Jaroslav Haslinger; Václav Horák
On identification of critical curves

Aplikace matematiky, Vol. 35 (1990), No. 3, 169–177

Persistent URL: http://dml.cz/dmlcz/104400

ON IDENTIFICATION OF CRITICAL CURVES

JAROSLAV HASLINGER, VÁCLAV HORÁK

(Received November 15, 1988)

Summary. The paper deals with the problem of finding a curve, going through the interior of the domain Ω, accross which the flux $\partial u/\partial n$, where u is the solution of a mixed elliptic boundary value problem solved in Ω, attains its maximum.

Keywords: Critical curves, mixed elliptic boundary value problem.

AMS Classification: 49A21.

INTRODUCTION

We look for a curve, going through the interior of the domain Ω, across which the flux $\partial u/\partial n$, where u is the solution of a mixed elliptic boundary value problem solved in Ω, attains its maximum. The existence of at least one curve is proved for an appropriate choice of the class of admissible curves. Sensitivity analysis is presented. By means of this approach, the mass movement problems having the importance in stability analysis of constructions, can be solved.

1. SETTING OF THE PROBLEM

Let

$$\Omega = \{(x_1, x_2) \in \mathbf{R}^2 \mid 0 < x_2 < h(x_1),\ x_1 \in (0, 1)\}$$

be a bounded domain, the Lipschitz boundary $\partial\Omega$ of which is decomposed as follows: $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, where

$$\Gamma_2 = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_2 = h(x_1),\ x_1 \in (0, 1)\}$$

and h is Lipschitz continuous in $[0, 1]$.

In Ω the following mixed boundary value problem $(\mathscr{P}')$ is given

$$(\mathscr{P}') \qquad \begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \Gamma_1 \\ \dfrac{\partial u}{\partial n} = g & \text{on} \quad \Gamma_2 \end{cases}$$

with $f \in L^2(\Omega)$, $g \in L^2(\Gamma_2)$. By V we denote the space

$$V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_1\} .$$

The variational form of $(\mathscr{P}')$ reads as follows:

$$(\mathscr{P}) \qquad \begin{cases} \text{Find } u \in V: \\ (\nabla u, \nabla v)_{0,\Omega} = (f, v)_{0,\Omega} + \int_{\Gamma_2} gv \, ds \quad \forall v \in V . \end{cases}$$

Here $\nabla v = (\partial v / \partial x_1, \partial v / \partial x_2)$ and $(\cdot, \cdot)_{0,\Omega}$ stands for the usual scalar product in $L^2(\Omega)$.

Let $0 < \bar{\alpha} < \bar{\beta} < 1, \delta > 0$ be given. By U_{ad} we denote a subset of Lipschitz continuous functions, defined as follows:

$$(1.1) \qquad \begin{aligned} U_{ad} = \{\varphi \mid \exists \alpha \in [0, \bar{\alpha}], \beta \in [\bar{\beta}, 1]: \varphi \in C^{0,1}([\alpha, \beta]) , \\ \varphi(\alpha) = h(\alpha), \varphi(\beta) = h(\beta), \delta \leqq \varphi \leqq h \text{ on } [\alpha, \beta] , \\ |\varphi(x_1) - \varphi(\bar{x}_1)| \leqq C_1 |x_1 - \bar{x}_1| \ \forall x_1, \bar{x}_1 \in [\alpha, \beta] , \\ \text{meas } \Omega(\varphi) = C_2\} , \end{aligned}$$

where

$$\bar{\Omega}(\varphi) = \{(x_1, x_2) \in \mathbf{R}^2 \mid 0 \leqq x_2 \leqq h(x_1) \quad x_1 \in [0, \alpha] \cup [\beta, 1] \\ 0 \leqq x_2 \leqq \varphi(x_1) \quad x_1 \in [\alpha, \beta]\}$$

(see Fig. 1) and C_1, C_2 are positive constants chosen in such a way that $U_{ad} \neq \emptyset$.

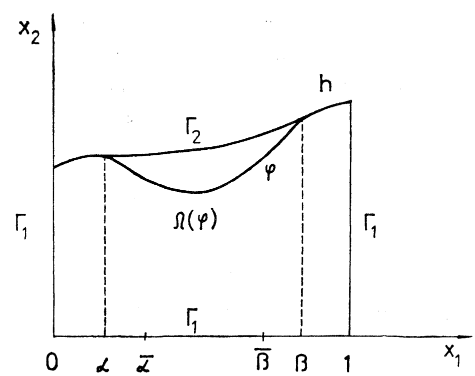

Fig. 1

Set

$$J(\varphi) = \left\langle \frac{\partial u}{\partial n}, 1 \right\rangle_{\partial\Omega(\varphi)} - \int_{\Gamma_2^1(\varphi)} g \, ds - \int_{\Gamma_2^2(\varphi)} g \, ds ,$$

where $\langle , \rangle_{\partial\Omega(\varphi)}$ denotes the duality pairing between $H^{-1/2}(\partial\Omega(\varphi))$ and $H^{1/2}(\partial\Omega(\varphi))$

(for the definition of these see [1] and

$$\Gamma_2^1(\varphi) = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_2 = h(x_1),\ x_1 \in (0, \alpha)\}$$
$$\Gamma_2^2(\varphi) = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_2 = h(x_1),\ x_1 \in (\beta, 1)\}\ .$$

Remark 1.1. If $\partial u/\partial n \in L^2(\partial\Omega(\varphi))$, then the duality pairing $\langle\ ,\ \rangle_{\partial\Omega(\varphi)}$ is represented by a scalar product in $L^2(\partial\Omega(\varphi))$ and

$$J(\varphi) = \int_{\Gamma(\varphi)} \frac{\partial u}{\partial n}\, \mathrm{d}s + \int_{\Gamma_1} \frac{\partial u}{\partial n}\, \mathrm{d}s + \int_{\Gamma_2{}^1(\varphi)} \frac{\partial u}{\partial n}\, \mathrm{d}s + \int_{\Gamma_2{}^2(\varphi)} \frac{\partial u}{\partial n}\, \mathrm{d}s\,,$$

where $\Gamma(\varphi) = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_2 = \varphi(x_1)\ \forall x_1 \in (\alpha, \beta)\}$. Note that the term $\int_{\Gamma_1} \partial u/\partial n\, \mathrm{d}s$ does not depend on φ.

Next we shall study the problem

$$(\mathbf{P}')\qquad \begin{cases}\text{Find } \varphi^* \in U_{\mathrm{ad}} \text{ such that}\\ J(\varphi^*) = \max\limits_{\varphi \in U_{\mathrm{ad}}} J(\varphi)\,.\end{cases}$$

Applying Green's formula

$$(1.2)\qquad (\nabla u, \nabla v)_{0,\Omega(\varphi)} = (-\Delta u, v)_{0,\Omega(\varphi)} + \left\langle \frac{\partial u}{\partial n}, v \right\rangle_{\partial\Omega(\varphi)} \qquad \forall v \in H^1(\Omega(\varphi))$$

with the special choice $v \equiv 1$ and using the fact that u solves $(\mathscr{P})$ we see that $J(\varphi)$ can be expressed as follows:

$$-\mathscr{I}(\varphi) \equiv J(\varphi) = -\int_{\Omega(\varphi)} f\, \mathrm{d}x - \int_{\Gamma_2{}^1(\varphi)} g\, \mathrm{d}s - \int_{\Gamma_2{}^2(\varphi)} g\, \mathrm{d}s\,.$$

Then $(\mathbf{P}')$ is equivalent to

$$(\mathbf{P})\qquad \begin{array}{l}\text{Find } \varphi^* \in U_{\mathrm{ad}} \text{ such that}\\ \mathscr{I}(\varphi^*) = \min\limits_{\varphi \in U_{\mathrm{ad}}} \mathscr{I}(\varphi)\,.\end{array}$$

2. EXISTENCE OF A SOLUTION OF (P)

The aim of this section is to establish the existence of at least one solution of $(\mathbf{P})$. We have

Theorem 2.1. *Let $U_{\mathrm{ad}} \neq \emptyset$. Then there exists at least one solution of* $(\mathbf{P})$.

Proof. Let $\{\varphi_n\}$, $\varphi_n \in U_{\mathrm{ad}}$ be a minimizing sequence of $(\mathbf{P})$, i.e.

$$q \equiv \inf_{\varphi \in U_{\mathrm{ad}}} \mathscr{I}(\varphi) = \lim_{n \to \infty} \mathscr{I}(\varphi_n)\,.$$

Functions φ_n are defined on $[\alpha_n, \beta_n]$, $\alpha_n \in [0, \bar{\alpha}]$, $\beta_n \in [\bar{\beta}, 1]$. There exist subsequences of $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ (denoted by the same symbol) and numbers α^*, β^*, $\alpha^* \in [0, \bar{\alpha}]$, $\beta^* \in [\bar{\beta}, 1]$ such that

$$(2.1) \qquad \alpha_n \to \alpha^* , \quad \beta_n \to \beta^* , \quad n \to \infty .$$

Denote

$$\bar{\Omega}_n = \{(x_1, x_2) \in \mathbf{R}^2 \mid 0 \leqq x_2 \leqq h(x_1) \;\; x_1 \in [0, \alpha_n] \cup [\beta_n, 1] \\ 0 \leqq x_2 \leqq \varphi(x_1) \;\; x_1 \in [\alpha_n, \beta_n]\} .$$

Let m be an integer and I_m the interval $I_m = [\alpha^* + 1/m, \beta^* - 1/m]$.

Let m be fixed. Then φ_n are defined on I_m for n sufficiently large. As $\varphi_n|_{I_m}$ satisfy all assumptions of the Ascoli-Arzela theorem one can find a subsequence $\{\varphi_{n^1}\}$ of $\{\varphi_n\}$ and a function $\varphi^{(m)} \in C(I_m)$ such that

$$(2.2) \qquad \varphi_{n^1} \rightrightarrows \varphi^{(m)} \quad \text{(uniformly) in } I_m .$$

Now, replacing m by $(m + 1)$, one can find a subsequence $\{\varphi_{n^2}\}$ of $\{\varphi_{n^1}\}$ and a function $\varphi^{(m+1)} \in C(I_{m+1})$ such that

$$(2.3) \qquad \varphi_{n^2} \rightrightarrows \varphi^{(m+1)} \quad \text{in} \quad I_{m+1} .$$

Clearly $\varphi^{(m+1)} = \varphi^{(m)}$ in I_m. Repeating the same procedure for any integer m and passing to the diagonal subsequence determined by means of $\{\varphi_{n^1}\}, \{\varphi_{n^2}\}, \ldots$ one can construct a sequence (denoted by $\{\varphi_n\}$) such that

$$\varphi_n \rightrightarrows \varphi^* , \quad n \to \infty \quad \text{in} \quad I_m ,$$

for any integer m, where

$$\varphi^* \equiv \varphi^{(m)} \quad \text{in} \quad I_m .$$

It is easy to see that $\varphi^* \in U_{\text{ad}}$. Indeed,

$$C_2 = \text{meas}\, \Omega_n = \\ = \int_\Omega \mathrm{d}x - \int_{\alpha_n}^{\beta_n} \int_{\varphi_n(x_1)}^{h_n(x_1)} \mathrm{d}x = \int_\Omega - \int_{I_m} \int_{\varphi_n(x_1)}^{h(x_1)} \mathrm{d}x + \int_{O_m} \int_{\varphi_n(x_1)}^{h(x_1)} \mathrm{d}x ,$$

where meas $O_m \to 0$ as $m \to \infty$. Keeping m fixed and $n \to \infty$, we have

$$(2.4) \qquad C_2 = \int_\Omega \mathrm{d}x - \int_{I_m} \int_{\varphi^*(x_1)}^{h(x_1)} \mathrm{d}x - \int_{O_m} \int_{\varphi^*(x_1)}^{h(x_1)} \mathrm{d}x .$$

Letting $m \to \infty$, we finally obtain

$$C_2 = \int_\Omega \mathrm{d}x - \int_{\alpha^*}^{\beta^*} \int_{\varphi^*(x_1)}^{h(x_1)} \mathrm{d}x = \text{meas}\, \Omega(\varphi^*) .$$

Further,

$$\varphi^*(\alpha^* + 1/m) = \lim_{n \to \infty} (\varphi_n(\alpha^* + 1/m) - \varphi_n(\alpha_n)) + \lim_{n \to \infty} \varphi_n(\alpha_n) = \\ = \lim_{n \to \infty} (\varphi_n(\alpha^* + 1/m) - \varphi_n(\alpha_n)) + \lim_{n \to \infty} h(\alpha_n) = c(m) + h(\alpha^*) ,$$

where $c(m) \to 0$ if $m \to \infty$. Thus $\varphi^*(\alpha^*) = h(\alpha^*)$ and similarly $\varphi^* (\beta^*) = h(\beta^*)$.

The other conditions, appearing in the definition of U_{ad}, are satisfied as well. Now we will show that

$$\lim_{n\to\infty} \mathscr{I}(\varphi_n) = \mathscr{I}(\varphi^*) . \tag{2.5}$$

Indeed,

$$\lim_{n\to\infty} \int_{\Gamma_2{}^1(\varphi_n)} g \, ds = \lim_{n\to\infty} \int_0^{\alpha_n} g \sqrt{(1 + (h')^2)} \, dx_1 = \tag{2.6}$$

$$= \int_0^{\alpha^*} g \sqrt{(1 + (h')^2)} \, dx_1 = \int_{\Gamma_2{}^1(\varphi^*)} g \, ds .$$

Similarly,

$$\lim_{n\to\infty} \int_{\Gamma_2{}^2(\varphi_n)} g \, ds = \int_{\Gamma_2{}^2(\varphi^*)} g \, ds .$$

Further,

$$\lim_{n\to\infty} \int_{\Omega(\varphi_n)} f \, dx = \int_\Omega f \, dx - \lim_{n\to\infty} \int_{\alpha_n}^{\beta_n} \int_{\varphi_n(x_1)}^{h(x_1)} f \, dx =$$

$$= \int_\Omega f \, dx - \int_{\alpha^*}^{\beta^*} \int_{\varphi^*(x_1)}^{h(x_1)} f \, dx = \int_{\Omega(\varphi^*)} f \, dx .$$

This together with (2.6) yields (2.5). □

Remark 2.1. The solution $\varphi^* \in U_{ad}$ of (**P**) is non-unique, in general. Indeed, let f be a constant in Ω and $g \equiv 0$ on Γ_2. Then

$$\mathscr{I}(\varphi) = \int_{\Omega(\varphi)} f \, dx = f \text{ meas } \Omega(\varphi) = C_2 f ,$$

i.e. $\mathscr{I}$ is constant on U_{ad}.

It is possible to assume another choice of U_{ad}, namely

$$\begin{aligned} U_{ad} = \{\varphi \mid \exists \alpha \in [0, \bar{\alpha}], \beta \in [\bar{\beta}, 1] &: \varphi \in C^{1,1}([\alpha, \beta]) \\ \varphi(\alpha) = h(\alpha), \ \varphi(\beta) = h(\beta), \ & \delta \leqq \varphi \leqq h \text{ on } [\alpha, \beta] , \\ |\varphi'(x_1)| \leqq C_1 \text{ on } (\alpha, \beta), \ & |\varphi''(x_1)| \leqq C_2 \text{ a.e.} \\ \text{in } (\alpha, \beta) \text{ and } l(\varphi) = & \text{ length } (\varphi) = C_3\} , \end{aligned} \tag{2.7}$$

i.e. U_{ad} is a subset of functions which are Lipschitz continuous together with their first derivatives in $[\alpha, \beta]$ and have a constant length. C_1, C_2, C_3 are positive constants chosen in such a way that $U_{ad} \neq \emptyset$. We assume the problem (**P**) with the same cost functional $\mathscr{I}$ but with U_{ad} given by (2.7). Using the same approach as before, one can prove

Theorem 2.2. *Let U_{ad}, given by (2.7), be non-empty. Then* (**P**) *has at least one solution φ^*.*

Proof. In the same way as in Theorem 2.1 one can find a sequence $\{\varphi_n\}_{n=1}^\infty$, $\varphi_n \in U_{ad}$ such that

$$\varphi_n \to \varphi^* , \quad n \to \infty \quad \text{in} \quad C^1(I_m)$$

for any integer m. Let us prove that $\varphi^* \in U_{ad}$. It is sufficient to show that $l(\varphi^*) = C_3$. Indeed,

$$(2.8) \qquad C_3 = l(\varphi_n) = \int_{\alpha_n}^{\beta_n} \sqrt{(1 + (\varphi_n')^2)}\,dx_1 = \\ = \int_{I_m} \sqrt{(1 + (\varphi_n')^2)}\,dx_1 + \int_{O_m} \sqrt{(1 + (\varphi_n')^2)}\,dx_1 = \\ = \int_{I_m} \sqrt{(1 + (\varphi_n')^2)}\,dx_1 + c(m),$$

where $c(m) \to 0$, $m \to \infty$ as meas $O_m \to 0$ for $m \to \infty$. Using the fact that $\varphi_n' \rightrightarrows \varphi^{*\prime}$, $n \to \infty$ in I_m for any integer m, one has

$$\int_{I_m} \sqrt{(1 + (\varphi_n')^2)}\,dx_1 \to \int_{I_m} \sqrt{(1 + (\varphi^{*\prime})^2)}\,dx_1 .$$

Finally, letting $m \to \infty$ we obtain from this and (2.8) that $l(\varphi^*) = C_3$. □

Sometimes one wishes to identify a curve $\varphi^* \in U_{ad}$ for which $\mathcal{I}(\varphi)$ is either equal to k or as close as possible to the given value k. In such a case we set

$$\tilde{\mathcal{I}}(\varphi) = (\mathcal{I}(\varphi) - k)^2$$

and define the problem

$$(\mathbf{P}_1) \qquad \begin{cases} \text{find} \quad \varphi^* \in U_{ad} \quad \text{such that} \\ \tilde{\mathcal{I}}(\varphi^*) \leqq \tilde{\mathcal{I}}(\varphi) \quad \forall \varphi \in U_{ad} \end{cases}$$

with U_{ad} given by (1.1), (2.7), respectively. Using exactly the same approach as before, one can prove

Theorem 2.3. *Let $U_{ad} \neq \emptyset$. Then $(\mathbf{P}_1)$ has at least one solution.*

3. SENSITIVITY ANALYSIS

Application of optimization methods for the minimization of $\mathcal{I}$ over U_{ad} usually requires the knowledge of the gradient of $\mathcal{I}$. The aim of this section is to derive the explicit form of the derivative of $\mathcal{I}$.

Let us assume a mapping F_t: $\mathbf{R}^2 \to \mathbf{R}^2$ given by

$$F_t(x_1, x_2) = (x_1, x_2) + t\mathcal{V}(x_1, x_2), \quad t > 0, \quad (x_1, x_2) \in \Omega, \quad \mathcal{V} \in \mathcal{M}.$$

Here $\mathcal{M}$ denotes the family of vector fields $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2)$ satisfying

$$\mathcal{M} = \{\mathcal{V} \in (H^1(\Omega))^2 \mid \mathcal{V} \equiv 0 \text{ on } \Gamma_1, \ F_t(\Gamma_2) = \Gamma_2 \text{ for } t > 0 \\ \text{sufficiently small}\}.$$

From the definition of $\mathcal{M}$ we easily deduce that $F_t(\Omega) = \Omega$ for $t > 0$ sufficiently small. Denote by $\Omega_t(\varphi) = F_t(\Omega(\varphi))$, $\Gamma_{2t}^i = F_t(\Gamma_2^i(\varphi))$ the images of $\Omega(\varphi)$ and $\Gamma_2^i(\varphi)$, $i = 1, 2$, respectively, and

$$\mathcal{I}_t(\varphi) = \int_{\Omega_t(\varphi)} f\,dx + \int_{\Gamma_{2t}^1(\varphi)} g\,ds + \int_{\Gamma_{2t}^2(\varphi)} g\,ds .$$

Our aim will be to calculate

$$\dot{\mathscr{I}}(\varphi, \mathscr{V}) = \frac{\mathrm{d}}{\mathrm{d}t} \mathscr{I}_t(\varphi)\Big|_{t=0} .$$

It is known (see [2]) that

$$(3.1) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_t(\varphi)} f \,\mathrm{d}x\right)\Big|_{t=0} = \int_{\Omega(\varphi)} \dot{f} \,\mathrm{d}x + \int_{\Omega(\varphi)} f \operatorname{div} \mathscr{V} \,\mathrm{d}x ,$$

where $\dot{f}$ denotes the material derivative of f, given by

$$\dot{f} = \frac{\partial f}{\partial t} + \nabla f \,.\, \mathscr{V} = \nabla f \,.\, \mathscr{V}$$

Here we have made use of the fact that f does not depend on t. Applying Green's formula to the second term of (3.1) and using the definition of $\mathscr{M}$, we see that (3.1) reduces to

$$(3.2) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_t(\varphi)} f \,\mathrm{d}x\right)\Big|_{t=0} = \int_{\Gamma(\varphi)} f \mathscr{V}_n \,\mathrm{d}s + \int_{\Gamma_2{}^1(\varphi)} f \mathscr{V}_n \,\mathrm{d}s + \int_{\Gamma_2{}^2(\varphi)} f \mathscr{V}_n \,\mathrm{d}s .$$

Let us calculate $\mathrm{d}/\mathrm{d}t \left(\int_{\Gamma_{2t}{}^1(\varphi)} g \,\mathrm{d}s\right)\big|_{t=0}$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Gamma_{2t}{}^1(\varphi)} g \,\mathrm{d}s\right)_{t=0} = \lim_{t \to 0+} \frac{1}{t}\left(\int_{\Gamma_{2t}{}^1(\varphi)} g \,\mathrm{d}s - \int_{\Gamma_2{}^1(\varphi)} g \,\mathrm{d}s\right) =$$

$$= \lim_{t \to 0+} \frac{1}{t}\left(\int_0^{\alpha + t\mathscr{V}_1(\alpha, h(\alpha))} g \sqrt{(1 + (h')^2)} \,\mathrm{d}x_1 - \int_0^\alpha g \sqrt{(1 + (h')^2)} \,\mathrm{d}x_1\right) =$$

$$= \lim_{t \to 0+} \frac{1}{t} \int_\alpha^{\alpha + t\mathscr{V}_1(\alpha, h(\alpha))} g \sqrt{(1 + (h')^2)} \,\mathrm{d}x_1 =$$

$$= g(\alpha) \sqrt{(1 + (h'(\alpha))^2)}\, \mathscr{V}_1(\alpha, h(\alpha)) .$$

Similarly

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Gamma_{2t}{}^2(\varphi)} g \,\mathrm{d}s\right) = \lim_{t \to 0+} \frac{1}{t}\left(_{\Gamma_{2t}{}^2(\varphi)} g \,\mathrm{d}s - \int_{\Gamma_2{}^2(\varphi)} g \,\mathrm{d}s\right) =$$

$$= \lim_{t \to 0+} \frac{1}{t}\left(\int_{\beta + t\mathscr{V}_1(\beta, h(\beta))}^1 g \sqrt{(1 + (h')^2)} \,\mathrm{d}x_1 - \int_\beta^1 g \sqrt{(1 + (h')^2)} \,\mathrm{d}x_1\right) =$$

$$= \lim_{t \to 0+} \frac{1}{t} \int_{\beta + t\mathscr{V}_1(\beta, h(\beta))}^\beta g \sqrt{(1 + (h')^2)} \,\mathrm{d}x_1 =$$

$$= -g(\beta) \sqrt{(1 + (h')^2 (\beta))}\, \mathscr{V}_1(\beta, h(\beta)) .$$

From this and (3.2) we finally obtain

$$(3.3)\qquad \dot{\mathscr{I}}(\varphi, \mathscr{V}) = \int_{\Gamma(\varphi)} f\mathscr{V}_n \,\mathrm{d}s + \\ + \int_{\Gamma_2{}^1(\varphi)} f\mathscr{V}_n \,\mathrm{d}s + \int_{\Gamma_2{}^2(\varphi)} f\mathscr{V}_n \,\mathrm{d}s - g(\beta)\sqrt{(1 + (h'(\beta))^2)}\,\mathscr{V}_1(\beta, h(\beta)) + \\ + g(\alpha)\sqrt{(1 + (h'(\alpha))^2)}\,\mathscr{V}_1(\alpha, h(\alpha)).$$

If Ω is a rectangle, then $h' = 0$ in $(0, 1)$, $\mathscr{V} = (\mathscr{V}_1, 0)$ and (3.3) takes the simpler form

$$(3.4)\qquad \dot{\mathscr{I}}(\varphi, \mathscr{V}) = -\int_{\Gamma(\varphi)} f\varphi'/\sqrt{(1 + (\varphi')^2)}\,\mathscr{V}_1 \,\mathrm{d}s - g(\beta)\mathscr{V}_1(\beta, h(\beta)) + \\ + g(\alpha)\,\mathscr{V}_1(\alpha, h(\alpha)),$$

as $n_1 = -\varphi'/\sqrt{(1 + (\varphi')^2)}$ on $\Gamma(\varphi)$ and $n_1 = 0$ on $\Gamma_2^1(\varphi) \cup \Gamma_2^2(\varphi)$.

Let U_{ad} be given by (1.1) and let a solution $\varphi^*\colon [\alpha^*, \beta^*] \to \mathbf{R}^1$ of the problem (**P**) be such that there exists a constant $0 < C_1' < C_1$:

$$|\varphi^*(x_1) - \varphi^*(\bar{x}_1)| \leqq C_1'|x_1 - \bar{x}_1| \quad \forall x_1, \bar{x}_1 \in [\alpha^*, \beta^*]$$

and

$$\delta < \varphi^*(x_1) < h(x_1) \quad \forall x_1 \in (\alpha^*, \beta^*).$$

The remaining constraint (constant volume) can be removed by introducing the lagrangian

$$(3.5)\qquad \mathscr{L}(\varphi) = \mathscr{I}(\varphi) - \lambda(\operatorname{meas}\Omega(\varphi) - C_2), \quad \lambda \in \mathbf{R}^1.$$

Denote

$$\mathscr{L}_t(\varphi) = \mathscr{I}_t(\varphi) - \lambda(\operatorname{meas}\Omega_t(\varphi) - C_2), \quad t > 0.$$

A necessary condition for φ^* to be a olution of (**P**) with the above mentioned property is

$$\dot{\mathscr{L}}(\varphi^*, \mathscr{V}) = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}_t(\varphi^*)\Big|_{t=0} = \dot{\mathscr{I}}(\varphi^*, \mathscr{V}) - \lambda\int_{\Gamma(\varphi^*)} \mathscr{V}_n \,\mathrm{d}s = 0$$

for any vector field $\mathscr{V} \in \mathscr{M}$ satisfying $\operatorname{supp}\mathscr{V}_i \subset [\alpha^*, \beta^*] \times [0, h(x_1)]$, $x_1 \in [\alpha^*, \beta^*]$, $i = 1, 2$. This and (3.3) lead to

$$f|_{\varphi^*} = \lambda = \text{const}.$$

References

[1] *J. Nečas:* Les méthodes directes en théorie des equations elliptiques. Academia, Praha, 1967.

[2] *J. Haslinger, P. Neittaanmäki:* Finite Element Approximation for Optimal Shape Design: Theory and Applications. John Wiley & Sons, 1988.

Souhrn

IDENTIFIKACE KRITICKÝCH KŘIVEK

JAROSLAV HASLINGER, VÁCLAV HORÁK

V práci je řešena úloha nalézt křivku φ z jisté množiny přípustných křivek, podle níž křivkový integrál z $\partial u/\partial n$, kde u je řešením smíšeného eliptického problému, nabývá svého maxima. Je ukázáno, že za jistých předpokladů alespoň jedna taková křivka existuje a je dána její charakterizace.

Резюме

ОТОЖДЕСТВЛЕНИЕ КРИТИЧЕСКИХ КРИВЫХ

JAROSLAV HASLINGER, VÁCLAV HORÁK

В работе изучается задача нахождения кривой φ из данного множества допустимых кривых, вдоль которой криволинейный интеграль от $\partial u/\partial n$, где u – решение эллиптической задачи, достигает своего максимального значения. Показано, что при некоторых предположениях такая кривая существует.

Authors' addresses: Doc. RNDr. *Jaroslav Haslinger,* CSc., KFK MFF UK, Malostranské nám. 25, 118 00 Praha 1; Doc. Ing. *Václav Horák,* DrSc., KFK MFF UK, Ke Karlovu 3, 120 00 Praha 2.