$$

using (2.15) and the following convergence

$$\operatorname{meas}\Delta(\alpha_h, \alpha) \to 0\,,$$

where

$$\Delta(\alpha_h, \alpha) = \left(D_h \dot{-} D(\alpha)\right) \cup \left(D(\alpha) \dot{-} D_h\right).$$

Passing to the limit with $h \to 0$ in (2.7) for $\mathbf{t}^h \equiv \mathbf{t}^{hn}$, on the basis of (2.16) and (2.17) we obtain

$$\text{(2.18)} \qquad (\boldsymbol{p}^0, \mathbf{t}^n)_{\mathscr{H}(\alpha)} = -(\boldsymbol{q}^*, \mathbf{t}^n)_{\mathscr{H}(\alpha)}\,.$$

Making use of (2.1) and (2.14), we may write for $n \to \infty$

$$\left\|\mathbf{t}^n - \tilde{\mathbf{t}}\right\|_{0,r,\hat{D}} = \left\|\operatorname{curl}\left(\varphi_n - \tilde{\varphi}\right)\right\|_{0,r,\hat{D}} \leqq 2^{1/2}\left\|\varphi_n - \tilde{\varphi}\right\|_{X_1(\hat{D})} \to 0\,.$$

Consequently, passing to the limit with $n \to \infty$ in (2.18), we obtain

$$(\boldsymbol{p}^0, \mathbf{t})_{\mathscr{H}(\alpha)} = -(\boldsymbol{q}^*, \mathbf{t})_{\mathscr{H}(\alpha)} .$$

Since the solution of the problem (1.11) is unique, $\boldsymbol{p}^0|_{D(\alpha)} = \boldsymbol{p}(\alpha)$ holds and the whole sequence $\{\boldsymbol{p}^{0h}\}$ converges weakly to $\boldsymbol{p}^0(\alpha)$ in $[L_r^2(\hat{D})]^2$.

5^0. It remains to prove the strong convergence. By virtue of (2.7) we have

$$\|\boldsymbol{p}^h\|^2_{\mathscr{H}(\alpha_h)} = -(\boldsymbol{q}^*, \boldsymbol{p}^h)_{\mathscr{H}(\alpha_h)} .$$

The weak convergence (2.11) and (1.11) yield

$$(2.19) \qquad \|\boldsymbol{p}^{0h}\|^2_{\mathscr{H}(\delta)} = -(\boldsymbol{q}^*, \boldsymbol{p}^{0h})_{\mathscr{H}(\delta)} \to -(\boldsymbol{q}^*, \boldsymbol{p}^0(\alpha))_{\mathscr{H}(\delta)} = -(\boldsymbol{q}^*, \boldsymbol{p}(\alpha))_{\mathscr{H}(\alpha)} =$$
$$= \|\boldsymbol{p}(\alpha)\|^2_{\mathscr{H}(\alpha)} = \|\boldsymbol{p}^0(\alpha)\|^2_{\mathscr{H}(\delta)} .$$

Combining the weak convergence and the convergence of norms (2.19), we arrive at the convergence in $\mathscr{H}(\delta)$, which is equivalent with the convergence in $[L_r^2(\hat{D})]^2$.

Q.E.D.

Instead of the Optimal Design Problem (1.12) we introduce the *Approximate Optimal Design Problem*

$$(2.20) \qquad \alpha^h = \arg\min_{\beta_h \in U_{ad}{}^h} J^*(\beta_h, \boldsymbol{q}^h(\beta_h)) ,$$

where $\boldsymbol{q}^h(\beta_h) = \boldsymbol{q}^* + \boldsymbol{p}^h(\beta_h)$.

Proposition 2. *Let* $\{\alpha_h\}$, $h \to 0$, *be a sequence of* $\alpha_h \in U^h_{ad}$, *converging to a function* α *in* $C([0, 1])$. *Then*

$$\lim_{h \to 0} J^*(\alpha_h, \boldsymbol{q}^h(\alpha_h)) = J^*(\alpha, \boldsymbol{q}(\alpha)) .$$

Proof. Let us denote $\boldsymbol{q}^{0h} = \boldsymbol{q}^* + \boldsymbol{p}^{0h}(\alpha_h)$, $\boldsymbol{q}^0 = \boldsymbol{q}^* + \boldsymbol{p}^0(\alpha)$. Obviously, we have

$$(2.21) \qquad J^*(\alpha_h, \boldsymbol{q}^h(\alpha_h)) = \int_{\hat{D}} F(\boldsymbol{q}^{0h})\, r \,\mathrm{d}r \,\mathrm{d}z - \int_{\hat{D} \dot{-} D_h} F(\boldsymbol{q}^*)\, r \,\mathrm{d}r \,\mathrm{d}z .$$

Proposition 1 implies that $\boldsymbol{q}^{0h} \to \boldsymbol{q}^0$ in $[L_r^2(\hat{D})]^2$. From the continuity of F we conclude

$$(2.22) \qquad \int_{\hat{D}} F(\boldsymbol{q}^{0h})\, r \,\mathrm{d}r \,\mathrm{d}z \to \int_{\hat{D}} F(\boldsymbol{q}^0)\, r \,\mathrm{d}r \,\mathrm{d}z .$$

Making use of (1.8), (1.9), we obtain

$$(2.23) \qquad \int_{\hat{D} \dot{-} D_h} F(\boldsymbol{q}^*)\, r \,\mathrm{d}r \,\mathrm{d}z \to \int_{\hat{D} \dot{-} D(\alpha)} F(\boldsymbol{q}^*)\, r \,\mathrm{d}r \,\mathrm{d}z .$$

It follows from (2.21), (2.22) and (2.23) that

$$J^*(\alpha_h, \boldsymbol{q}^h(\alpha_h)) \to \int_{\hat{D}} F(\boldsymbol{q}^0)\, r \,\mathrm{d}r \,\mathrm{d}z - \int_{\hat{D} \dot{-} D(\alpha)} F(\boldsymbol{q}^*)\, r \,\mathrm{d}r \,\mathrm{d}z =$$
$$= \int_{D(\alpha)} r\, F(\boldsymbol{q}(\alpha)) \,\mathrm{d}r \,\mathrm{d}z = J^*(\alpha, \boldsymbol{q}(\alpha)) .$$

Q.E.D.

Proposition 3. *The Approximate Design Problem* (2.20) *has at least one solution for any* $h = 1/N$.

Proof. It is readily seen that denoting by $a \in R^{N+1}$ the vector of $\beta_h(jh)$, $j = 0, 1, \ldots, N$, we have $\beta_h \in U_{ad}^h$ if and only if $a \in \mathscr{A}$, where $\mathscr{A}$ is a compact subset of R^{N+1}.

One can show that the nodal values of $\mathbf{p}^h(\beta_h) \in S_h$ depend continuously on a, using e.g. the results of Pironneau [5]. Then for the extensions we have

$$\mathbf{p}^{0h}(\beta_h) \to \mathbf{p}^{0h}(\alpha_h) \quad \text{in} \quad \mathscr{H}(\delta) \quad \text{if} \quad \beta_h \to \alpha_h \quad \text{in} \quad C([0, 1]) .$$

Arguing as in the proof of Proposition 2, we obtain that

$$J^*(\beta_h, \mathbf{q}^* + \mathbf{p}^h(\beta_h)) = j(a)$$

depends continuously on a. Consequently, the minimum is attained in the set $\mathscr{A}$.

Theorem 1. *Let* $\{\alpha_h\}$, $h \to 0$, *be a sequence of solutions of the Approximate Optimal Design Problem* (2.20). *Then a subsequence* $\{\alpha_{\hat{h}}\}$ *exists, such that*

$$\alpha_{\hat{h}} \to \alpha^0 \quad \text{in} \quad C([0, 1]) , \tag{2.24}$$

where α^0 *is a solution of the Optimal Design Problem* (1.1).

The approximate solutions $\mathbf{q}^{\hat{h}}(\alpha_{\hat{h}})$ *converge in accordance with Proposition* 1 *to the solution* $\mathbf{q}(\alpha^0)$. *Any uniformly convergent subsequence of* $\{\alpha_h\}$ *has the properties mentioned above.*

Proof. Consider an arbitrary $\beta \in U_{ad}$. There exists a sequence $\{\beta_h\}$, $h \to 0$, $\beta_h \in U_{ad}^h$ such that $\beta_h \to \beta$ in $C([0, 1])$ (for the proof – see the Appnedix in [2]).

Since U_{ad} is compact in $C([0, 1])$, there exists a subsequence $\{\alpha_{\hat{h}}\} \subset \{\alpha_h\}$, such that (2.24) holds and $\alpha^0 \in U_{ad}$. By definition, we have

$$J^*(\alpha_{\hat{h}}, \mathbf{q}(\alpha_{\hat{h}})) \leqq J^*(\beta_{\hat{h}}, \mathbf{q}(\beta_{\hat{h}})) \quad \forall \hat{h} .$$

Passing to the limit with $\hat{h} \to 0$ and using Proposition 2 on both sides, we arive at the inequality

$$J^*(\alpha^0, \mathbf{q}(\alpha^0)) \leqq J^*(\beta, \mathbf{q}(\beta)) .$$

Consequently, α^0 is a solution of the problem (1.12), which is equivalent with (1.1). The rest of the Theorem follows from Proposition 1.

Corollary. *There exists at least one solution of the Optimal Design Problem* (1.1).

Proof follows from Proposition 3 and Theorem 1.

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Souhrn

OPTIMALIZACE OBLASTI V OSOVĚ SYMETRICKÝCH ELIPTICKÝCH ÚLOHÁCH DUÁLNÍ METODOU KONEČNÝCH PRVKŮ

IVAN HLAVÁČEK

V práci se uvažuje osově symetrická eliptická úloha druhého řádu s kombinovanými okrajovými podmínkami. Je třeba najít tvar oblasti, pro který nabývá minima účelový funkcionál, vyjádřený prostřednictvím gradientu řešení. K přibližnému řešení stavové úlohy se používá nové duální metody konečných prvků. Dokazuje se existence optimální oblasti a konvergence přibližných řešení.

Резюме

ОПТИМИЗАЦИЯ ОБЛАСТИ В ОСЕСИММЕТРИЧЕСКИХ ЭЛЛИПТИЧЕСКИХ ЗАДАЧАХ ДВОЙСТВЕННЫМ МЕТОДОМ КОНЕЧНЫХ ЭЛЕМЕНТОВ

IVAN HLAVÁČEK

Рассматривается осесимметрическая эллиптическая задача второго порядка. Требуется найти форму области, для которой целевой функционал, заданный посредством градиента решения, достигает своево минимума. Применяется новый двойственный метод конечных элементов и доказывается существование и сходимость приближенных решений.

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